Tschirnhaus Transformations, Sylvester's Matrices and Reducing the Quintic

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Abstract

I provide a step by step presentation of the Tschirnhaus transformation using the Sylvester matrices for reducing terms in polynomial equations. A specific example converts a quintic to Bring's normal form.

The Sylvester Matrix

Sylvester's matrix is an interesting study of the vector space spanned by the product of two polynomials. Let me have two polynomials, usually of different degree, such as

$$\begin{aligned} A &= a_n p^n + a_{n-1} p^{n-1} + \ldots + a_0 \\ B &= b_m q^m + b_{m-1} q^{m-1} + \ldots + b_0 \end{aligned}$$

The product polynomial will be of degree $n + m$, and can be thought of as a linear combination of each polynomial multiplied by each the terms of the *other* polynomial.

$$\begin{aligned} AB &= a_0(b_m q^m + b_{m-1} q^{m-1} + \ldots + b_0) + \\ & a_1(b_m q^{m+1} + b_{m-1} q^{m} + \ldots + b_0 q) + \\ & a_2(b_m q^{m+2} + b_m q^{m+1} + \ldots + b_0 q^2) + \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & a_n(b_m q^{m+n} + b_m q^{m+n-1} + \ldots + b_0 q^n) \end{aligned}$$

We have a vector space, where the n equations above have a basis vector made from B, scaled by the polynomial variable.

In a similar fashion, we have m equations representing the product as a linear combination using scaled values of A as the basis set.

$$\begin{aligned}
AB &= b_0(a_n p^n + a_{n-1}p^{n-1} + \ldots + a_0) + \\
&\quad b_1(a_n p^{n+1} + a_n p^{n-1} + \ldots + a_0 q) + \\
&\quad b_2(a_n p^{n+2} + a_{n+1}p^{n-1} + \ldots + a_0 q^2) + \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad b_n(a_n p^{n+m} + a_{n+m-1}p^{n-1} + \ldots + a_0 q^m)
\end{aligned}$$

If, for some reason, we wish to look at the linear independence of these basis vectors, we can make a matrix in the following fashion, and see if it has a determinant of zero. I will illustrate the process with two polynomials of degree 3 and 4.

$$\begin{aligned}
A &= a_3 q^3 + a_2 q^2 + a_1 q + a_0 \\
B &= b_4 q^4 + b_3 q^3 + b_2 q^2 + b_1 q + b_0
\end{aligned}$$

$$M = \begin{pmatrix}
a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 \\
0 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\
0 & 0 & a_3 & a_2 & a_1 & a_0 & 0 \\
0 & 0 & 0 & a_3 & a_2 & a_1 & a_0 \\
b_4 & b_3 & b_2 & b_1 & b_0 & 0 & 0 \\
0 & b_4 & b_3 & b_2 & b_1 & b_0 & 0 \\
0 & 0 & b_4 & b_3 & b_2 & b_1 & b_0
\end{pmatrix}$$

This determinant is called the resultant, (or sometimes the eliminant), of these two polynomials. If the two polynomials share a common root, this determinant will be zero. More interestingly, if we want these these polynomials to be linearly dependent, set this determinant to zero.

Polynomial Substitution

We use this Sylvester matrix for efficient polynomial substitutions. When solving polynomial equations, one of the most common substitutions is to center the roots to eliminate the second term in a polynomial. For example, the cubic equation $x^3 + ax^2 + bx + c = 0$, upon substitution $y = x - (a/3)$ becomes the reduced cubic $y^3 + By + C$, where B and C have slightly complicated formulas. The quintic equation $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$, upon substitution $y = x - (a/5)$, becomes the reduced quintic $y^5 + By^3 + Cy^2 + Dy + E$, where B, C, D and E have somewhat more complicated formulas than the reduced cubic.

For the simple root centering, we have a linear change of variable, and simple substitution and collection of terms gives us the new polynomial. But what about a more complicated substitution, such as $y = x^2 + ax + b$? We could solve the equation above for $x(y)$, choose a solution, and carefully substitute, hoping to cancel the square root terms, but this gets harder still for higher order polynomial substitutions. We have an easier way, and that way uses the Sylvester matrix.

We write out our source polynomial and substitution polynomials, and form the Sylvester matrix. Take for example, a quartic polynomial $A = ax^4 + bx^3 + cx^2 + dx + e = 0$ and a quadratic substitution $y = fx^2 + gx + h$. Our second polynomial is $fx^2 + gx + (h - y) = 0$. Our matrix will be 6x6, with two copies of our quartic, and four copies of our quadratic. I put the higher number of copies term at the top of the matrix, as it is close to triangular form and reduces the number of row operations necessary to compute the determinant.

$$M = \begin{pmatrix} f & g & (h-y) & 0 & 0 & 0 \\ 0 & f & g & (h-y) & 0 & 0 \\ 0 & 0 & f & g & (h-y) & 0 \\ 0 & 0 & 0 & f & g & (h-y) \\ a & b & c & d & e & 0 \\ 0 & a & b & c & d & e \end{pmatrix}$$

Our determinant, set to zero, is a quartic equation in y, where the old roots have been moved in accordance with quadratic relationship above.

Let us illustrate this with a numerical example. Build a quartic equation with roots 3, 5, 7, and 11.

$$(x-3)(x-5)(x-7)(x-11) = x^4 - 26x^3 + 236x^2 - 886x + 1155$$

Make the substitution $y = 3x^2 + 7x + 13$. Our second polynomial is $3x^2 + 7x + (13 - y)$. Our matrix is

$$M = \begin{pmatrix} 3 & 7 & (13-y) & 0 & 0 & 0 \\ 0 & 3 & 7 & (13-y) & 0 & 0 \\ 0 & 0 & 3 & 7 & (13-y) & 0 \\ 0 & 0 & 0 & 3 & 7 & (13-y) \\ 1 & -26 & 236 & -886 & 1155 & 0 \\ 0 & 1 & -26 & 236 & -886 & 1155 \end{pmatrix}$$

Our determinant equation is

$$y^4 - 846 * y^3 + 223988 * y^2 - 22387554 * y + 710361531$$

Our original roots are 3, 5, 7, and 11. Our shifted roots, using $y = 3x^2 + 7x + 13$ are 61, 123, 209, and 453. Our equation from the shifted roots is $(y - 61)(y - 123)(y - 209)(y - 453)$, which, when expanded, matches the determinant equation above.

Tschirnhaus' Transform and the Cubic

Ehrenfried Walter von Tschirnhaus, in his paper of 1683, *A method for removing all intermediate terms from a given equation*, featured the elimination of quadratic and linear terms from a cubic equation, effectively solving the cubic. I am using the translation by R. F. Green from Ohio State University.

He begins with the reduced cubic, $y^3 - qy - r = 0$ as suppressing the quadratic term was well known, and the zero term simplifies the remaining expressions. He wished to suppress quadratic and linear terms. He introduces the second polynomial $z = y^2 - by - a$, which is $y^2 - by - (a + z) = 0$. He skips over the substitution process. He was not using the Sylvester method, as James Joseph Sylvester had not yet been born (by a hundred thirty years, or so.)

I use Sylvester's method below to get the transformation equations. Our original polynomial of order three is $y^3 - qy - r = 0$. Our substitution polynomial is order two, so we need a 5x5 matrix, with three copies of the substitution equation, and two copes of the original equation.

$$M = \begin{pmatrix} 1 & -b & -(a+z) & 0 & 0 \\ 0 & 1 & -b & -(a+z) & 0 \\ 0 & 0 & 1 & -b & -(a+z) \\ 1 & 0 & -q & -r & 0 \\ 0 & 1 & 0 & -q & -r \end{pmatrix}$$

The determinant of M is

$$\begin{aligned}\det(M) &= +z^3\\ &+(-2q+3a)z^2\\ &+(+3a^2-4qa+q^2-qb^2+3rb)z\\ &+(+a^3-2qa^2+q^2a-qab^2+3arb-r^2-qrb+rb^3)\end{aligned}$$

To eliminate the quaratic term, we find $a = (2q)/3$. To eliminate the linear term, we solve the quadratic for b,

$$\begin{aligned}0 &= +3a^2-4qa+q^2-qb^2+3rb\\ a &= 2q/3 \quad \text{substitute for } a\\ 0 &= +3b^2q-9rb+q^2\\ b &= \frac{9r \pm \sqrt{81r^2-12q^3}}{6q}\end{aligned}$$

Defining $M = \sqrt{81 * r^2 - 12 * q^3}$, we have the doubly reduced cubic

$$z^3 + \left(\frac{3Mr^3+27r^4}{2q^3} - 4r^2 + \frac{8q^3-6rM}{27}\right) = 0$$

The three roots for z are trivial. Once z is found, we solve (and examine for validity) the defining equation for y, which is $y^2 - by - (a+z) = 0$.

In this fashion, Tschirnhaus solved the cubic, and expected to similarly solve the quartic and quintic.

Tschirnhaus' Transform and the Quintic

I have no references from Tschinhaus for his work on the quintic. Instead, I follow the pleasant work of Felix Klein from "Lectures on the icosahedron and the solution of equations of the fifth degree" as reported in [4].

We start with the reduced quintic, and we wish to get the double reduced quintic. We will use two copies of our incoming equation is $y^5 + cy^3 + dy^2 + ey + f = 0$. We will use five copies of our substitution equation$z = y^2 + gy + h$ or $y^2 + gy + (h - z) = 0$.

We form our 7x7 Sylvester matrix.

$$M = \begin{pmatrix} 1 & g & (h-z) & 0 & 0 & 0 & 0 \\ 0 & 1 & g & (h-z) & 0 & 0 & 0 \\ 0 & 0 & 1 & g & (h-z) & 0 & 0 \\ 0 & 0 & 0 & 1 & g & (h-z) & 0 \\ 0 & 0 & 0 & 0 & 1 & g & (h-z) \\ 1 & 0 & c & d & e & f & 0 \\ 0 & 1 & 0 & c & d & e & f \end{pmatrix}$$

We form our determinant, and from the factor for z^4, learn that $h = ((2c)/5)$. We make this substitution, and run the determinant again. For the cubic term, we have the coefficient $cg^2 + 3dg + (2e - ((3c^2)/5))$.

Solving for g, we have

$$g = \frac{-3d \pm \sqrt{9d^2 - 8ce + (24c^2/5)}}{2c}$$

This term will usually be complex.

Here is a demonstration code fragment using the C programming language.

```
B = b_in/a_in;
C = c_in/a_in;
D = d_in/a_in;
E = e_in/a_in;
F = f_in/a_in; // normalize

// first substitution to go to simple reduced form

c = (5.0*C - 2.0*B*B)/5.0;
d = (25.0*D - 15.0*C*B + 4.0*B*B*B)/25.0;
e = -(3.0*B*B*B*B - 125.0*E - 15.0*C*B*B + 50.0*D*B)/125.0;
f = +((4.0*B*B*B*B*B)/3125.0) - ((C*B*B*B)/125.0) + ((D*B*B)/25.0) - ((E*B)/5.0) + F ;

// second stage substitution to go to double reduced form

h = ((2.0*c)/5.0);

M = csqrt( ((12.0*c*c*c)/5.0) - 8.0*c*e + 9.0*d*d) ;
g = (-3.0*d + M)/(2.0*c) ;

complex c2,c3,c4,c5,cm1,cm2,cm3,cm4,cm5;
complex d2,d3,d4,d5, e2,e3;

c2 = c*c; c3 = c2*c; c4 = c3*c; c5 = c4*c;
cm1 = 1.0/c; cm2 = cm1*cm1; cm3 = cm2*cm1; cm4 = cm3*cm1; cm5 = cm4*cm1;
```

```
d2 = d*d; d3 = d2*d; d4 = d3*d; d5 = d4*d;
e2 = e*e; e3 = e2*e;

M = csqrt( ((12.0*c*c*c)/5.0) - 8.0*c*e + 9.0*d*d) ;

p = + (1.0/50.0)*(225*cm3*M*d3 + 1350*cm2*d2*e - 32*c3 - 400*cm1*e2 + 125*cm1*M*f
        - 375*cm1*d*f - 675*cm3*d4 + 220*c*e + 40*M*d - 260*d2 - 350*cm2*M*d*e) ;

q = - (1.0/250.0)*(1250*cm2*M*e*f + 3500*d*f + 160*c*M*d + 640*c2*e - 450*cm1*M*d*e
       + 16875*cm3*d3*f - 1040*c*d2 - 750*e2 + 9000*cm3*d2*e2 + 1200*cm1*d2*e
       - 1000*cm2*e3 - 10125*cm4*d4*e - 5625*cm3*M*d2*f + 900*cm2*M*d3 - 96*c4
       - 11250*cm2*d*e*f - 1500*cm3*M*d*e2 - 250*M*f + 3375*cm4*M*d3*e - 2700*cm2*d4) ;

r = - (  (12.0/5.0)*d2*e - (27.0/5.0)*cm3*M*d3*e - (8.0/25.0)*c3*e + (54.0/25.0)*cm1*d4
       - (19.0/2.0)*cm1*d*e*f + (243.0/2.0)*cm5*d5*f - (72.0/5.0)*cm2*d2*e2
       + 30*cm3*d*e2*f - 135*cm4*d3*e*f + (8.0/25.0)*c*M*f+ (81.0/5.0)*cm3*d4*e
       - (2.0/5.0)*M*d*e - (18.0/25.0)*cm1*M*d3 + (8.0/5.0)*cm1*e3 - (81.0/2.0)*cm5*M*d4*f
       - 2*cm3*M*e2*f - (4.0/5.0)*c*d*f + 27*cm4*M*d2*e*f - (3.0/10.0)*cm1*M*e*f
       + (104.0/125.0)*c2*d2 + (192.0/3125.0)*c5 + (12.0/5.0)*cm2*M*d*e2 + f*f
       - (2.0/25.0)*c*e2 - (51.0/10.0)*cm2*M*d2*f - (16.0/125.0)*c2*M*d + (63.0/2.0)*cm2*d3*f );

printf("\nDouble suppressed quartic and cubic coefficients follow: \n");
printf("p = (%lf + I %lf) \n", creal(p), cimag(p));
printf("q = (%lf + I %lf) \n", creal(q), cimag(q));
printf("r = (%lf + I %lf) \n", creal(r), cimag(r));
```

Bring Normal Form

When solving the quintic using Klein's method, the double reduced form above works well. If, however, you want to follow the hypergeometric function solution, you will need to further reduce to Bring's normal form. Here I follow the example and explanation from [3].

Our original polynomial is $y^5 + 0y^4 + 0y^3 + py^2 + qy + r = 0$, and we wish to eliminate the quadratic term. Naively, we would expect to use a cubic Tschirnhaus substitution, but unfortunately, this does not work. Erland Samuel Bring did a quartic substitution, with some clever algebra, to achieve the desired result.

Our substitution polynomial will be $y^4 + ay^3 + by^2 + cy + (d - z) = 0$. We will have a 9x9 Sylvester matrix.

$$M = \begin{pmatrix} 1 & a & b & c & (d-z) & 0 & 0 & 0 & 0 \\ 0 & 1 & a & b & c & (d-z) & 0 & 0 & 0 \\ 0 & 0 & 1 & a & b & c & (d-z) & 0 & 0 \\ 0 & 0 & 0 & 1 & a & b & c & (d-z) & 0 \\ 0 & 0 & 0 & 0 & 1 & a & b & c & (d-z) \\ 1 & 0 & 0 & p & q & r & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & p & q & r & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & p & q & r & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & p & q & r \end{pmatrix}$$

When we take the determinant of M, we have a fairly complicated new polynomial. Illustrating with a snip of computer code, for the polynomial $z^5 + P_4 z^4 + P_3 z^3 + P_2 z^2 + P_1 z + P_0 = 0$

```
P4 = ( - 5*d + 4*q + 3*p*a ) ;

P3 = ( + 5*a*b*r + 3*p^2*a^2 + 3*p*b*c - 3*p^2*b - 12*p*d*a + 10*d^2
 + 4*a*q*c + 6*q^2 - 4*p*r - 16*d*q + 5*r*c + 5*p*a*q + 2*q*b^2 ) ;

P2 = ( - 15*d*a*b*r - 4*a^2*q^2*b - 3*a^3*q*r + 7*p*a^2*b*r + 4*q^2*b^2
 - 3*p^3*a*b + p*c^3 + 2*a*q*b*r - 2*p^2*q*b + 5*a*r*c^2 + 5*b^2*r*c
 - 8*p*b^2*r + 11*q*r*c + 24*d^2*q - 15*p*d*a*q - 10*d^3 + p^3*a^3
 - 2*p*q*b*c + 12*p*d*r + 8*a*q^2*c - p^2*b^3 + 5*p*a^2*q*c + 18*p*d^2*a
 - 9*p*d*b*c - p*a*r*c - 12*d*a*q*c - p*a*q*b^2 + p*a*q^2 + 4*q^3
 - 18*d*q^2 + 3*p^2*a*b*c - 8*p*q*r - 5*b*r^2 + 3*p^3*c + p^2*a^2*q
 + 9*p^2*d*b - p^2*a*r - 9*p^2*d*a2 - 3*p^2*c2 + 4*q*b*c^2 - 15*d*r*c
 - p^4 - 5*a^2*r^2 - 6*d*q*b^2 ) ;
```

Our known values are p, q, and r. Our free parameters are a, b, c, and d. We see that to eliminate P_4, the quartic term, we need to relate a and d as in $d = ((4q + 3ap)/5)$.

Looking at the P_3 and P_2 terms, we see what looks like a mess of coupled non-linear terms. I personally would give up at this point, and use a numerical root solver, but Erlang Bring did not have computers back in 1786, and was much more stubborn and clever than I am.

Bring suggested a change of variables. Instead of the two variables b and c, he substituted three, A, E and N. (I am using case sensitive variables, so $a \neq A$.) Our initial free parameters b, c and d are expressed in terms of

a, A, E and N.

$$\begin{aligned} b &= a + N \\ c &= Aa + E \\ d &= \left(\frac{4q + 3ap}{5}\right) \end{aligned}$$

Substituting these expressions in the formulas for P_3 and P_4, and collecting terms as polynomials in a, gives a quadratic equation in a for P_3, and a cubic for a in P_4.

We don't try to solve the quadratic equation in P_3. Instead, we eliminate the coefficients of this polynomial by choices of A, E, and N. Our cubic term from above, with indicated substitutions is

```
-1/5*( + 15*E*p*a + 25*r*A*a + 25*r*a^2 - 23*p*a*q - 15*p^2*a
 + 10*a^2*q + 20*E*a*q + 20*N*a*q + 15*A*p*a^2
 - 3*p^2*a^2 - 15*N*p^2 + 25*r*N*a - 2*q^2 - 20*r*p
 + 15*A*N*p*a + 25*r*E + 15*N*E*p + 10*N^2*q + 20*A*a^2*q)*z^3
```

Notice the polynomial is a quadratic equation in 'a'. We don't know what value 'a' is, but we can still eliminate the cubic term by making the coefficients of a^2, a^1, a^0 all zero. This is the clever insight of Bring.

Collecting terms in powers of a, we have

```
a^2*(  + A*(15*p + 20*q) + (25*r + 10*q - 3*p^2) )
 +
a*( + 15*E*p + 20*E*q + 20*N*q + 25*r*N + 15*A*N*p + 25*r*A - 23*p*q - 15*p^2 )
 +
( + 25*r*E + 15*N*E*p + 10*N^2*q - 15*N*p^2 - 2*q^2 - 20*r*p )
```

To eliminate the a^2 term, solve for the value A which makes this term zero.

```
A = (3p*p - 10*q - 25*r)/(15*p + 20*q).}
```

Having a value for A, we look at the linear equation (middle equation), and find E in terms of N.

```
E*(15*p + 20*q) + N*(20*q + 25*r + 15*A*p) + (25*r*A - 23*p*q - 15*p^2) ==>

E = -( N*(20*q + 25*r + 15*A*p) + (25*r*A - 23*p*q - 15*p^2) )/(15*p + 20*q)
```

We now substitute for E in the constant term (third equation), and find a quadratic equation for N.

```
 + (25*(6*p*q + 15*r*p + 9*A*p^2 - 8*q^2) )*N^2

 + (5*(125*r^2 - 9*p^2*q + 100*r*q + 150*r*A*p) )*N

 + (625*r^2*A + 30*p*q^2 - 75*r*p^2 - 175*r*p*q + 40*q^3)
```

Solve for N, then backsolve for E. I am using the positive square root.

```
ex A1, B1, C1; // quadratic

A1 =  (25*(6*p*q + 15*r*p + 9*A*p*p - 8*q*q) ) ;

B1 = (5*(125*r*r - 9*p*p*q + 100*r*q + 150*r*A*p) ) ;

C1 =  (625*r*r*A + 30*p*q*q - 75*r*p*p - 175*r*p*q + 40*q*q*q) ;

N = (-B1 + sqrt(B1*B1 - 4*A1*C1) )/(2*A1) ; // use positive square root

E = -( N*(20*q + 25*r + 15*A*p) + (25*r*A - 23*p*q - 15*p^2) )/(15*p + 20*q) ;
```

We have now eliminated the cubic term, and set values for A, N, and E.

We now look at the quadratic term P_2, and see that it is a cubic in a.

```
a^3*(
 + 75*q*r - 125*r*A + 55*p*q*A - 25*p*A^3 + 115*p*q - 125*r*A^2 + 25*p^2
 + 50*p*r + 60*p^2*A - 100*q*A^2 + 100*q^2 + 2*p^3
)
+
a^2*(
 + 60*p^2*A*N - 250*E*r*A + 50*p*r*N + 250*p*r*A + 125*r^2 + 60*E*p^2
 + 75*p^2*A^2 + 20*q^2 + 230*p*q*A + 40*q^2*A + 75*p^2*N - 250*r*A*N
 + 250*q*r + 55*E*p*q - 200*E*q*A - 100*q*A^2*N - 125*E*r - 52*p^2*q
 - 75*E*p*A^2 - 60*p^3 + 230*p*q*N + 100*q^2*N + 200*p*r )
+
a*(
 + 75*p^2*N^2 + 230*p*q*A*N + 25*q*r*A - 125*r*A*N^2 + 230*E*p*q
 + 115*p*q*N^2 + 250*q*r*N - 130*p^2*q + 40*q^2*N - 100*E^2*q
 - 31*p*q^2 + 40*E*q^2 + 125*r^2 + 250*E*p*r - 75*p^3*A - 250*E*r*N
 + 150*E*p^2*A - 60*p^3*N - 200*E*q*A*N + 60*E*p^2*N
 - 155*p^2*r + 400*p*r*N - 75*E^2*p*A - 125*E^2*r )
+
(
 + 4*q^3 - 100*E^2*q*N - 25*E^3*p - 130*p^2*q*N + 125*r^2*N
 + 75*E^2*p^2 - 40*p*q*r + 230*E*p*q*N + 200*p*r*N^2 + 25*p^4
 + 25*E*q*r - 125*E*r*N^2 + 20*q^2*N^2 + 25*p^2*N^3 - 75*E*p^3 )
```

We solve this cubic for any root of a, and thus ascertain values for a, b, c, and d. Using these values, we carry out our initial transformation, and obtain a quintic with suppressed quartic, cubic and quadratic terms.

Our triply reduced equation is in the form $x^5 + P_1 x + P_0 = 0$. We now do a simple scaling to bring the linear coefficient to negative one.

$$y = \frac{x}{\sqrt[4]{-P_1}}$$

We now have the Bring normal form

$$y^5 - y + \left(\frac{P_0}{(-P_1)\sqrt[4]{-P_1}} \right) = 0$$

Numerical Walkthrough

We start by building a quartic with known roots 3,5,7,11, and 13.

$$\begin{aligned} (x-3)(x-5)(x-7)(x-11)(x-13) &= 0 \\ x^5 - 39x^4 + 574x^3 - 3954x^2 + 12673x - 15015 &= 0 \end{aligned}$$

Next, we eliminate the quartic term by centering the roots. We use $y = x - 7.8$.

$$y^5 - 34.4y^3 - 13.44y^2 + 234.496y + 178.913280 = 0$$

Our offset roots are

```
(-4.800000 + I 0.000000)
(-2.800000 + I 0.000000)
(-0.800000 + I 0.000000)
( 3.200000 + I 0.000000)
( 5.200000 + I 0.000000)
```

Now, we eliminate the quartic and cubic terms using a quadratic transformation. $z = y^2 + gy + h$.

```
g = ( -0.586047 + I (-2.581291))
h = (-13.760000 + I ( 0.000000))
```

```
z[0] = ( y[0]*y[0] + g*y[0] + h ) = ( 12.093023 + I ( 12.390195))
z[1] = ( y[1]*y[1] + g*y[1] + h ) = ( -4.279070 + I (  7.227614))
z[2] = ( y[2]*y[2] + g*y[2] + h ) = (-12.651163 + I (  2.065032))
z[3] = ( y[3]*y[3] + g*y[3] + h ) = ( -5.395349 + I ( -8.260130))
z[4] = ( y[4]*y[4] + g*y[4] + h ) = ( 10.232558 + I (-13.422711))
```

Our transformed equation is $z^5 + pz^2 + qz + r = 0$.

```
p = (  2861.254883 + I (  -860.089934))
q = ( 36765.593952 + I ( -4510.628714))
r = (293888.517937 + I (-99933.654841))
```

Now we go to Bring form, using a quartic transformation.

$$w = z^4 + az^3 + bz^2 + cz + d = (((z + a) * z + b) * z + c) * z + d$$

```
A = (  20.514913 + I (  -12.986892))
N = (  39.998593 + I (    0.851016))
E = (2624.027206 + I ( -709.028957))
a = ( -15.694205 + I (    3.296287))
b = (  24.304388 + I (    4.147303))
c = (2344.870489 + I ( -437.586971))
d = (4170.465178 + I (10149.463818))

P4 = ( 0.000000 + I ( 0.000000))
P3 = ( 0.000002 + I ( 0.000000))
P2 = (-0.128418 + I (-0.164551))
P1 = (     152155023288725760.000000 + I (    -501602358933399040.000000))
P0 = (3972273920358288562585 6.000000 + I (2348950060199443431424 0.000000))

roots
( -5215.272078 + I (-31185.080729))
(-15302.631653 + I ( 33169.657472))
( 26950.335451 + I (-17940.118911))
(-34177.418115 + I ( -6229.575465))
( 27744.986394 + I ( 22185.117633))
```

Our triple reduced equation is $w^5 + P_1 * w + P_0 = 0$.

Divide w by the scale factor $\sqrt[4]{-P_1} = (24034.183983 + 12097.731226I)$ to get Bring's Normal Form: $z^5 - z + (-0.735805 - 3.188186I) = 0$, with roots

```
(-0.694220 + I (-0.948091))
( 0.046259 + I ( 1.356819))
( 0.594884 + I (-1.045880))
(-1.238665 + I ( 0.364292))
( 1.291742 + I ( 0.272860))
```

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