Unary Geometric Algebra Operators and Determinants

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Abstract

Using real matrix implementations of three dimensional Euclidean geometric algebra and Minkowski geometric algebra, we can define the determinant of a multivector, which can be used as a measure of magnitude. Likewise, we can define unary operators which change the sign of various multivector components in a fashion similar to complex conjugation, which preserve the value of the determinant, but zero out substantial portions of the multivector product with the original multivector. This note documents the determinants in Euclidean and Minkowski space, lists the interesting unary operators which preserve determinant, then documents efficient calculation of the determinant using these unary operators.

Minkowski Geometric Algebra

Minkowski geometric algebra is a spacetime algebra using east coast metric $(+,+,+,-)$. Our components are one scalar, e_q, three space directions e_x, e_y, and e_z, a time dimension e_t, and higher order products of the previous, yielding three pure space bivectors e_xe_y, e_xe_z, and e_ye_z, three spacetime bivectors e_xe_t, e_ye_t, and e_ze_t, four trivectors $e_xe_ye_z, e_xe_ye_t, e_xe_ze_t$, and $e_ye_ze_t$, and a quadvector $e_xe_ye_ze_t$.

A generic multivector is the scaled sum of the previous elements, which I usually write as

$$
\begin{aligned}
&Ae_q + \\
&Be_x + Ce_y + De_z + Ee_t + \\
&Fe_{xy} + Ge_{xz} + He_{yz} + Je_{xt} + Ke_{yt} + Le_{zt} + \\
&Me_{xyz} + Ne_{xyt} + Pe_{xzt} + Re_{yzt} + \\
&Se_{xyzt}
\end{aligned}
$$

We can define this algebra by a multiplication table. In condensed format, to fit on one page, we have Table 1.

Minkowski Matrix Implementation

In text format, one set of real 4x4 matrices implementing Minkowski geometric algebra are

```
     Unity                                xyzt
[  1  0  0  0]                       [  0  0  0  1]
[  0  1  0  0]                       [  0  0 -1  0]
[  0  0  1  0]                       [  0  1  0  0]
[  0  0  0  1]                       [ -1  0  0  0]

        x              y              z              t
[  1  0  0  0] [  0  1  0  0] [  0  0  0 -1] [  0 -1  0  0]
[  0 -1  0  0] [  1  0  0  0] [  0  0  1  0] [  1  0  0  0]
[  0  0  1  0] [  0  0  0  1] [  0  1  0  0] [  0  0  0  1]
[  0  0  0 -1] [  0  0  1  0] [ -1  0  0  0] [  0  0 -1  0]

       xy             xz             yz             xt             yt             zt
[  0  1  0  0] [  0  0  0 -1] [  0  0  1  0] [  0 -1  0  0] [  1  0  0  0] [  0  0  1  0]
[ -1  0  0  0] [  0  0 -1  0] [  0  0  0 -1] [ -1  0  0  0] [  0 -1  0  0] [  0  0  0  1]
[  0  0  0  1] [  0  1  0  0] [ -1  0  0  0] [  0  0  0  1] [  0  0 -1  0] [  1  0  0  0]
[  0  0 -1  0] [  1  0  0  0] [  0  1  0  0] [  0  0  1  0] [  0  0  0  1] [  0  1  0  0]

       xyz            xyt            xzt            yzt
[  0  0  1  0] [  1  0  0  0] [  0  0  1  0] [  0  0  0  1]
[  0  0  0  1] [  0  1  0  0] [  0  0  0 -1] [  0  0  1  0]
[ -1  0  0  0] [  0  0 -1  0] [  1  0  0  0] [  0  1  0  0]
[  0 -1  0  0] [  0  0  0 -1] [  0 -1  0  0] [  1  0  0  0]
```

q	x	y	z	t	xy	xz	yz	xt	yt	zt	xyz	xyt	xzt	yzt	xyzt
x	q	xy	xz	xt	y	z	xyz	t	xyt	xzt	yz	yt	zt	xyzt	yzt
y	-xy	q	yz	yt	-x	-xyz	z	-xyt	t	yzt	-xz	-xt	-xyzt	zt	-xzt
z	-xz	-yz	q	zt	xyz	-x	-y	-xzt	-yzt	t	xy	xyzt	-xt	-yt	xyt
t	-xt	-yt	-zt	-q	xyt	xzt	yzt	x	y	z	-xyzt	-xy	-xz	-yz	xyz
xy	-y	x	xyz	xyt	-q	-yz	xz	-yt	xt	xyzt	-z	-t	-yzt	xzt	-zt
xz	-z	-xyz	x	xzt	yz	-q	-xy	-zt	-xyzt	xt	y	yzt	-t	-xyt	yt
yz	xyz	-z	y	yzt	-xz	xy	-q	xyzt	-zt	yt	-x	-xzt	xyt	-t	-xt
xt	-t	-xyt	-xzt	-x	yt	zt	xyzt	q	xy	xz	-yzt	-y	-z	-xyz	yz
yt	xyt	-t	-yzt	-y	-xt	-xyzt	zt	-xy	q	yz	xzt	x	xyz	-z	-xz
zt	xzt	yzt	-t	-z	xyzt	-xt	-yt	-xz	-yz	q	-xyt	-xyz	x	y	xy
xyz	yz	-xz	xy	xyzt	-z	y	-x	yzt	-xzt	xyt	-q	-zt	yt	-xt	-t
xyt	yt	-xt	-xyzt	-xy	-t	-yzt	xzt	-y	x	xyz	zt	q	yz	-xz	-z
xzt	zt	xyzt	-xt	-xz	yzt	-t	-xyt	-z	-xyz	x	-yt	-yz	q	xy	y
yzt	-xyzt	zt	-yt	-yz	-xzt	xyt	-t	xyz	-z	y	xt	xz	-xy	q	-x
xyzt	-yzt	xzt	-xyt	-xyz	-zt	yt	-xt	yz	-xz	xy	t	z	-y	x	-q

Table 1: Minkowski Geometric Algebra Multiplication Table

For a generic Minkowski multivector,

$$
\begin{aligned}
&Ae_q + \\
&Be_x + Ce_y + De_z + Ee_t + \\
&Fe_{xy} + Ge_{xz} + He_{yz} + Je_{xt} + Ke_{yt} + Le_{zt} + \\
&Me_{xyz} + Ne_{xyt} + Pe_{xzt} + Re_{yzt} + \\
&Se_{xyzt}
\end{aligned}
$$

the associated matrix representation is

$$
\begin{pmatrix}
+A+B+K+N & +C-E+F-J & +H+L+M+P & -D-G+R+S \\
+C+E-F-J & +A-B-K+N & +D-G+R-S & -H+L+M-P \\
-H+L-M+P & +D+G+R+S & +A+B-K-N & +C+E+F+J \\
-D+G+R-S & +H+L-M-P & +C-E-F+J & +A-B+K-N
\end{pmatrix}
$$

To recover the multivector components from a generic 4x4 matrix $W[4][4]$, we have the trace-like formulas

$$
\begin{aligned}
A &= (+W[0][0] + W[1][1] + W[2][2] + W[3][3])/4 \\
B &= (+W[0][0] - W[1][1] + W[2][2] - W[3][3])/4 \\
C &= (+W[0][1] + W[1][0] + W[2][3] + W[3][2])/4 \\
D &= (-W[0][3] + W[1][2] + W[2][1] - W[3][0])/4 \\
E &= (-W[0][1] + W[1][0] + W[2][3] - W[3][2])/4 \\
F &= (+W[0][1] - W[1][0] + W[2][3] - W[3][2])/4 \\
G &= (-W[0][3] - W[1][2] + W[2][1] + W[3][0])/4 \\
H &= (+W[0][2] - W[1][3] - W[2][0] + W[3][1])/4 \\
J &= (-W[0][1] - W[1][0] + W[2][3] + W[3][2])/4 \\
K &= (+W[0][0] - W[1][1] - W[2][2] + W[3][3])/4 \\
L &= (+W[0][2] + W[1][3] + W[2][0] + W[3][1])/4 \\
M &= (+W[0][2] + W[1][3] - W[2][0] - W[3][1])/4 \\
N &= (+W[0][0] + W[1][1] - W[2][2] - W[3][3])/4 \\
P &= (+W[0][2] - W[1][3] + W[2][0] - W[3][1])/4 \\
R &= (+W[0][3] + W[1][2] + W[2][1] + W[3][0])/4 \\
S &= (+W[0][3] - W[1][2] + W[2][1] - W[3][0])/4
\end{aligned}
$$

In this fashion, we can convert from multivector to matrix, use matrix tools such as Sage, Mathematica, Sympy and Ginac, and convert the results back to multivector format. For this work, I have used Ginac.

Determinant for Minkowski Spacetime

Using matrix tools, we can write a formula for the Minkowski determinant. Grouping terms, we can find suggestive factorings, such as the following.

$$
\begin{aligned}
\det \;=\;& +(A^2+B^2+C^2+D^2-E^2-F^2-G^2-H^2 \\
& +J^2+K^2+L^2-M^2+N^2+P^2+R^2-S^2))^2 \\
& +4*(\\
& -(AB+(-HM+KN+LP))^2 \\
& -(AC+(-GM+JN-LR))^2 \\
& -(AD+(+FM+JP+KR))^2 \\
& +(AE+(+FN+GP+HR))^2 \\
& +(AF+(+DM-EN+LS))^2 \\
& \\
& +(AG+(-CM-EP-KS))^2 \\
& +(AH+(-BM+ER-JS))^2 \\
& -(AJ+(-CN-DP-HS))^2 \\
& -(AK+(-BN+DR-GS))^2 \\
& -(AL+(-BP-CR+FS))^2 \\
& \\
& +(AM+(-BH+CG-DF))^2 \\
& -(AN+(-BK+CJ-EF))^2 \\
& -(AP+(-BL+DJ-EG))^2 \\
& -(AR+(+CL-DK+EH))^2 \\
& +(AS+(-FL+GK-HJ))^2)=0
\end{aligned}
$$

However, by using unary operators, we can find much simpler forms.

Determinant Preserving Unary Operators

Unary operators are widely used in complex numbers, quaternions and geometric algebra. These operators change the sign of various components of the geometric element according to some simple rule. For complex numbers, we change the sign of the imaginary component. For geometric alebra, the parity operator changes the sign of the spatial basis, while the reverse changes the sign of each term based upon the sign of the multivector basis elements multiplied in reverse order. Being simple sign reversal of specific components, applying the operator twice recovers the original term.

Given that Minkowski spacetime has sixteen multivector components, there are $2^{16} = 65536$ different unary operators possible. My first task was to see how many of these preserve the Minkowski determinant. Only 64 of the 65536 preserved the determinant, and these 64 are 32 unique functions and their negates, leaving in effect, 32 functions of interest. These are listed below, along with their index in the search, and a score parameter to be discussed shortly.

```
(A,  B, C, D, E,  F, G, H, J, K, L,  M, N, P, R,  S) score =  0 N(    0)
(A,  B, C, D, E, -F,-G,-H,-J,-K,-L, -M,-N,-P,-R,  S) score = 10 N( 2046)
(A,  B, C, D,-E,  F, G, H,-J,-K,-L,  M,-N,-P,-R, -S) score =  0 N( 2287)
(A,  B, C, D,-E, -F,-G,-H, J, K, L, -M, N, P, R, -S) score =  6 N( 3857)
(A,  B, C,-D, E,  F,-G,-H, J, K,-L, -M, N,-P,-R, -S) score =  0 N( 4919)
(A,  B, C,-D, E, -F, G, H,-J,-K, L,  M,-N, P, R, -S) score =  6 N( 5321)
(A,  B, C,-D,-E,  F,-G,-H,-J,-K, L, -M,-N, P, R,  S) score =  0 N( 7128)
(A,  B, C,-D,-E, -F, G, H, J, K,-L,  M, N,-P,-R,  S) score =  6 N( 7206)
(A,  B,-C, D, E,  F,-G, H,-J, K,-L,  M, N,-P, R, -S) score =  6 N( 8869)
(A,  B,-C, D, E, -F, G,-H, J,-K, L, -M,-N, P,-R, -S) score =  0 N( 9563)
(A,  B,-C, D,-E,  F,-G, H, J,-K, L,  M,-N, P,-R,  S) score =  6 N(10826)
(A,  B,-C, D,-E, -F, G,-H,-J, K,-L, -M, N,-P, R,  S) score =  0 N(11700)
(A,  B,-C,-D, E,  F, G,-H,-J, K, L, -M, N, P,-R,  S) score =  6 N(12690)
(A,  B,-C,-D, E, -F,-G, H, J,-K,-L,  M,-N,-P, R,  S) score =  0 N(13932)
(A,  B,-C,-D,-E,  F, G,-H, J,-K,-L, -M,-N,-P, R, -S) score = 10 N(14717)
(A,  B,-C,-D,-E, -F,-G, H,-J, K, L,  M, N, P,-R, -S) score =  0 N(16003)
(A, -B, C, D, E,  F, G,-H, J,-K,-L,  M, N, P,-R, -S) score =  6 N(16739)
(A, -B, C, D, E, -F,-G, H,-J, K, L, -M,-N,-P, R, -S) score =  0 N(18077)
(A, -B, C, D,-E,  F, G,-H,-J, K, L,  M,-N,-P, R,  S) score =  6 N(18828)
(A, -B, C, D,-E, -F,-G, H, J,-K,-L, -M, N, P,-R,  S) score =  0 N(20082)
(A, -B, C,-D, E,  F,-G, H, J,-K, L, -M, N,-P, R,  S) score =  6 N(21076)
(A, -B, C,-D, E, -F, G,-H,-J, K,-L,  M,-N, P,-R,  S) score =  0 N(21930)
(A, -B, C,-D,-E,  F,-G, H,-J, K,-L, -M,-N, P,-R, -S) score = 10 N(23227)
(A, -B, C,-D,-E, -F, G,-H, J,-K, L,  M, N,-P, R, -S) score =  0 N(23877)
```

```
(A, -B,-C, D, E,  F,-G,-H,-J,-K, L,  M, N,-P,-R,  S) score =  0 N(25542)
(A, -B,-C, D, E, -F, G, H, J, K,-L, -M,-N, P, R,  S) score =  6 N(25656)
(A, -B,-C, D,-E,  F,-G,-H, J, K,-L,  M,-N, P, R, -S) score =  0 N(27433)
(A, -B,-C, D,-E, -F, G, H,-J,-K, L, -M, N,-P,-R, -S) score = 10 N(27863)
(A, -B,-C,-D, E,  F, G, H,-J,-K,-L, -M, N, P, R, -S) score =  0 N(28913)
(A, -B,-C,-D, E, -F,-G,-H, J, K, L,  M,-N,-P,-R, -S) score = 10 N(30479)
(A, -B,-C,-D,-E,  F, G, H, J, K, L, -M,-N,-P,-R,  S) score =  0 N(30750)
(A, -B,-C,-D,-E, -F,-G,-H,-J,-K,-L,  M, N, P, R,  S) score = 10 N(32736)
```

When we take the magnitude of a complex number, we multiply that complex number by its complex conjugate, and obtain a single, real number with zero imaginary component. In a similar fashion, if I multiply a multivector by the unary transform of that multivector as defined above, I often find many components are eliminated by the symmetries and antisymmetries of the geometric product. The score parameter is the count of zeroed out components.

Of these 32 forms, four apply sign changes uniformly across blades. These map to the identity element at N(0), reversion at N(2046), grade involution at N(30750), and Clifford conjugation at N(32736), where the names match those in Dorst's Geometric Algebra for Computer Scientists, [1] page 604. Also of great interest is N(3857), which corresponds to the transpose of the matrix format.

When we take the product of a multivector and its reverse N(2046), the product has zero in the bivector and trivector components.

```
(A,  B, C, D, E,  F, G, H, J, K, L,  M, N, P, R,  S)*
(A,  B, C, D, E, -F,-G,-H,-J,-K,-L, -M,-N,-P,-R,  S) =
(A^2+M^2+D^2-R^2+G^2+B^2-K^2-S^2-N^2+H^2-E^2-L^2+C^2-P^2-J^2+F^2,

+2*C*F-2*K*N-2*L*P-2*R*S+2*D*G+2*M*H-2*E*J+2*B*A,
-2*R*L+2*P*S-2*E*K-2*M*G+2*H*D-2*B*F+2*A*C+2*N*J,
+2*P*J+2*A*D+2*R*K-2*H*C-2*E*L-2*B*G-2*N*S+2*M*F,
-2*K*C+2*R*H+2*E*A-2*B*J-2*M*S+2*N*F+2*P*G-2*L*D,

0,0,0,0,0,0,   // bivector is zeroed

0,0,0,0,       // trivector is zeroed

-2*L*F+2*P*C+2*M*E+2*G*K-2*H*J-2*R*B-2*D*N+2*A*S)
```

Likewise, the product of a multivector and its Clifford conjugate N(32736) has ten zero elements, with the vector and bivector eliminated.

```
(A,  B, C, D, E,  F, G, H, J, K, L,  M, N, P, R,  S)*
(A, -B,-C,-D,-E, -F,-G,-H,-J,-K,-L,  M, N, P, R,  S) =
(A^2-M^2-D^2+R^2+G^2-B^2-K^2-S^2+N^2+H^2+E^2-L^2-C^2+P^2-J^2+F^2,

0,0,0,0,       // vector is zeroed

0,0,0,0,0,0,   // bivector is zeroed

+2*M*A-2*B*H+2*G*C-2*R*J-2*L*N-2*D*F+2*E*S+2*P*K,
+2*C*J+2*H*P-2*E*F-2*R*G+2*D*S+2*A*N-2*M*L-2*B*K,
-2*E*G+2*R*F-2*B*L+2*M*K-2*S*C+2*P*A+2*D*J-2*H*N,
+2*B*S-2*L*C+2*D*K-2*P*F+2*G*N+2*R*A-2*E*H-2*M*J,

-2*L*F-2*P*C-2*M*E+2*G*K-2*H*J+2*R*B+2*D*N+2*A*S)
```

Simpler Minkowski Determinant

As the unary operators for reversion and Clifford conjugation preserve the determinant, we now have an opportunity for simplifying this calculation. Because ten terms in the product of a multivector and its reverse are zero, the determinant for this expression is greatly simplied, and happens to be a complete square. As the determinant of a product is the product of the determinant, and the product is a square, undoing the square neatly provides the desired determinant.

Using the reverse operator as an example, I want to find det(V).

```
V = Mink(A,  B, C, D, E,  F, G, H, J, K, L,  M, N, P, R,  S)
U = Mink(A,  B, C, D, E, -F,-G,-H,-J,-K,-L, -M,-N,-P,-R,  S)
W = U*V

a = W.q = A^2+B^2+C^2+D^2-E^2+F^2+G^2+H^2-J^2-K^2-L^2+M^2-N^2-P^2-R^2-S^2

b = W.x =  + 2*A*B + 2*C*F + 2*D*G - 2*E*J + 2*H*M - 2*K*N - 2*L*P - 2*R*S
c = W.y =  + 2*A*C - 2*B*F + 2*D*H - 2*E*K - 2*G*M + 2*J*N - 2*L*R + 2*P*S
d = W.z =  + 2*A*D - 2*B*G - 2*C*H - 2*E*L + 2*J*P + 2*K*R + 2*M*F - 2*N*S
e = W.t =  + 2*A*E - 2*B*J - 2*C*K - 2*D*L + 2*F*N + 2*G*P + 2*H*R - 2*M*S

s = W.xyzt =  + 2*A*S - 2*B*R + 2*C*P - 2*D*N + 2*E*M - 2*F*L + 2*G*K - 2*H*J

det(W) = det(V)*det(U) = (det(V))^2 = (a*a - b*b - c*c - d*d + e*e + s*s)^2
det(V) = a*a - b*b - c*c - d*d + e*e + s*s (sign verified)
```

Repeating the process using the Clifford conjugate, we have

```
U = Mink(A,  B, C, D, E,  F, G, H, J, K, L,  M, N, P, R,  S)
V = Mink(A, -B,-C,-D,-E, -F,-G,-H,-J,-K,-L,  M, N, P, R,  S)
W = U*V;

a = W.q = A^2-B^2-C^2-D^2+E^2+F^2+G^2+H^2-J^2-K^2-L^2-M^2+N^2+P^2+R^2-S^2

m = W.xyz =  + 2*A*M - 2*B*H + 2*C*G - 2*D*F + 2*E*S - 2*J*R + 2*K*P - 2*L*N
n = W.xyt =  + 2*A*N - 2*B*K + 2*C*J + 2*D*S - 2*E*F - 2*G*R + 2*H*P - 2*M*L
p = W.xzt =  + 2*A*P - 2*B*L - 2*C*S + 2*D*J - 2*E*G + 2*F*R - 2*H*N + 2*K*M
r = W.yzt =  + 2*A*R + 2*B*S - 2*C*L + 2*D*K - 2*E*H - 2*F*P + 2*G*N - 2*M*J

s = W.xyzt =  + 2*A*S + 2*B*R - 2*C*P + 2*D*N - 2*E*M - 2*F*L + 2*G*K - 2*H*J

det(V) = a*a + m*m - n*n - p*p - r*r + s*s  (sign verified)
```

Both formulas are different in component detail, have the same computational cost, and are a major improvement over the simple matrix determinant formula.

Euclidean Geometric Algebra Subspace

The three dimensional Euclidean geometric algebra is a subspace of Minkowski spacetime.

```
U_3D = Mink(A, B,C,D,0, F,G,H,0,0,0, M,0,0,0, 0)
```

Direct substitution in the Clifford formulas above yield the simple result

```
V_3D = Mink(A, B,C,D,0, F,G,H,0,0,0, M,0,0,0, 0)
a = W.q = A^2 - B^2 - C^2 - D^2 + F^2 + G^2 + H^2 - M^2
m = W.xyz = + 2*A*M - 2*B*H + 2*C*G - 2*D*F
det(V_3D) = a*a + m*m
```

When using Euclidean 3D geometric algebra, I commonly use a bivector basis of e_{zx}, rather than the e_{xz} used by default in my Mink implementation. This leads to a sign change for **G** in the formula for **m** above. (Always check these details when comparing formulas.)

Euclidean Unary Operators

In the Clifford conjugate product, the vector and bivector components disappear, leaving a simple scalar and trivector combination, which mimics a complex number.

References

[1] Leo Dorst, Daniel Fontune and Stephen Mann, *Geometric Algebra for Computer Science* Morgan Kaufmann Publishers, ISBN 978-0-12-374942-0