Two Views of Electromagnetism

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Abstract

This note is a simple equation dump for Maxwell equations in geometric algebra format. First, is a three dimensional summary of electromagnetism including monopoles, followed by a four dimensional summary. These equations are now fairly standard, as seen in the references at the end of note.

EM in Euclidean 3D Geometric Algebra

In 3D Euclidean space, we have a metric of (1,1,1), meaning that the dot product squares of the spatial basis vectors equal one. Time is a parameter, not a coordinate.

Conventional Maxwell equations in SI units are

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \frac{\rho_e}{\epsilon} \\
\vec{\nabla} \cdot \vec{B} &= 0 \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\vec{\nabla} \times \vec{B} &= \mu\epsilon \frac{\partial \vec{E}}{\partial t} + \mu \vec{j}_e
\end{aligned}
$$

Extending these equations to include magnetic monopoles, we have

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \frac{\rho_e}{\epsilon} \\
\vec{\nabla} \cdot \vec{B} &= \mu \rho_m \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} - \mu \vec{j}_m \\
\vec{\nabla} \times \vec{B} &= \mu \epsilon \frac{\partial \vec{E}}{\partial t} + \mu \vec{j}_e
\end{aligned}
$$

The elements in the geometric product for three space are

- scalars
- three basis vector x, y, and z
- three bivectors yz, xz, and xy, dual to the vectors above, and
- trivector xyz which acts as the pseudo-scalar and imaginary element for three space.

The multiplication table among these elements is

```
********* Three Dimensional (1,1,1) metric ****************
```

q	x	y	z	xy	xz	yz	xyz
x	q	xy	xz	y	z	xyz	yz
y	-xy	q	yz	-x	-xyz	z	-xz
z	-xz	-yz	q	xyz	-x	-y	xy
xy	-y	x	xyz	-q	-yz	xz	-z
xz	-z	-xyz	x	yz	-q	-xy	y
yz	xyz	-z	y	-xz	xy	-q	-x
xyz	yz	-xz	xy	-z	y	-x	-q

The component level expressions for $c = ab$ are

```
c.q   = a.q*b.q   + a.x*b.x   + a.y*b.y   - a.xy*b.xy
      + a.z*b.z   - a.xz*b.xz  - a.yz*b.yz  - a.xyz*b.xyz ;
c.x   = a.q*b.x   + a.x*b.q   - a.y*b.xy  + a.xy*b.y
      - a.z*b.xz  + a.xz*b.z   - a.yz*b.xyz - a.xyz*b.yz  ;
c.y   = a.q*b.y   + a.x*b.xy  + a.y*b.q   - a.xy*b.x
      - a.z*b.yz  + a.xz*b.xyz + a.yz*b.z   + a.xyz*b.xz  ;
c.z   = a.q*b.z   + a.x*b.xz  + a.y*b.yz  - a.xy*b.xyz
      + a.z*b.q   - a.xz*b.x   - a.yz*b.y   - a.xyz*b.xy  ;
c.xy  = a.q*b.xy  + a.x*b.y   - a.y*b.x   + a.xy*b.q
      + a.z*b.xyz - a.xz*b.yz  + a.yz*b.xz  + a.xyz*b.z   ;
c.xz  = a.q*b.xz  + a.x*b.z   - a.y*b.xyz + a.xy*b.yz
      - a.z*b.x   + a.xz*b.q   - a.yz*b.xy  - a.xyz*b.y   ;
c.yz  = a.q*b.yz  + a.x*b.xyz + a.y*b.z   - a.xy*b.xz
      - a.z*b.y   + a.xz*b.xy  + a.yz*b.q   + a.xyz*b.x   ;
c.xyz = a.q*b.xyz + a.x*b.yz  - a.y*b.xz  + a.xy*b.z
      + a.z*b.xy  - a.xz*b.y   + a.yz*b.x   + a.xyz*b.q   ;
```

The pure wedge product in 3D is defined as

```
************** Three Dimensional Wedge Only **************

       x      y      z      xy     xz     yz     xyz
 x     0      xy     xz     0      0      xyz    0
 y     -xy    0      yz     0      -xyz   0      0
 z     -xz    -yz    0      xyz    0      0      0
 xy    0      0      xyz    0      0      0      0
 xz    0      -xyz   0      0      0      0      0
 yz    xyz    0      0      0      0      0      0
 xyz   0      0      0      0      0      0      0

c.q   = 0  ;     c.x   = 0  ;
c.y   = 0  ;     c.z   = 0  ;
c.xy  = a.x*b.y  - a.y*b.x  ;
c.xz  = a.x*b.z  - a.z*b.x  ;
c.yz  = a.y*b.z  - a.z*b.y  ;
c.xyz = a.x*b.yz - a.y*b.xz + a.xy*b.z
      + a.z*b.xy - a.xz*b.y + a.yz*b.x;
```

EM equations in GA format copied from [1] follow:

$$
\begin{aligned}
\partial &= \frac{1}{c}\frac{\partial}{\partial t} + \nabla \\
\bar{\partial} &= \frac{1}{c}\frac{\partial}{\partial t} - \nabla \\
j &= e_1 e_2 e_3 \quad \text{trivector, not current} \\
j^2 &= -1 \\
J &= (\rho/\epsilon_0) - c\mu_0 \mathbf{J} \\
\partial \cdot J &= 0 \\
\partial\bar{\partial} A &= J \\
A &= \frac{\mu_0}{r\pi}\int_{\text{Volume}} \frac{J'}{r'} d\tau' \\
A &= \phi - c\mathbf{A} \\
F &= \bar{\partial} A \\
F &= \mathbf{E} + jc\mathbf{B} \\
\partial F &= J \\
\partial \cdot J &= 0 \\
U &= \frac{1}{2}\epsilon_0 F F^\dagger \\
\partial \cdot U &= 0 \\
\text{Action} &= J \cdot A \\
\mathbf{S} &= \frac{1}{\mu_0}\mathbf{B} \wedge \mathbf{E} \\
V &= c - \mathbf{v} \\
K &= \frac{q}{c} V F
\end{aligned}
$$

The electric field is a vector, while magnetic field is a bivector. This matches the parity properties for electric and magnetic fields. However, mixing of electric and magnetic fields will require a factor which accomplishes the grade mixing. I recall that this will involve the velocity term.

The magnetic monopole extensions, also copied from [1], are

$$
\begin{aligned}
\left(\frac{1}{c\partial t} + \nabla\right)(\mathbf{E} + jc\mathbf{B}) &= \rho - c\mu_0\mathbf{J} - j\mu_0\mathbf{J}_m + jc\mu_0\rho_m \\
\nabla \cdot \mathbf{B} &= \rho_m \\
\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= -\mathbf{J}_m \\
F &= l + \mathbf{E} + jc\mathbf{B} + jc \\
l &= c\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t}
\end{aligned}
$$

Main observation is that the monopole is associated with the trivector or pseudo-scalar component. I want to compare this with Jackson's Electrodynamics for the parity violating aspects.

EM in Minkowski 4D Geometric Algebra

The elements in the geometric product for four space are

- scalars
- four basis vector x, y, z and t
- six bivectors: xy, xz, xt, yz, yt, and zt
- four trivectors xyz, xyt, xzt and yzt,
- one quadvector $xyzt$

We have an even element subspace, with terms

- scalars
- six bivectors xy dual to zt, xz dual to yt, and yz dual to xt
- one quadvector $xyzt$

q	x	y	z	t	xy	xz	yz	xt	yt	zt	xyz	xyt	xzt	yzt	xyzt
x	-q	xy	xz	xt	-y	-z	xyz	-t	xyt	xzt	-yz	-yt	-zt	xyzt	-yzt
y	-xy	q	yz	yt	-x	-xyz	z	-xyt	t	yzt	-xz	-xt	-xyzt	zt	-xzt
z	-xz	-yz	q	zt	xyz	-x	-y	-xzt	-yzt	t	xy	xyzt	-xt	-yt	xyt
t	-xt	-yt	-zt	q	xyt	xzt	yzt	-x	-y	-z	-xyzt	xy	xz	yz	-xyz
xy	y	x	xyz	xyt	q	yz	xz	yt	xt	xyzt	z	t	yzt	xzt	zt
xz	z	-xyz	x	xzt	-yz	q	-xy	zt	-xyzt	xt	-y	-yzt	t	-xyt	-yt
yz	xyz	-z	y	yzt	-xz	xy	-q	xyzt	-zt	yt	-x	-xzt	xyt	-t	-xt
xt	t	-xyt	-xzt	x	-yt	-zt	xyzt	q	-xy	-xz	yzt	-y	-z	xyz	yz
yt	xyt	-t	-yzt	y	-xt	-xyzt	zt	xy	-q	-yz	xzt	-x	-xyz	z	xz
zt	xzt	yzt	-t	z	xyzt	-xt	-yt	xz	yz	-q	-xyt	xyz	-x	-y	-xy
xyz	-yz	-xz	xy	xyzt	z	-y	-x	-yzt	-xzt	xyt	q	zt	-yt	-xt	t
xyt	-yt	-xt	-xyzt	xy	t	yzt	xzt	-y	-x	-xyz	-zt	q	yz	xz	-z
xzt	-zt	xyzt	-xt	xz	-yzt	t	-xyt	-z	xyz	-x	yt	-yz	q	-xy	y
yzt	-xyzt	zt	-yt	yz	-xzt	xyt	-t	-xyz	z	-y	xt	-xz	xy	-q	x
xyzt	yzt	xzt	-xyt	xyz	zt	-yt	-xt	yz	xz	-xy	-t	z	-y	-x	-q

Table 1: Four Dimensional (-1,1,1,1) Metric Multiplication Table

c.q	=	a.q*b.q - a.x*b.x + a.y*b.y + a.xy*b.xy + a.z*b.z + a.xz*b.xz - a.yz*b.yz + a.xyz*b.xyz
		+ a.t*b.t +a.xt*b.xt - a.yt*b.yt + a.xyt*b.xyt - a.zt*b.zt + a.xzt*b.xzt - a.yzt*b.yzt - a.xyzt*b.xyzt ;
c.x	=	a.q*b.x + a.x*b.q - a.y*b.xy + a.xy*b.y - a.z*b.xz + a.xz*b.z - a.yz*b.xyz - a.xyz*b.yz
		- a.t*b.xt +a.xt*b.t - a.yt*b.xyt - a.xyt*b.yt - a.zt*b.xzt - a.xzt*b.zt + a.yzt*b.xyzt - a.xyzt*b.yzt ;
c.y	=	a.q*b.y - a.x*b.xy + a.y*b.q + a.xy*b.x - a.z*b.yz - a.xz*b.xyz + a.yz*b.z - a.xyz*b.xz
		- a.t*b.yt -a.xt*b.xyt + a.yt*b.t - a.xyt*b.xt - a.zt*b.yzt + a.xzt*b.xyzt - a.yzt*b.zt - a.xyzt*b.xzt ;
c.z	=	a.q*b.z - a.x*b.xz + a.y*b.yz + a.xy*b.xyz + a.z*b.q + a.xz*b.x - a.yz*b.y + a.xyz*b.xy
		- a.t*b.zt -a.xt*b.xzt + a.yt*b.yzt - a.xyt*b.xyzt + a.zt*b.t - a.xzt*b.xt + a.yzt*b.yt + a.xyzt*b.xyt ;
c.t	=	a.q*b.t - a.x*b.xt + a.y*b.yt + a.xy*b.xyt + a.z*b.zt + a.xz*b.xzt - a.yz*b.yzt + a.xyz*b.xyzt
		+ a.t*b.q +a.xt*b.x - a.yt*b.y + a.xyt*b.xy - a.zt*b.z + a.xzt*b.xz - a.yzt*b.yz - a.xyzt*b.xyz ;
c.xy	=	a.q*b.xy + a.x*b.y - a.y*b.x + a.xy*b.q + a.z*b.xyz - a.xz*b.yz + a.yz*b.xz + a.xyz*b.z
		+ a.t*b.xyt -a.xt*b.yt + a.yt*b.xt + a.xyt*b.t - a.zt*b.xyzt - a.xzt*b.yzt + a.yzt*b.xzt - a.xyzt*b.zt ;
c.xz	=	a.q*b.xz + a.x*b.z - a.y*b.xyz + a.xy*b.yz - a.z*b.x + a.xz*b.q - a.yz*b.xy - a.xyz*b.y
		+ a.t*b.xzt -a.xt*b.zt + a.yt*b.xyzt + a.xyt*b.yzt + a.zt*b.xt + a.xzt*b.t - a.yzt*b.xyt + a.xyzt*b.yt ;
c.yz	=	a.q*b.yz - a.x*b.xyz + a.y*b.z + a.xy*b.xz - a.z*b.y - a.xz*b.xy + a.yz*b.q - a.xyz*b.x
		+ a.t*b.yzt +a.xt*b.xyzt - a.yt*b.zt + a.xyt*b.xzt + a.zt*b.yt - a.xzt*b.xyt + a.yzt*b.t + a.xyzt*b.xt ;
c.xt	=	a.q*b.xt + a.x*b.t - a.y*b.xyt + a.xy*b.yt - a.z*b.xzt + a.xz*b.zt - a.yz*b.xyzt - a.xyz*b.yzt
		- a.t*b.x +a.xt*b.q - a.yt*b.xy - a.xyt*b.y - a.zt*b.xz - a.xzt*b.z + a.yzt*b.xyz - a.xyzt*b.yz ;
c.yt	=	a.q*b.yt - a.x*b.xyt + a.y*b.t + a.xy*b.xt - a.z*b.yzt - a.xz*b.xyzt + a.yz*b.zt - a.xyz*b.xzt
		- a.t*b.y -a.xt*b.xy + a.yt*b.q - a.xyt*b.x - a.zt*b.yz + a.xzt*b.xyz - a.yzt*b.z - a.xyzt*b.xz ;
c.zt	=	a.q*b.zt - a.x*b.xzt + a.y*b.yzt + a.xy*b.xyzt + a.z*b.t + a.xz*b.xt - a.yz*b.yt + a.xyz*b.xyt
		- a.t*b.z -a.xt*b.xz + a.yt*b.yz - a.xyt*b.xyz + a.zt*b.q - a.xzt*b.x + a.yzt*b.y + a.xyzt*b.xy ;
c.xyz	=	a.q*b.xyz + a.x*b.yz - a.y*b.xz + a.xy*b.z + a.z*b.xy - a.xz*b.y + a.yz*b.x + a.xyz*b.q
		- a.t*b.xyzt +a.xt*b.yzt - a.yt*b.xzt - a.xyt*b.zt + a.zt*b.xyt + a.xzt*b.yt - a.yzt*b.xt + a.xyzt*b.t ;
c.xyt	=	a.q*b.xyt + a.x*b.yt - a.y*b.xt + a.xy*b.t + a.z*b.xyzt - a.xz*b.yzt + a.yz*b.xzt + a.xyz*b.zt
		+ a.t*b.xy -a.xt*b.y + a.yt*b.x + a.xyt*b.q - a.zt*b.xyz - a.xzt*b.yz + a.yzt*b.xz - a.xyzt*b.z ;
c.xzt	=	a.q*b.xzt + a.x*b.zt - a.y*b.xyzt + a.xy*b.yzt - a.z*b.xt + a.xz*b.t - a.yz*b.xyt - a.xyz*b.yt
		+ a.t*b.xz -a.xt*b.z + a.yt*b.xyz + a.xyt*b.yz + a.zt*b.x + a.xzt*b.q - a.yzt*b.xy + a.xyzt*b.y ;
c.yzt	=	a.q*b.yzt - a.x*b.xyzt + a.y*b.zt + a.xy*b.xzt - a.z*b.yt - a.xz*b.xyt + a.yz*b.t - a.xyz*b.xt
		+ a.t*b.yz +a.xt*b.xyz - a.yt*b.z + a.xyt*b.xz + a.zt*b.y - a.xzt*b.xy + a.yzt*b.q + a.xyzt*b.x ;
c.xyzt	=	a.q*b.xyzt + a.x*b.yzt - a.y*b.xzt + a.xy*b.zt + a.z*b.xyt - a.xz*b.yt + a.yz*b.xt + a.xyz*b.t
		- a.t*b.xyz + a.xt*b.yz - a.yt*b.xz - a.xyt*b.z + a.zt*b.xy + a.xzt*b.y - a.yzt*b.x + a.xyzt*b.q ;

Table 2: Four Dimensional (-1,1,1,1) metric component equations

In the 4D model, electro-magnetism occurs in an even grade subspace. The spatial coordinates we see, are actually bivectors composed of time/space products.

$$
\begin{aligned}
\sigma_1 &= \gamma_0\gamma_1 \qquad \text{time is embedded in our coordinates} \\
\sigma_2 &= \gamma_0\gamma_2 \\
\sigma_3 &= \gamma_0\gamma_3 \\
\sigma_i\sigma_i &= \gamma_0\gamma_i\gamma_0\gamma_i \\
&= -\gamma_0\gamma_0\gamma_i\gamma_i \\
&= \gamma_i\gamma_i \\
&= 1 \qquad \text{positive unit norm} \\
\sigma_i\sigma_j &= \gamma_0\gamma_i\gamma_0\gamma_j \\
&= -\gamma_0\gamma_0\gamma_i\gamma_j \\
&= \gamma_0\gamma_0\gamma_j\gamma_i \\
&= -\gamma_0\gamma_j\gamma_0\gamma_i \\
&= -\sigma_j\sigma_i \qquad \text{anti} - \text{commutative between basis}
\end{aligned}
$$

The pseudovector for our bicoordinates, is also the pseudovector for fourspace.

$$
\begin{aligned}
\sigma_1\sigma_2\sigma_3 &= \gamma_0\gamma_1\gamma_0\gamma_2\gamma_0\gamma_3 \\
&= -\gamma_0\gamma_1\gamma_0\gamma_0\gamma_2\gamma_3 \\
&= \gamma_0\gamma_1\gamma_2\gamma_3
\end{aligned}
$$

Verify that the four-space pseudovector squares to negative one.

$$
\begin{aligned}
(\gamma_0\gamma_1\gamma_2\gamma_3)(\gamma_0\gamma_1\gamma_2\gamma_3) &= \gamma_0\gamma_1\gamma_2\gamma_3\gamma_0\gamma_1\gamma_2\gamma_3 \\
&= -\gamma_0\gamma_0\gamma_1\gamma_2\gamma_3\gamma_1\gamma_2\gamma_3 \\
&= -\gamma_0\gamma_0\gamma_1\gamma_1\gamma_2\gamma_3\gamma_2\gamma_3 \\
&= \gamma_0\gamma_0\gamma_1\gamma_1\gamma_2\gamma_2\gamma_3\gamma_3 \\
&= (-1)(1)(1)(1) = -1 \\
I &= \gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_1\sigma_2\sigma_3
\end{aligned}
$$

The fourspace fields relate to three space fields by

$$
\begin{aligned}
\mathbf{E} &= E^k \sigma_k \\
\mathbf{B} &= B^k \sigma_k \\
F &= \mathbf{E} + I\mathbf{B} \\
\mathbf{E} &= \frac{1}{2}\left(F - \gamma_0 F \gamma_0\right) \\
I\mathbf{B} &= \frac{1}{2}\left(F + \gamma_0 F \gamma_0\right) \\
\nabla &= \gamma^\mu \partial_m u \\
J &= (\rho + \mathbf{J})\gamma_0 \\
\nabla F &= J
\end{aligned}
$$

References

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