Extreme Utility of Nilpotents and Idempotents

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June 30, 2015

Abstract

Idempotents (mathematical objects which square to themselves) and Nilpotents (mathematical objects which square to zero) are powerful mathematical tools found in quantum mechanics, as well as in algebraic descriptions of spacetime. This note discusses idempotents and nilpotents in the context of fundamentals of mathematics, and illustrates their utility in finite dimensional spacetime, as well as their role linearizing complex operations.

Nilpotents and Idempotents are Required for Finite Mathematical Vocabularies

Physics and mathematics make use of sets, groups, rings and other structures which interact among themselves, yet remain finite in scope. Such finiteness is quite unlikely in the most general sense, as will be demonstrated below. Idempotents and nillpotents are the terminating elements in strings of definitions which create finite structures.

Fundamental Mathematical Vocabulary

Peter Rowlands is a physicist with the Oliver Lodge laboratory at the University of Liverpool. He is a man with diverse interests, ranging from history of physics, to information theory, to biology, as well as fundamentals of mathematics. Bernard Diaz is a computer scientist and information theorist, also at the University of Liverpool. My generic reference for this section is *Zero to Infinity*, by Peter Rowlands.

Peter Rowlands and Bernard Diaz have spent quite some time pondering the generic structure of math at an abstract level before functions or even numbers are defined. As such, they are dealing with symbols, and lists of symbols, and very generic interactions. In their articles and the book cited above, these scholars use a computer science Turing machine approach to explaining the hierarchy of math symbols. They present a nice, two step algorithm for extending their model. The computer science model is not necessary for explaining their model. It is merely one method, which matches their common background. In my presentation of their work below, I will not use the computer science frame, but jump directly into symbols and vocabularies.

Start With Zero

The starting point for their mathematical vocabulary, is zero, as in the null set, or the number zero. At this initial point, we don't necessarily have numbers defined, but we need a starting point, so we *define* 0 as the initial symbol in our mathematical vocabulary.

I point out at this point, that zero is our first nilpotent as well as idempotent, as $0^2 = 0$.

Our dictionary at this point, is $\{0\}$.

Keep Zero in the Center of Mathematical Structures

Physics and mathematics *love* equations. If we take our current vocabulary, which consists only of zero, we can write the trivial equation

$$0 = 0$$

When we start adding symbols to our mathematical vocabulary, if we want to have equations (which we do), we will necessarily need to have a negation operator to provide negative values.

In earlier writings, Rowlands and Diaz use a minus sign (as I will) to indicate negation. In their later writings, Rowlands and Diaz use a conjugation symbol, to emphasize that, at this vocabulary level of discussion, neither addition nor subtraction have yet been defined.

First Extension and Real Numbers

We will now add a new symbol to our mathematical vocabulary. Let's use the letter R, suggested by real numbers. If we look at a trivial equation using R, we have

$$\begin{aligned} R &= R \\ R + (-R) &= 0 \end{aligned}$$

We see that we actually introduced two symbols, R and $(-R)$. Keeping zero in the center of our mathematical hierarchy will always result in dual, complementary symbols introduced into our vocabulary at every extension.

At this stage, we now have a new dictionary $\{(-R), 0, R\}$. Unlike our first stage $\{0\}$, we now have direction. However, we cannot assign positive or negative to R or $(-R)$. We can only say that they are in opposite directions with respect to each other.

Peter makes an interesting point about integers versus reals. When we learn numbers as children, we learn integers first, then fractions, then real numbers. In reality, real numbers are the most fundamental of these classes. Integers and fractions are subsets of real numbers. In a bit, we will see how we get integers from reals, via idempotents. In our present vocabulary, we don't yet have integers.

Simplest Interactions, and Potents

We have extended our vocabulary to $\{(-R), 0, R\}$. We now check the interactions between these elements. We have not defined functions, such as addition or multiplication or exponentiation. However, we can use a multiplication table format to identify the composite symbols which will arise due to interactions between our symbols. In general, terms will be order sensitive. Using subtraction as an example, $(a - b)$ is usually different from $(b - a)$.

In the table below, prefactors are on left and postfactors are on top.

	$(-R)$	0	R
$(-R)$	$(-R)(-R)$	$(-R)0$	$(-R)R$
0	$0(-R)$	$0\ 0$	$0\ R$
R	$R(-R)$	$R\ 0$	$R\ R$

We look at this table, and have the immediate realization, that in the most general case, we have just expanded our vocabulary with nine new terms. If we add these nine new symbols to our existing three, and look at the next level of interactions, we find $12^2 = 144$ new terms in our vocabulary. In this most general of cases, the symbol table recursively inflates to infinity (and beyond)[1].

At this point, we usually shift from most general vocabulary, to specific implementation. For each entry in the interaction table above, we can either assign a symbol from our existing vocabulary, or 'kick the can' and deal with the new symbol in a later iteration.

Specific Case: Real Numbers

In this case, we are now manually assigning an existing meaning to our new symbols. Using real numbers and multiplication as a template, we assign

	$(-R)$	0	R
$(-R)$	$(-R)(-R) = R$	$(-R)0 = 0$	$(-R)R = (-R)$
0	$0(-R) = 0$	$0\ 0 = 0$	$0\ R = 0$
R	$R(-R) = (-R)$	$R\ 0 = 0$	$R\ R = R$

In our assignment process, we have created two idempotents: $R\ R = R$ and $0\ 0 = 0$.

[1]Buzz Lightyear

Our symbol table, or alphabet, if you will, is $\{(-R), 0, R\}$. Our manually assigned interaction table is

	$(-R)$	0	R
$(-R)$	R	0	$(-R)$
0	0	0	0
R	$(-R)$	0	R

Everything is neat, tidy and self-contained. For this particular implementation, we have a few nice features. If we define our interaction as multiplication, from the idempotent equation $R^2 = R$, we have the equivalent equations $R^2 - R = 0$, or $R(R-1) = 0$, which has the solutions $R = 0$ and $R = 1$, both of which are idempotents. Furthermore, the products of these two idempotents is zero, a feature we will seek among other idempotents later. Further still, our idempotent equation has defined the integer distance between 0 and 1. Further even more, we have now provided a method to distinguish between positive versus negative numbers, via our manually defined relationships $R^2 = R$ and $(-R)^2 = R$.

Using the interaction of multiplication, and the result $R = 1$, we now have a multiplication table for unit basis real numbers.

	(-1)	0	1
(-1)	1	0	(-1)
0	0	0	0
1	(-1)	0	1

This particular table has been handpicked, based upon prior knowledge of real numbers. In the more generic case, where each table element is chosen from one of our three symbols, we would have $3^9 = 19683$ different interaction tables. If we allow deferred evaluation symbols, our number of tables and algebra increases. Are there any other interesting algebras among the 19683 potential self-contained interaction tables? (Of course, there are.)

Specific Case: Trinary Saturating OR Logic

Table 1 is an example of a different interaction table, among our 19683 candidates, chosen to complete a mathematical vocabulary at a finite level. This case is a saturating three level (trinary) logic based upon saturating addition. Notice that by closing out our interaction table using existing symbols, we have automatically generated idempotents.

	$(-R)$	0	R
$(-R)$	$(-R)$	$(-R)$	0
0	$(-R)$	0	R
R	0	R	R

Table 1: Trinary OR Logic Specific Implementation

	$(-a)$	(-R)	0	R	a
$(-a)$	b	a	0	$(-a)$	$(-b)$
$(-R)$	a	R	0	$(-R)$	$(-a)$
0	0	0	0	0	0
R	$(-a)$	$(-R)$	0	R	a
a	$(-b)$	$(-a)$	0	a	b

Table 2: Deferred Completion Table

Specific Case: Deferred Completion

Can we have an extension which is not self-contained at this level, but becomes complete later? (Certainly.) Table 2 is such an example. I start with the initial symbol 0, extended with R just as before. My vocabulary is $\{(-R), 0, R\}$. I flatten this interaction table for real numbers just as previously.

I then extend this with the new symbol a and it's complement. My vocabulary is now $\{(-a), (-R), 0, R, a\}$. I create my interaction table, and flatten with a twist: I introduce a new symbol b not yet defined.

As I have undefined symbols, I need to extend the interaction once again, including b, it's complement, as well as mutual interaction terms such as ab, ba, bR and so on. This process of definition ends only when I have nilpotents or idempotents closing out my consistent set of definitions.

Specific Case: Wedge Product

The previous examples have only had zero as a nilpotent. We now use the example of Grassman algebras and the wedge product to show how nilpotents also effect a closure of our mathematical vocabulary. Start as before with a vocabulary $\{(-a), (-R), 0, R, a\}$. In this new interaction table, have the

	$(-a)$	(-R)	0	R	a
$(-a)$	0	a	0	$(-a)$	0
$(-R)$	a	R	0	$(-R)$	$(-a)$
0	0	0	0	0	0
R	$(-a)$	$(-R)$	0	R	a
a	0	$(-a)$	0	a	0

Table 3: Grassman Wedge Product Table for Scalars and One Vector

square of a be zero. Once again, we have a neat, tidy table with no undefined elements, courtesy of both idempotents, and nilpotents.

I conclude this section with the statement that idempotents and nilpotents allow the formation of finite vocabulary mathematical systems.

Nilpotents and Idempotents in Calculations

Math and physics make great use of polynomials to represent non-linear functions, as well as solutions to differential equations. When nilpotents and idempotents enter into polynomials, some amazing simplifications occur.

Start with a nilpotent z which squares to zero. A real number r combined with z has the following powers.

$$\begin{aligned}
(r+z)^2 &= r^2 + 2rz + z^2 \\
&= r^2 + 2rz \\
(r+z)^3 &= (r+z) * (r^2 + 2rz) \\
&= r^3 + 3r^2 z \\
(r+z)^4 &= r^4 + 4r^3 z
\end{aligned}$$

What is happening here, is that in our binomial expansion, all terms with the power of z greater than one, drop out. We have a structure which is stubbornly linear in z, no matter how we try to non-linearly manipulate it.

How about taking the exponential of this combination?

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \\
e^{(r+z)} &= 1 + (r+z) + \frac{(r+z)^2}{2!} + \frac{(r+z)^3}{3!} + \frac{(r+z)^4}{4!} + \dots \\
&= 1 + (r+z) + \frac{r^2 + 2rz}{2!} + \frac{r^3 + 3r^2 z}{3!} + \frac{r^4 + 4r^3 z}{4!} + \dots \\
e^{(r+z)} &= \left[1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \frac{r^4}{4!} + \dots\right] + \\
& \quad z\left[1 + \frac{2r}{2!} + \frac{3r^2}{3!} + \frac{4r^3}{4!} + \dots\right] \\
e^{(r+z)} &= \left[1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \frac{r^4}{4!} + \dots\right] + \\
& \quad z\left[1 + \frac{r}{1} + \frac{r^2}{2!} + \frac{r^3}{3!} + \dots\right] \\
e^{(r+z)} &= e^r + z e^r = (1+z)e^r
\end{aligned}
$$

This same result is consistent with the series expansion, $e^z = (1+z)$, with all terms zero.

Now we look at powers of real and idempotent combinations. I will use the letter P for my idempotent, with $P^2 = P$ the defining condition.

$$
\begin{aligned}
(r+P)^2 &= r^2 + 2rP + P^2 \\
&= r^2 + (2r+1)P \\
(r+P)^3 &= r^3 + 3r^2 P + 3rP^2 + P^3 \\
&= r^3 + (3r^2 + 3r + 1)P \\
(r+P)^4 &= r^4 + 4r^3 P + 6r^2 P^2 + 4rP^3 + P^4 \\
&= r^4 + (4r^3 + 6r^2 + 4r + 1)P
\end{aligned}
$$

We see again, stubbornly linear behavior, this time in P.

Let's take the exponential of P.

$$\begin{aligned} e^P &= 1 + P + \frac{P^2}{2!} + \frac{P^3}{3!} + \frac{P^4}{4!} + \dots \\ &= 1 + P + \frac{P}{2!} + \frac{P}{3!} + \frac{P}{4!} + \dots \\ &= 1 + P\left(1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots\right) \\ &= 1 + P(e-1) \end{aligned}$$

With this as guidance, we expect

$$e^{(r+P)} = e^r + Pe^r(e-1)$$

Next, examine the case of two orthogonal idempotents. In geometric algebra, these terms arise in the simplest form as expressions involving unit directions, such as a_x, in the form

$$\begin{aligned} P_+ &= \frac{1}{2} + \frac{a_x}{2} \\ P_- &= \frac{1}{2} - \frac{a_x}{2} \qquad \text{where} \\ a_x^2 &= 1 \end{aligned}$$

Verifying,

$$\begin{aligned} (P_+)^2 &= \left(\frac{1}{2} + \frac{a_x}{2}\right)^2 = \frac{1}{4} + 2\frac{1}{2}\frac{a_x}{2} + \frac{a_x^2}{4} = \frac{1}{2} + \frac{a_x}{2} = P_+ \\ (P_-)^2 &= \left(\frac{1}{2} - \frac{a_x}{2}\right)^2 = \frac{1}{4} - 2\frac{1}{2}\frac{a_x}{2} + \frac{a_x^2}{4} = \frac{1}{2} - \frac{a_x}{2} = P_- \\ (P_+)(P_-) &= \left(\frac{1}{2} + \frac{a_x}{2}\right)\left(\frac{1}{2} - \frac{a_x}{2}\right) = \frac{1}{4} - \frac{1}{4} = 0 \end{aligned}$$

Now look at a general combination of two orthogonal idempotents.

$$\begin{aligned} (aP_+ + bP_-)^2 &= a^2P_+^2 + 2abP_+P_- + b^2P_-^2 \\ &= a^2P_+ + b^2P_- \\ (aP_+ + bP_-)^3 &= a^3P_+^3 + 3a^2P_+^2bP_- + 3aP_+b^2P_-^2 + b^3P_-^3 \\ &= a^3P_+ + b^3P_- \\ (aP_+ + bP_-)^n &= a^nP_+ + b^nP_- \end{aligned}$$

Let's take the exponential of this combination.

$$
\begin{aligned}
e^{(aP_+ + bP_-)} &= 1 + (aP_+ + bP_-) + \frac{(aP_+ + bP_-)^2}{2!} + \frac{(aP_+ + bP_-)^3}{3!} + \dots \\
&= 1 + (aP_+ + bP_-) + \frac{(a^2P_+ + b^2P_-)}{2!} + \frac{(a^3P_+ + b^3P_-)}{3!} + \dots \\
&= 1 + P_+(-1 + e^a) + P_-(-1 + e^b) \\
&= 1 - (P_+ + P_-) + e^aP_+ + e^bP_- \\
&= e^aP_+ + e^bP_-
\end{aligned}
$$

We see again, that idempotents and nilpotents keep the system linear with regard to these potents, regardless of our nonlinear polynomial abuse.

Potents and Quantum Mechanics

Nilpotents, idempotents and anti-idempotents (which square like $N^2 = -N$) are fundamental to quantum mechanics.

Quantum mechanics originated in atomic spectroscopy, by explaining the spectrum of light emitted by the hydrogen atom. Empirically, light emitted by hydrogen atoms was known to be at frequencies given by

$$
\begin{aligned}
f(n, m) &= \frac{13.6\ eV}{h\ (\text{ in eV sec})}\left(\frac{1}{n^2} - \frac{1}{m^2}\right) \\
&= 3.2884 \cdot 10^{15}\left(\frac{1}{n^2} - \frac{1}{m^2}\right)\ \text{Hz}
\end{aligned}
$$

where $n \geq 1$ and $m > n$.

When $n = 1$, we have the Lyman series, at ultraviolet frequencies of 2.46654e+15 Hz, 2.92331e+15 Hz, 3.08317e+15 Hz and so on. When $n = 2$, we have the Balmer series, in the visible range at frequencies of 4.56766e+14 Hz, 6.16635e+14 Hz, 6.90631e+14 Hz and so on. When $n = 3$, we have the Paschen series, falling in the red to infrared, at frequencies of 1.59868e+14 Hz, 2.33864e+14 Hz, 2.7406e+14 Hz and so forth.

In electronics, single frequency oscillators are easy to create, for example using LC tanks and amplifiers, and are characterized by a second order linear differential equation. However, in the presense of non-linear media, such as diodes or non-linear optics (such as in green laser points), harmonics are generated at frequencies of 2x, 3x, 4x, and so on times the fundamental frequency.

Given that electric fields have a $1/r^2$ dependency, and electric potentials have a $1/r$ dependency, we expect non-linearities to be easily expressed. However, as seen in the frequency list above, harmonics really aren't present. We seem to start from a highest possible frequency, and define lower possible frequencies, rather than having a fundamental frequency, and generating higher multiple harmonics.

The absense of harmonics in atomic spectra is a clue that, somehow, this system enforced strict linearity in the describing mathematics, and avoided non-linearities. This clue indicates that we should express our physics in terms of nilpotents, idempotents and anti-idempotents, and let the non-linearities fall out of the math.

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