Vector Curvature And Torsion In 3D

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Abstract

I initially present standard formulas for curvature and torsion as a function of pathlength in 3D space. These standard formulas have sign ambiguity when recovering curvature and torsion from numerical data. I then extend those formulas to present curvature and torsion as vectors, eliminating the sign ambiguity.

Left Hand Frenet-Serret Formulas in 3D Space

The usual Frenet-Serret formulas parameterize a curve using pathlength, and specify the curve using curvature (a measure of deviation from a line) and torsion (deviation from a plane).

The curve is $\vec{x}(s)$ using a conventional right hand coordinate system. However, at each point along the curve, I build a local, left handed coordinate system, with the tangent $\vec{u}$, normal $\vec{n}$ and binormal $\vec{b} = -\vec{u} \times \vec{n}$. My motivation for using a lefthanded local coordiate system is that the left-hand coordinate system extends to four space and beyond with a much simpler sign scheme than a right handed system.

Usually, the Frenet-Serret equations treat curvature and torsion as signed numbers. For creating curves, this is no problem, as an S-shaped curve demonstrates both positive and negative curvature. However, when deducing curvature from an existing curve, the formulas involve square roots, and commonly require the author to use judgement when choosing the sign. An alternative, which I use here, is to treat curvature and torsion as vectors. As the derivative of a unit vector is always normal to the original vector, I can express the derivatives of the unit vectors in three-space as the cross product of the original unit vector and a combination of curvature and torsion vectors as shown below.

$$
\begin{aligned}
(ds)^2 &= d\vec{x} \cdot d\vec{x} \\
\frac{d\vec{x}}{ds} &= \vec{u} \\
\frac{d\vec{u}}{ds} &= \kappa\vec{n} = \vec{u} \times \vec{\kappa} \\
\frac{d\vec{n}}{ds} &= \tau\vec{b} - \kappa\vec{u} = \vec{n} \times (\vec{\tau} + \vec{\kappa}) \\
\frac{d\vec{b}}{ds} &= -\tau\vec{n} = \vec{b} \times \vec{\tau}
\end{aligned}
$$

Now we parameterize a curve by t, rather than by s. We have several formulas of convenience to present.

$$
\begin{aligned}
(ds)^2 &= \vec{v} \cdot \vec{v}(dt)^2 \\
v &= \sqrt{\vec{v} \cdot \vec{v}} \\
v^2 &= \vec{v} \cdot \vec{v} \\
\frac{d}{dt}v^2 &= 2v\frac{dv}{dt} \\
\frac{d}{dt}\vec{v} \cdot \vec{v} &= 2\vec{v} \cdot \vec{a} \\
\frac{dv}{dt} &= \frac{\vec{v} \cdot \vec{a}}{v} \\
ds &= vdt
\end{aligned}
$$

We now solve for κ and $\vec{\kappa}$.

$$
\begin{aligned}
\vec{u} &= \frac{\vec{v}}{v} \\
\kappa\vec{n} &= \frac{d\vec{u}}{ds} = \frac{1}{v}\frac{d}{dt}\left(\frac{\vec{v}}{v}\right) \\
&= \frac{1}{v}\left(\frac{\vec{a}}{v} - \vec{v}\frac{1}{v^2}\frac{dv}{dt}\right) \\
&= \frac{1}{v}\left(\frac{\vec{a}}{v} - \frac{\vec{v}}{v^2}\frac{\vec{v}\cdot\vec{a}}{v}\right) \\
&= \frac{1}{v^4}\left(\vec{a}v^2 - \vec{v}\vec{v}\cdot\vec{a}\right) \\
&= \frac{\vec{v}\times(\vec{a}\times\vec{v})}{v^4} \\
&= \vec{u}\times\left(\frac{\vec{a}\times\vec{v}}{v^3}\right) \\
&= \vec{u}\times\vec{\kappa} \\
\vec{\kappa} &= \frac{\vec{a}\times\vec{v}}{v^3} \\
\vec{n} &= \frac{\vec{u}\times(\vec{a}\times\vec{v})}{|\vec{a}\times\vec{v}|}
\end{aligned}
$$

From the definition of the local frame as a left handed system, we know $\vec{n} = \vec{u}\times\vec{b}$. By inspection with the formula above, we see $\vec{b}||\vec{\kappa}$, and we find (as $\vec{b}$ is mutually perpendicular to $\vec{u}$ and $\vec{n}$)

$$
\vec{b} = \frac{\vec{a}\times\vec{v}}{|\vec{a}\times\vec{v}|}
$$

Solving for τ and $\vec{\tau}$, we find

$$
\begin{aligned}
\vec{b} &= \frac{\vec{a}\times\vec{v}}{|\vec{a}\times\vec{v}|} \\
\frac{d\vec{b}}{ds} &= \frac{1}{v}\frac{d}{dt}\left(\frac{\vec{a}\times\vec{v}}{|\vec{a}\times\vec{v}|}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\vec{b}}{ds} &= \frac{1}{v}\frac{d}{dt}\left(\frac{\vec{a}\times\vec{v}}{|\vec{a}\times\vec{v}|}\right) = \vec{b}\times\vec{\tau} \\
&= \frac{1}{v}\left(\frac{\vec{j}\times\vec{v}}{|\vec{a}\times\vec{v}|} + (\vec{a}\times\vec{v})\,\frac{d}{dt}\frac{1}{\sqrt{(\vec{a}\times\vec{v})\cdot(\vec{a}\times\vec{v})}}\right) \\
&= \frac{1}{v}\left(\frac{\vec{j}\times\vec{v}}{|\vec{a}\times\vec{v}|} + (\vec{a}\times\vec{v})\left[-\frac{1}{2}\frac{2(\vec{j}\times\vec{v})\cdot(\vec{a}\times\vec{v})}{|\vec{a}\times\vec{v}|^3}\right]\right) \\
&= \frac{1}{v|\vec{a}\times\vec{v}|^3}\left[(\vec{j}\times\vec{v})\left[(\vec{a}\times\vec{v})\cdot(\vec{a}\times\vec{v})\right] - (\vec{a}\times\vec{v})\left[(\vec{a}\times\vec{v})\cdot(\vec{a}\times\vec{v})\right]\right] \\
&= \frac{1}{v|\vec{a}\times\vec{v}|^3}\left[(\vec{a}\times\vec{v})\times\left[(\vec{j}\times\vec{v})\times(\vec{a}\times\vec{v})\right]\right] \\
&= \left[\frac{\vec{a}\times\vec{v}}{|\vec{a}\times\vec{v}|}\right]\times\frac{(\vec{j}\times\vec{v})\times(\vec{a}\times\vec{v})}{v(\vec{a}\times\vec{v})\cdot(\vec{a}\times\vec{v})} \\
\vec{\tau} &= \frac{(\vec{j}\times\vec{v})\times(\vec{a}\times\vec{v})}{v(\vec{a}\times\vec{v})\cdot(\vec{a}\times\vec{v})} \\
&= \frac{\vec{v}\left(\vec{j}\cdot(\vec{a}\times\vec{v})\right)}{v\,(\vec{a}\times\vec{v})\cdot(\vec{a}\times\vec{v})} \\
\tau &= \frac{\left(\vec{j}\cdot(\vec{a}\times\vec{v})\right)}{(\vec{a}\times\vec{v})\cdot(\vec{a}\times\vec{v})}
\end{aligned}
$$

We note $\vec{\tau}\perp\vec{\kappa}$, $\vec{\kappa}=\kappa\vec{b}$ and $\vec{\tau}=\tau\vec{u}$. Checking consistency, we examine

$$
\begin{aligned}
\frac{d\vec{n}}{ds} &= \vec{n}\times(\vec{\tau}+\vec{\kappa}) \\
&= \vec{n}\times(\tau\vec{u}+\kappa\vec{b}) \\
&= \tau\vec{b}-\kappa\vec{u}
\end{aligned}
$$

Summarizing, we need velocity and acceleration to find curvature, and we need velocity, acceleration and jerk to find torsion.

$$
\begin{aligned}
\vec{\kappa} &= \frac{\vec{a}\times\vec{v}}{v^3} \\
\vec{\tau} &= \frac{\vec{v}\left(\vec{j}\cdot(\vec{a}\times\vec{v})\right)}{v\,(\vec{a}\times\vec{v})\cdot(\vec{a}\times\vec{v})}
\end{aligned}
$$