The Parson Ring Model for the Electron

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Abstract

In 1915, Alfred Lauck Parson published "A Magneton Theory of the Structure of the Atom" in Smithsonian Miscellaneous Collection, Pub 2371. In his paper, he proposed a spinning ring model of the electron, where the magnetic force associated with the current ring provided the binding forces of chemical bonds. This model received attention from Arthur H. Compton, Clinton Davisson, Lars O. Grondahl, David L. Webster, and H. Stanley Allen, but fell out of favor due to the rise of quantum mechanics. This model has been periodically rediscovered, most recently by Suichi Iida, W. Bostick, David Bergman, J. Paul Wesley, and Philip Kanarev.

Based upon the works of the previously cited authors, this note provides a closed form solution for the electric and magnetic potentials of a spinning charged ring, and discusses the strengths and weaknesses of the Parson model.

Complete Elliptic Integrals E(k) and K(k)

The solutions presented are based upon complete elliptic integrals of the first and second kind. Elliptic integrals have a variety of nomenclature, and I want to present the version I used before proceeding.

Complete Elliptic Integrals of the First Kind

$$K(k) = F(1,k) \quad = \quad \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

Using a substitution of $t = \sin(\theta)$, we have an equivalent form.

$$\begin{aligned} K(k) = F(1,k) &= \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \\ &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \\ &= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \end{aligned}$$

We now substitute $\sin^2\theta = 0.5 - 0.5 * \cos(2\theta)$, and change our variable of integration.

$$\begin{aligned} K(k) &= \frac{1}{4}\int_{-\pi/2}^{\theta=\pi/2} \frac{d(2\theta)}{\sqrt{1-k^2\left(\frac{1}{2}-\frac{1}{2}\cos(2\theta)\right)}} \\ &= \frac{1}{4}\int_{-\pi}^{\phi=\pi} \frac{d\phi}{\sqrt{1-k^2\left(\frac{1}{2}-\frac{1}{2}\cos\phi\right)}} \\ &= \frac{1}{4}\oint \frac{d\phi}{\sqrt{1-k^2\left(\frac{1}{2}-\frac{1}{2}\cos\phi\right)}} \end{aligned}$$

We do a little cleanup, and introduce b^2.

$$\begin{aligned} K(k) &= \frac{1}{4}\oint \frac{d\phi}{\sqrt{1-k^2\left(\frac{1}{2}-\frac{1}{2}\cos\phi\right)}} \\ &= \frac{1}{4}\oint \frac{d\phi}{\sqrt{\frac{2-k^2}{2}-\frac{k^2}{2}\cos\phi}} \\ &= \frac{1}{4}\sqrt{\frac{2}{2-k^2}}\oint \frac{d\phi}{\sqrt{1-\frac{k^2}{2-k^2}\cos\phi}} \\ &= \frac{1}{4}\sqrt{1+b^2}\oint \frac{d\phi}{\sqrt{1-b^2\cos\phi}} \end{aligned}$$

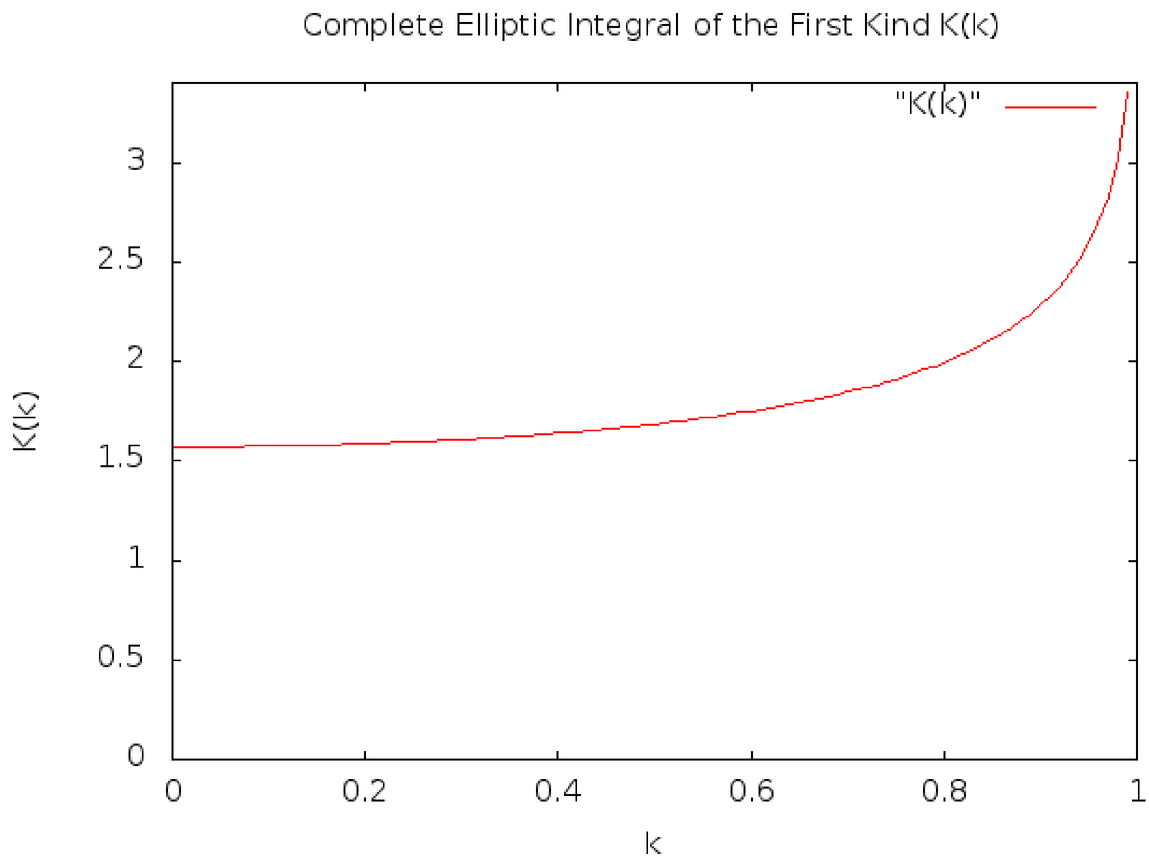

Figure 1: K(k) - Complete Elliptic Integral of the First Kind

$$b^2 = \frac{k^2}{2-k^2}$$

$$k = \sqrt{\frac{2b^2}{1+b^2}}$$

$K(k)$ has a pole to infinity at $k=1$. We will have fun cancelling infinities associated with $K(k)$ as we work through this paper. A graph of $K(k)$ is shown in Figure 1.

Complete Elliptic Integrals of the Second Kind

$$E(k) = E(1,k) \quad = \quad \int_0^1 \sqrt{\frac{(1-k^2t^2)}{(1-t^2)}}dt$$

$$\begin{aligned}
E(k) &= \int_0^1 \sqrt{\frac{(1-k^2t^2)}{(1-t^2)}}dt \\
&= \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}d\theta \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \sqrt{1-k^2\sin^2\theta}d\theta \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \sqrt{1-k^2\left(\frac{1}{2}-\frac{1}{2}\cos(2\theta)\right)}d\theta \\
&= \frac{1}{4}\int_{-\pi/2}^{\theta=\pi/2} \sqrt{1-k^2\left(\frac{1}{2}-\frac{1}{2}\cos(2\theta)\right)}d(2\theta) \\
&= \frac{1}{4}\int_{-\pi}^{\phi=\pi} \sqrt{1-k^2\left(\frac{1}{2}-\frac{1}{2}\cos\phi\right)}d\phi \\
&= \frac{1}{4}\oint \sqrt{1-k^2\left(\frac{1}{2}-\frac{1}{2}\cos\phi\right)}d\phi
\end{aligned}$$

Now, we are going to clean up and re-arrange some terms. The range for k is $0 \leq k^2 \leq 1$. We will introduce $b^2 = k^2/(2-k^2)$ shortly which will also have the range $0 \leq b^2 \leq 1$.

$$\begin{aligned}
E(k) &= \frac{1}{4}\oint \sqrt{1-k^2\left(\frac{1}{2}-\frac{1}{2}\cos\phi\right)}d\phi \\
&= \frac{1}{4}\oint \sqrt{\left(\frac{2-k^2}{2}\right)-\frac{k^2}{2}\cos\phi d\phi} \\
&= \frac{1}{4}\sqrt{\frac{2-k^2}{2}}\oint \sqrt{1-\left(\frac{k^2}{2-k^2}\right)\cos\phi d\phi}
\end{aligned}$$

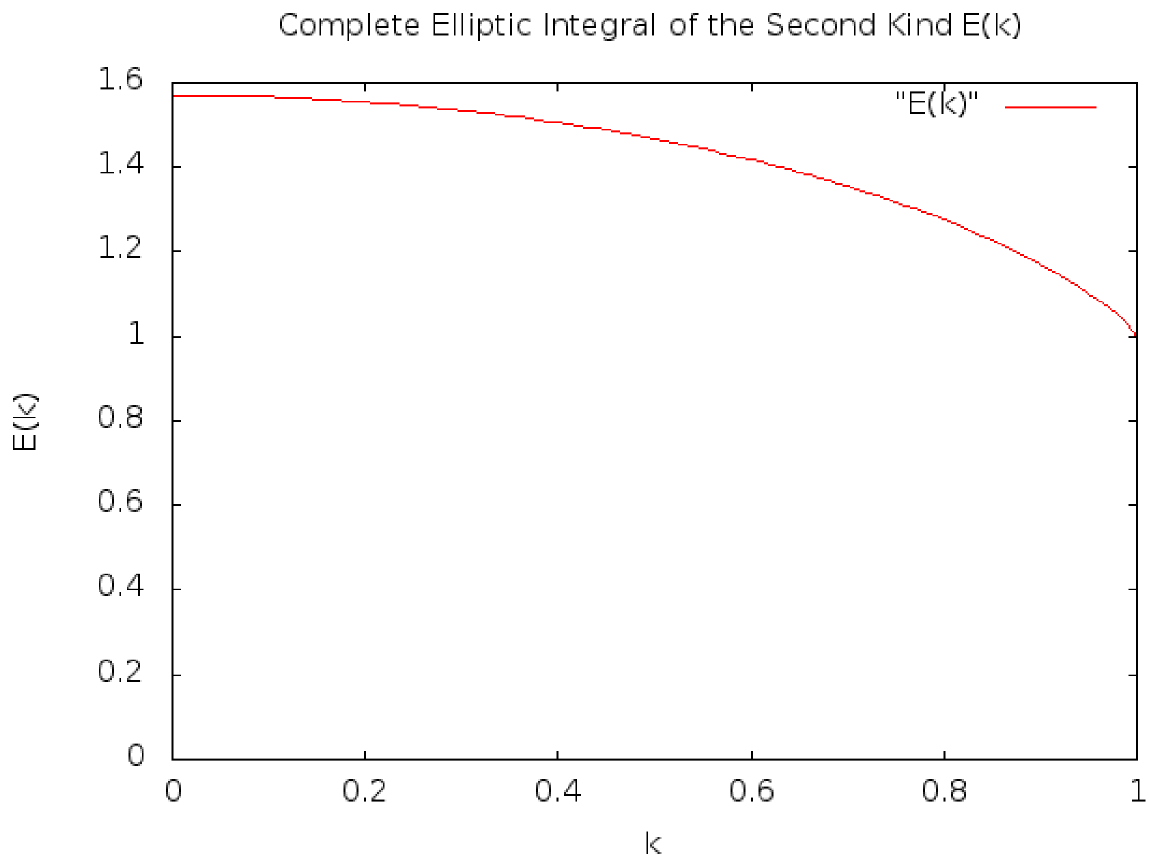

Figure 2: E(k) - Complete Elliptic Integral of the Second Kind

We now introduce b^2, and re-arrange some more.

$$
\begin{aligned}
E(k) &= \frac{1}{4}\sqrt{\frac{1}{1+b^2}} \oint \sqrt{1-b^2\cos\phi}\,d\phi \\
\oint \sqrt{1-b^2\cos\phi}\,d\phi &= 4\sqrt{1+b^2}E(k) \\
\text{with} \quad k &= \sqrt{\frac{2b^2}{1+b^2}}
\end{aligned}
$$

$E(k)$ is boring compared to $K(k)$. A graph of $E(k)$ is shown in Figure 2.

Source code for calculating both K(k) and E(k) is posted at `http://www.kurtnalty.com`.

Potential of the Charged Ring

In this section, we will calculate the electrostatic potential associated with the charged ring. Just as a mathematical point charge has an infinity associated with the field at the charge point, we will likewise have an infinity for the field at the ring radius as well. Traditionally, we don't try to measure the voltage at the ring, itself. We will see that the origin of the infinity is with the $K(k)$ term. We will later try to cancel this electrostatic infinity with a negative magnetic infinity of the same magnitude.

For these calculations, I will be using MKS units. I will put a ring of charge q and radius R in the XY plane at $z = 0$. I will place my observation point in cylindrical coordinates at (ρ, ϕ, z), where $\rho = \sqrt{x^2 + y^2}$. Due to cylindrical symmetry, this same potential will be found all around the circle at height z and radial distance same as our test point x coordinate.

My path of integration around the ring will be parameterized by angle θ, starting at the x axis and going counter-clockwise as seen from above. The potential V at an observation point (ρ, ϕ, z) is given by

$$
\begin{aligned}
q' &= \frac{q}{2\pi R} \quad \text{charge density} \\
r &= \sqrt{(x - R\cos\theta)^2 + (y - R\sin\theta)^2 + z^2} \quad \text{distance from ring to observation point} \\
&= \sqrt{x^2 + y^2 + z^2 + R^2 - 2x\cos\theta - 2y\sin\theta} \quad \text{expanded} \\
&= \sqrt{\rho^2 + z^2 + R^2 - 2\rho R\cos\theta} \quad \text{aligning } \theta \text{ and } \phi \\
V &= \frac{1}{4\pi\epsilon} \oint \left(\frac{q}{2\pi R}\right) \frac{1}{r} R d\theta \\
&= \frac{q}{8\pi^2\epsilon} \oint \frac{1}{r} d\theta \\
&= \frac{q}{8\pi^2\epsilon} \oint \frac{d\theta}{\sqrt{\rho^2 + z^2 + R^2 - 2\rho R\cos\theta}} \\
&= \frac{q}{8\pi^2\epsilon} \frac{1}{\sqrt{\rho^2 + z^2 + R^2}} \oint \frac{d\theta}{\sqrt{1 - b^2\cos\theta}} \\
b^2 &= \frac{2\rho R}{\rho^2 + z^2 + R^2}
\end{aligned}
$$

We look at this, and recognize our $K(k)$ integral in disguise.

$$\begin{aligned}
K(k) &= \frac{1}{4}\sqrt{1+b^2}\oint \frac{d\theta}{\sqrt{1-b^2\cos\theta}} \\
\oint \frac{d\theta}{\sqrt{1-b^2\cos\theta}} &= \frac{4}{\sqrt{1+b^2}}K(k) \\
&= \frac{4}{\sqrt{1+b^2}}K\left(\sqrt{\frac{2b^2}{1+b^2}}\right)
\end{aligned}$$

$$\begin{aligned}
b^2 &= \frac{2\rho R}{\rho^2+z^2+R^2} \\
1+b^2 &= \frac{2\rho R+\rho^2+z^2+R^2}{\rho^2+z^2+R^2} \\
V &= \frac{q}{8\pi^2\epsilon}\frac{1}{\sqrt{\rho^2+z^2+R^2}}\oint \frac{d\theta}{\sqrt{1-b^2\cos\theta}} \\
&= \frac{q}{8\pi^2\epsilon}\frac{1}{\sqrt{\rho^2+z^2+R^2}}\frac{4}{\sqrt{1+b^2}}K\left(\sqrt{\frac{2b^2}{1+b^2}}\right) \\
&= \frac{q}{2\pi^2\epsilon}\frac{1}{\sqrt{2\rho r+\rho^2+z^2+R^2}}K\left(\sqrt{\frac{2b^2}{1+b^2}}\right)
\end{aligned}$$

We now substitute for b^2 to get a nice, standalone expression.

$$\begin{aligned}
V(\rho,\phi,z) &= \frac{q}{2\pi^2\epsilon}\frac{1}{\sqrt{2\rho R+\rho^2+z^2+R^2}}K\left(\sqrt{\frac{4\rho R}{2\rho R+\rho^2+z^2+R^2}}\right) \\
&= \frac{q}{2\pi^2\epsilon}\frac{1}{\sqrt{(\rho+R)^2+z^2}}K\left(\sqrt{\frac{4\rho R}{(\rho+R)^2+z^2}}\right) \\
V(\rho,\phi,z) &= \frac{q}{2\pi^2\epsilon}\frac{1}{\sqrt{(\rho+R)^2+z^2}}K(k) \\
k &= \sqrt{\frac{4\rho R}{(\rho+R)^2+z^2}}
\end{aligned}$$

Illustrations for the Voltage Field

Figure 3 provides an illustration of the voltage along a line from $r = 0$ to $r = 2.5m$ at $z = 0$ for a $1m$ radius filament charged to $1\mu C$. Figure 4 provides a contour map of isolines of voltage in the r, z plane.

The Vector Potential A of a Current Loop

The rotating ring has a current flow $I = q/T = qf = q\omega/(2\pi)$. This current induces a vector potential A. Due to the cylindrical symmetry of the ring, the only nonzero component will be A_θ. Assume a current loop in the XY plane at $Z = 0$ of radius R carrying a current I. The observation point (ρ, ϕ, z) is in cylindrical coordinates, with $\rho = \sqrt{x^2 + y^2}$, and z being the height above the coil.

$$\begin{aligned}
\vec{A}(\rho, \phi, z) &= \vec{a}_\phi \frac{\mu I R}{4\pi} \oint \frac{\cos\theta d\theta}{\sqrt{z^2 + \rho^2 + R^2 - 2R\rho\cos\theta}} \\
A_\phi &= \frac{\mu I R \sqrt{a+b}}{\pi b} \left[\left(1 - \frac{k^2}{2}\right) K(k) - E(k) \right]
\end{aligned}$$

where K and E are complete elliptic integrals, and

$$\begin{aligned}
a &= z^2 + \rho^2 + R^2 \\
b &= 2R\rho \\
k &= \sqrt{\frac{2b}{a+b}} = \sqrt{\frac{4R\rho}{z^2 + (R+\rho)^2}} \\
a + b &= z^2 + (\rho + R)^2
\end{aligned}$$

We see the k argument matches that for the electrostatic potential.

Expanding k in the some of the expressions for A_ϕ, we have

$$A_\phi = \frac{\mu I}{2\pi} \frac{\sqrt{z^2 + (\rho+R)^2}}{\rho} \left[\left(\frac{z^2 + \rho^2 + R^2}{z^2 + (\rho + R)^2} \right) K(k) - E(k) \right]$$

Illustrations for the Vector Potential

Due to circular symmetry, the only non-zero vector potential component is A_θ. Figure 3 provides an illustration of the voltage along a line from $r = 0$

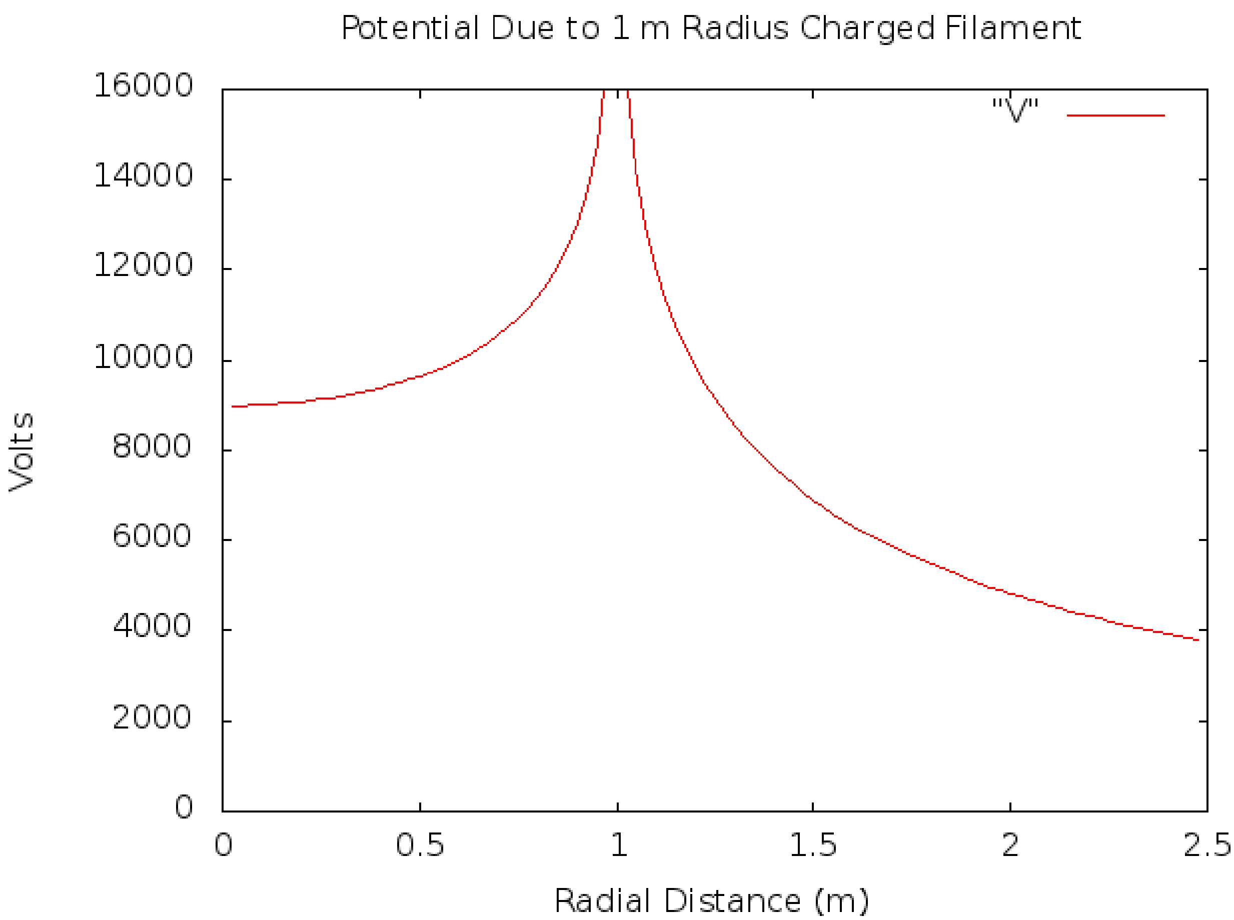

Figure 3: Voltage along Z=0, radial axis for 1uC, 1m radius filament

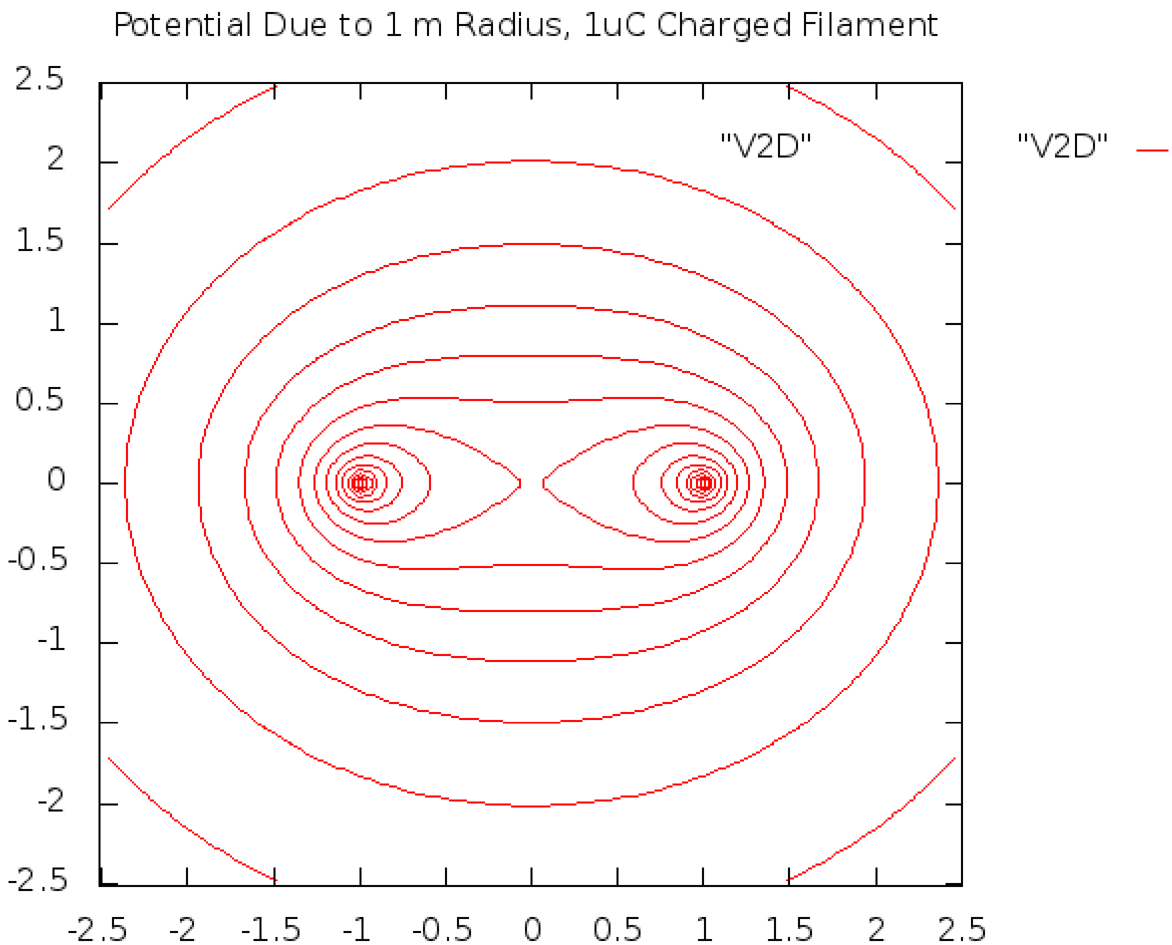

Figure 4: Contour Plot for Voltage Equipotentials

to $r = 2.5m$ at $z = 0$ for a $1m$ radius filament charged to $1\mu C$. Figure 4 provides a contour map of isolines of voltage in the r, z plane. On the contour plot, the right side is up, out of the page, while the left side is downwards, into the page.

Combined Potential

Repeating our expressions for V and A_ϕ

$$\begin{aligned} V(\rho,\phi,z) &= \frac{q}{2\pi^2\epsilon}\frac{1}{\sqrt{(R+\rho)^2+z^2}}K(k) \\ A_\phi(\rho,\phi,z) &= \frac{\mu I}{2\pi}\frac{\sqrt{z^2+(\rho+R)^2}}{\rho}\left[\left(\frac{z^2+\rho^2+R^2}{z^2+(\rho+R)^2}\right)K(k)-E(k)\right] \end{aligned}$$

The combined electrodynamic field experienced by the charge is $V - \vec{v}\cdot\vec{A}$, where the first term is the voltage due to electrostatic charges, and the second term is generator term due to motion in a magnetic field. Both terms will have singularities at $k = 1$ which occurs at $r = R, z = 0$. These singularities are due to $K(1) = \infty$. Our plan is to find where these singularities from the electrostatic and magnetic terms cancel each other. The well behaved function $E(k)$ has no singularities, and has $E(1) = 1$.

$$\begin{aligned} V - \vec{v}\cdot\vec{A} &= \frac{q}{2\pi^2\epsilon}\frac{1}{\sqrt{(R+\rho)^2+z^2}}K(k) \\ &\quad -\frac{\mu v I}{2\pi}\frac{\sqrt{z^2+(\rho+R)^2}}{\rho}\left[\left(\frac{z^2+\rho^2+R^2}{z^2+(\rho+R)^2}\right)K(k)-E(k)\right] \end{aligned}$$

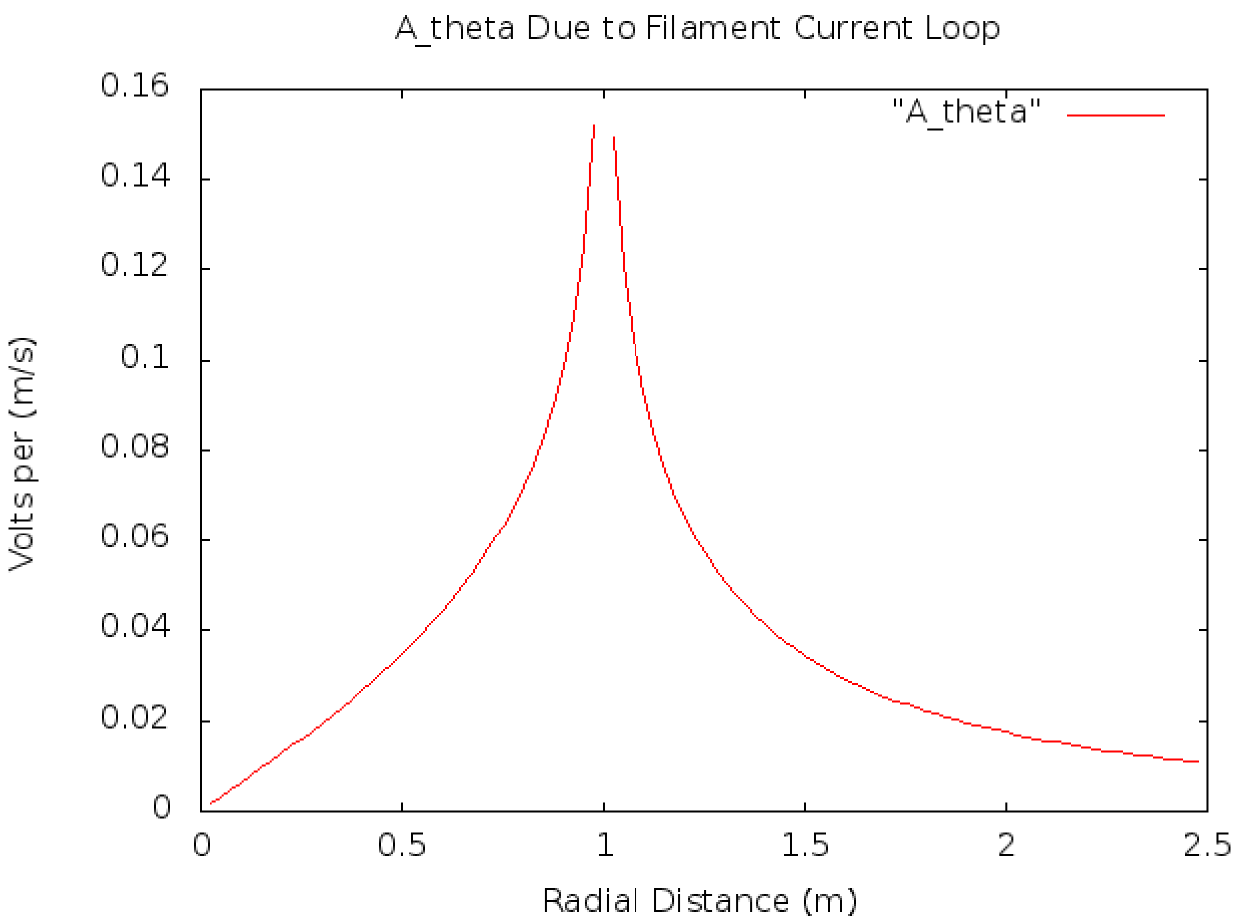

Figure 5: A_θ along z=0, radial axis for 200kA, 1m radius current filament

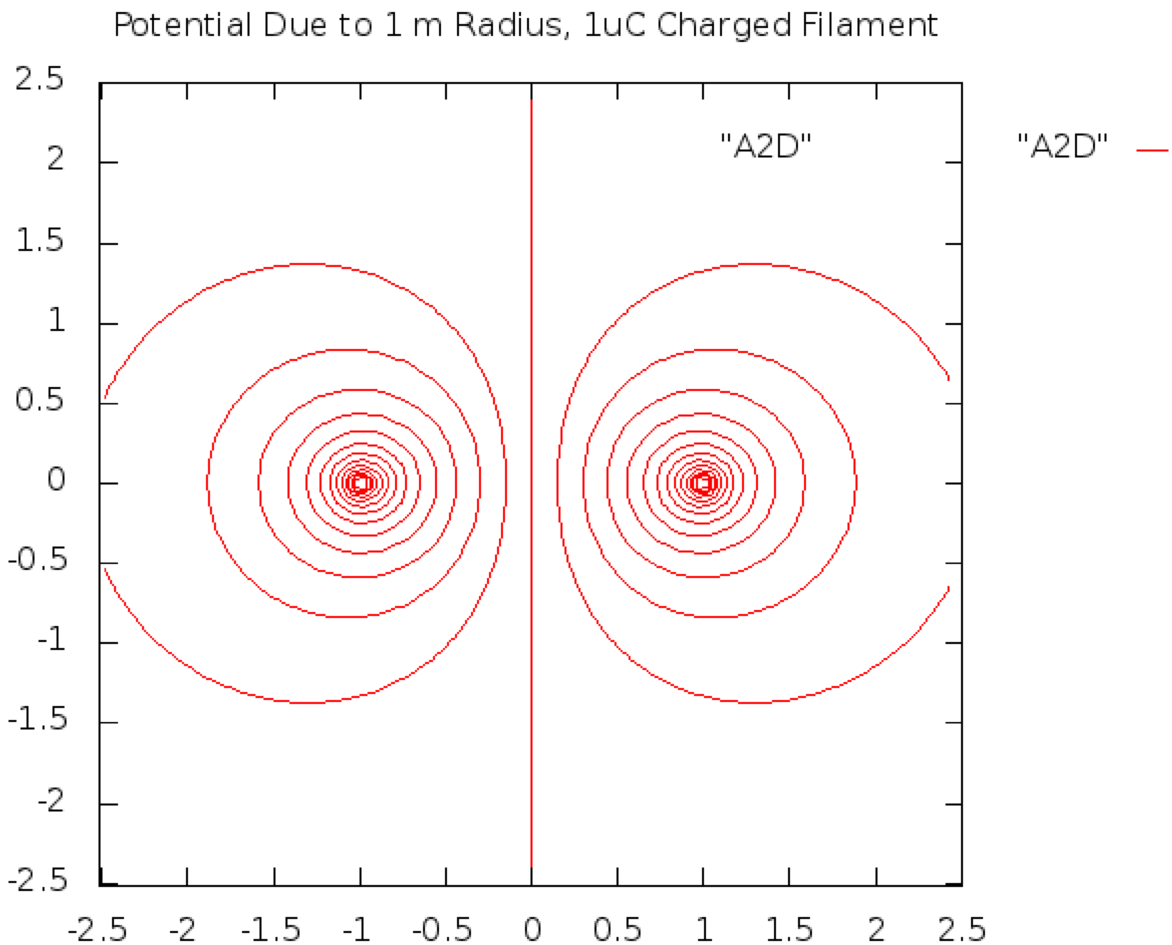

Figure 6: Contour Plot for A_θ Equipotentials

Setting $z = 0$ and $\rho = R$, we specialize to

$$\begin{aligned}
V - \vec{v}\cdot\vec{A} &= \frac{q}{2\pi^2\epsilon}\frac{1}{\sqrt{(2R)^2}}K(k) \\
&\quad -\frac{\mu v I}{2\pi}\frac{\sqrt{(2R)^2}}{R}\left[\left(\frac{2R^2}{(2R)^2}\right)K(k) - E(k)\right] \\
&= \frac{q}{4R\pi^2\epsilon}K(k) - \frac{\mu v I}{\pi}\left[\left(\frac{1}{2}\right)K(k) - E(k)\right] \\
&= K(k)\left[\frac{q}{4R\pi^2\epsilon} - \frac{\mu v I}{2\pi}\right] - E(k)\left[\frac{\mu v I}{\pi}\right]
\end{aligned}$$

To kill the singularity at $K(k = 1)$ requires the factor of K to go to zero.

$$\left[\frac{q}{4R\pi^2\epsilon} - \frac{\mu v I}{2\pi}\right] = 0$$

$$\frac{q}{4R\pi^2\epsilon} = \frac{\mu v I}{2\pi}$$

The current due to a circulating ring is $I = qv/2\pi R$, so

$$\begin{aligned}
\frac{q}{4R\pi^2\epsilon} &= \frac{\mu v I}{2\pi} \\
I &= \frac{qv}{2\pi R} \\
\frac{q}{4R\pi^2\epsilon} &= \frac{\mu q v^2}{4\pi^2 R} \\
v^2 &= \frac{1}{\mu\epsilon} \\
v^2 &= c^2 \\
v &= \pm c
\end{aligned}$$

We see that the criteria for elimination of infinities, is simply that the ring *must* rotate at c. I suggest the $\pm$ corresponds to clockwise or counter-clockwise current choices.

Given that the ring rotates at c, it is now trivial to find the effective potential at the ring. On the ring, we have $z = 0$, $\rho = R$, $k = 1$, and

$E(1) = 1$.

$$\begin{aligned} V - \vec{v} \cdot \vec{A} &= K(k) \left[\frac{q}{4R\pi^2\epsilon} - \frac{\mu v I}{2\pi} \right] - E(k) \left[\frac{\mu v I}{\pi} \right] \\ &= -\frac{\mu c I}{\pi} \\ &= -\frac{\mu c^2 q}{2\pi^2 R} \\ &= -\frac{q}{2\epsilon\pi^2 R} \end{aligned}$$

Illustrations of the Combined Potential

To give a feel for the $K(k = 1)$ infinities, and the nice cancellation at c, I have plotted the combined field as a function of radial distance along the $z = 0$ line for cases of $0.95c$, $1.0c$ and $1.05c$ in Figures 7, 8 and 9. Figure 10 provides a contour plot of the combined potential at $v = c$, showing the lack of singulaties, and suggesting how easy it may be to mistake a charged luminal ring for an ideal point source, when all we see are circular contours once we are a few diameters away from the ring.

B and E Fields

We can try to get the forces on the ring elements by using the Lorentz force law.

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

For this stationary arrangement of charge and current, we will try

$$\begin{aligned} \vec{E} &= -\vec{\nabla} V \\ \vec{B} &= \vec{\nabla} \times \vec{A} \end{aligned}$$

We will see later that we must include the convective derivative in E.

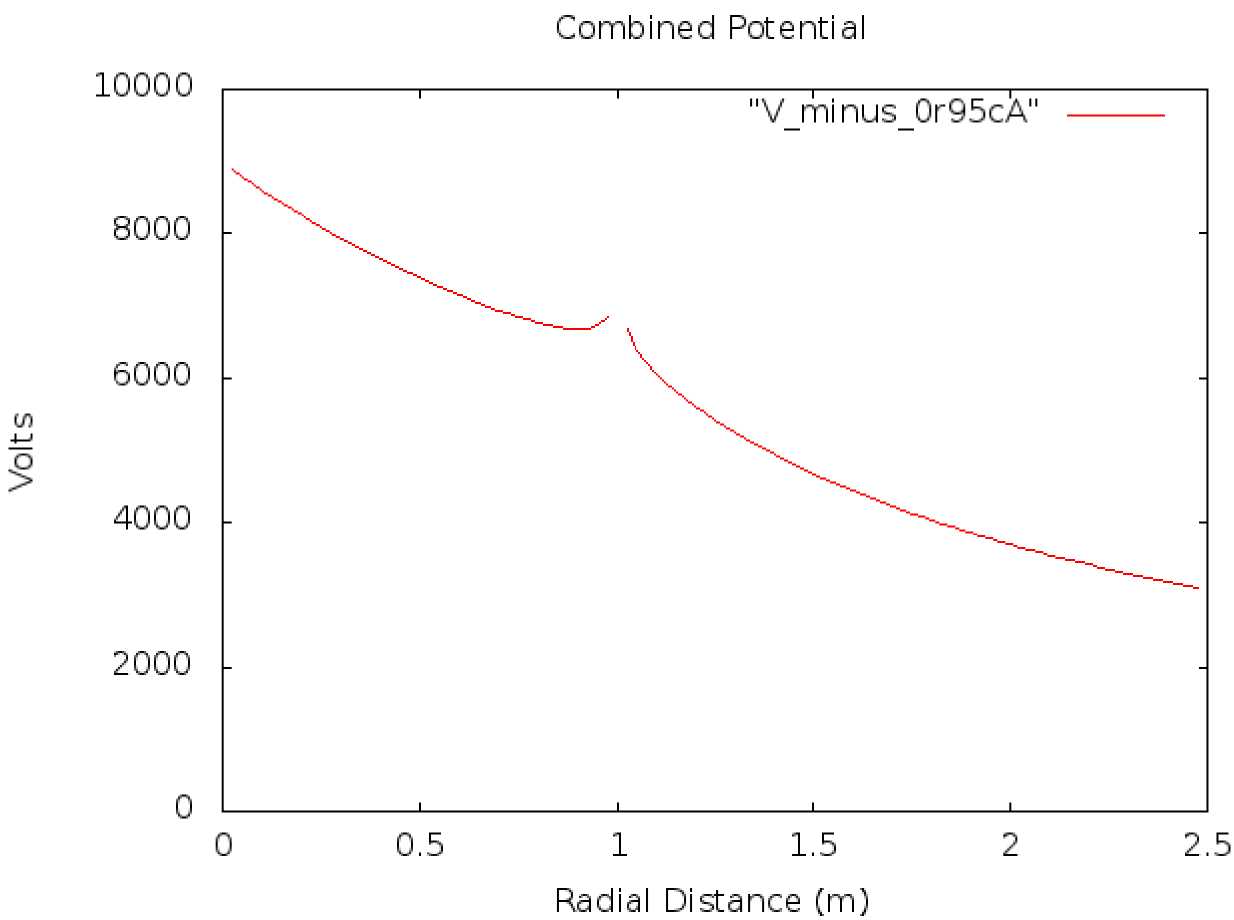

Figure 7: Combined Potential at 0.95c. Notice Positive Spike.

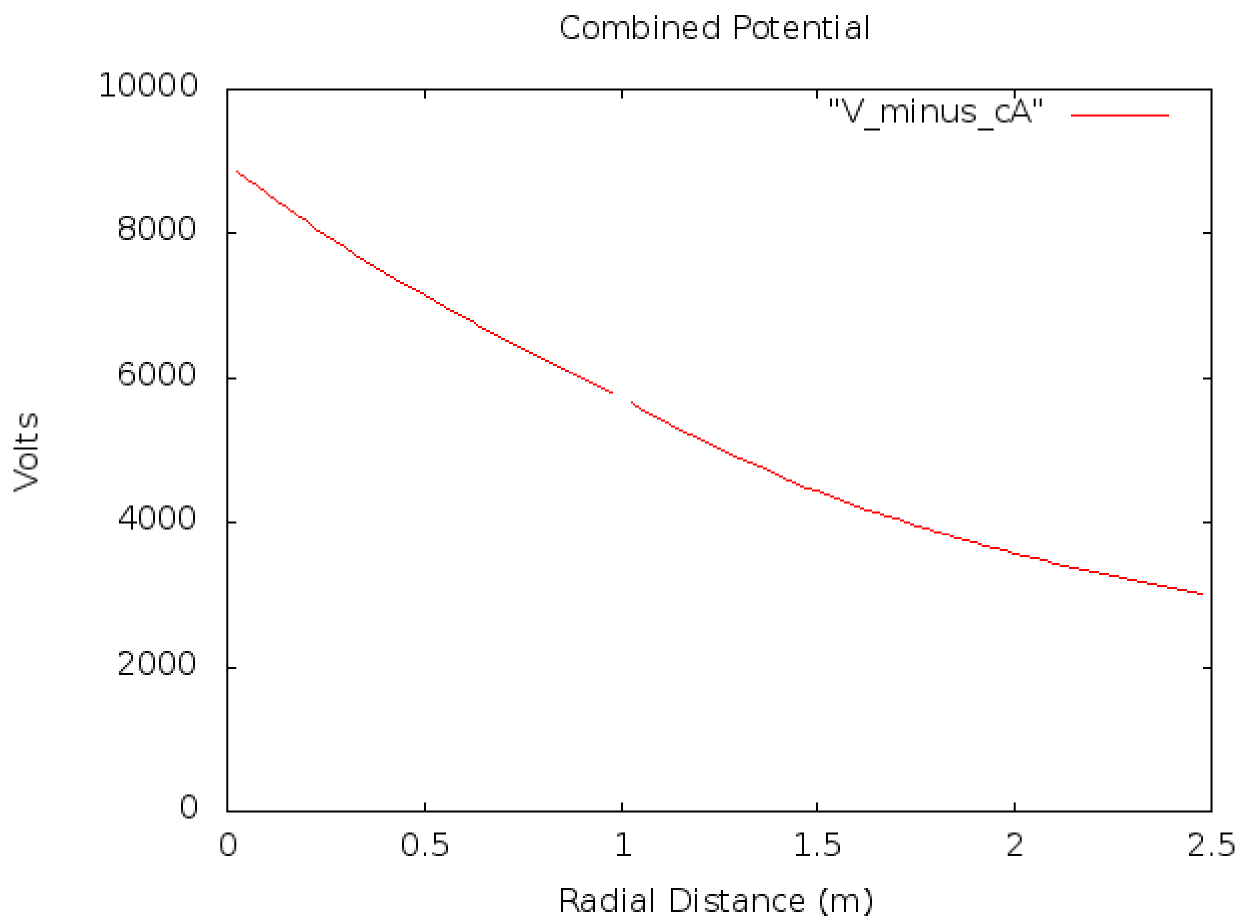

Figure 8: Combined Potential at c. Notice Smooth Curve.

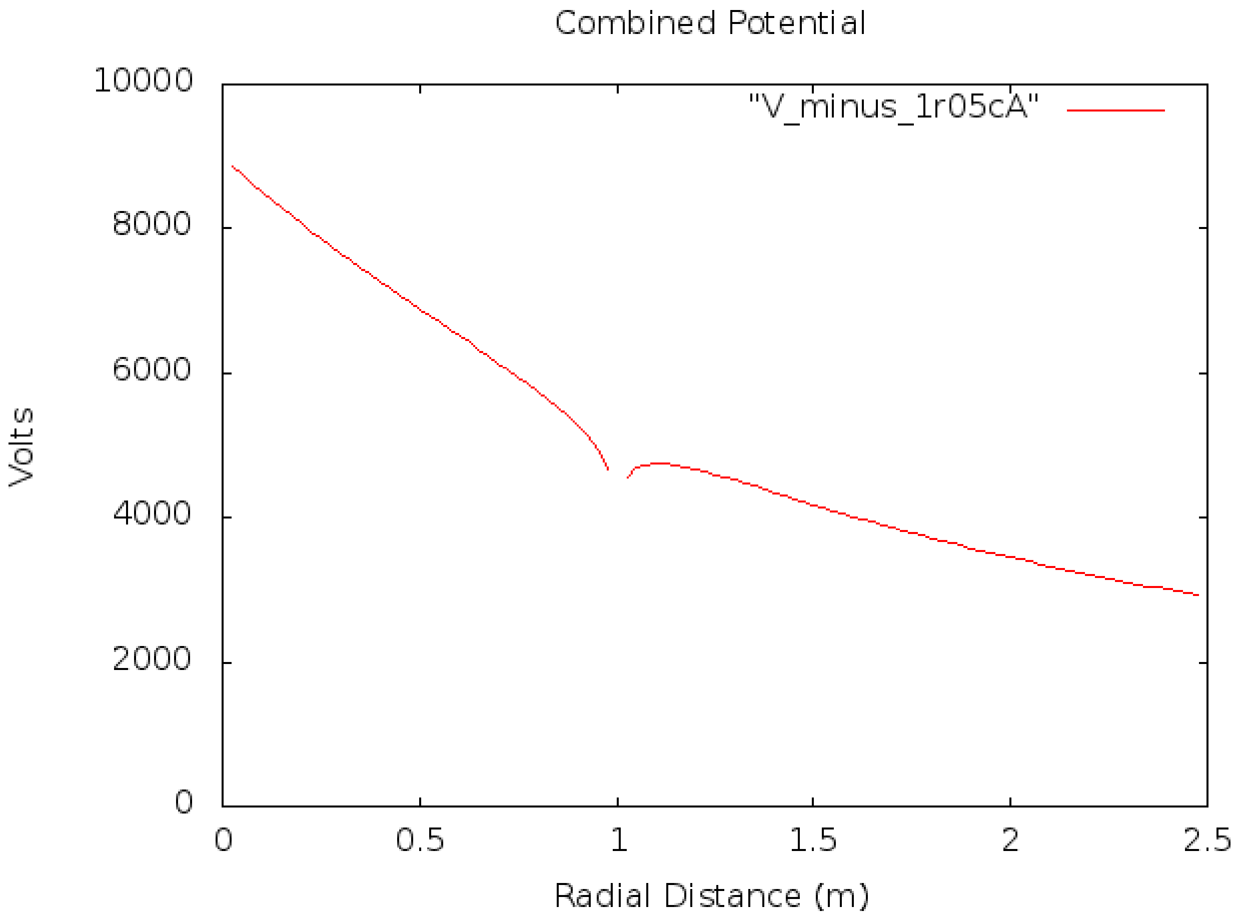

Figure 9: Combined Potential at 1.05c. Notice Negative Spike.

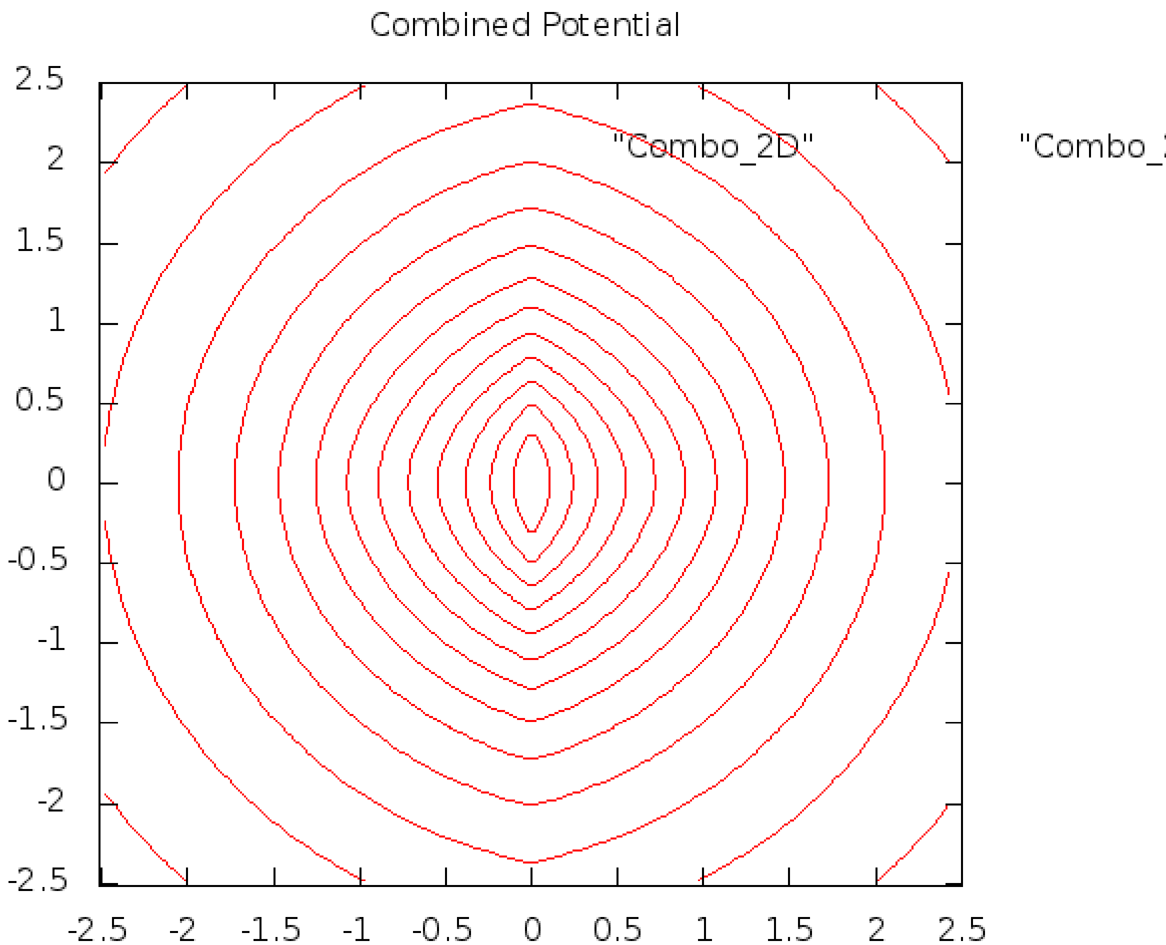

Figure 10: Contour Plot Combined Potentials. Notice *NO* Infinities.

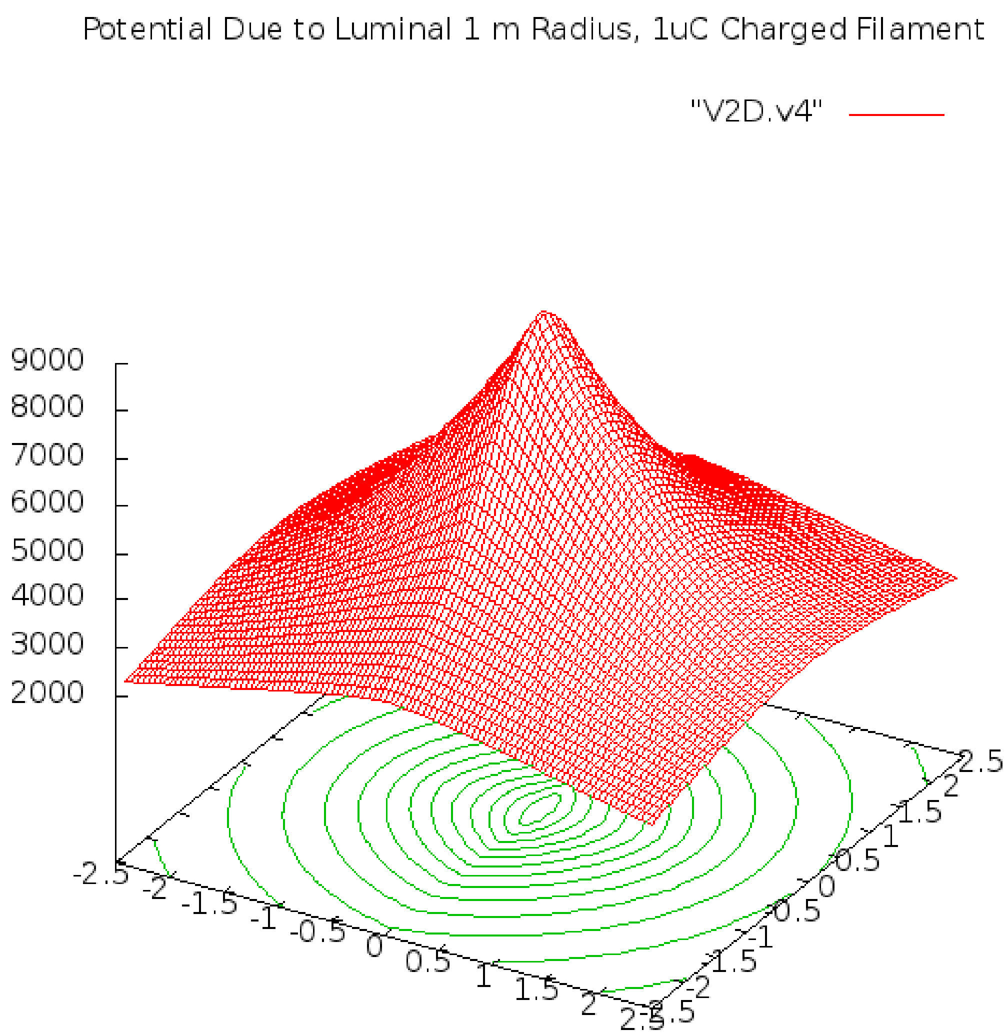

Figure 11: Combined Potential Contour to 2.5 Radii

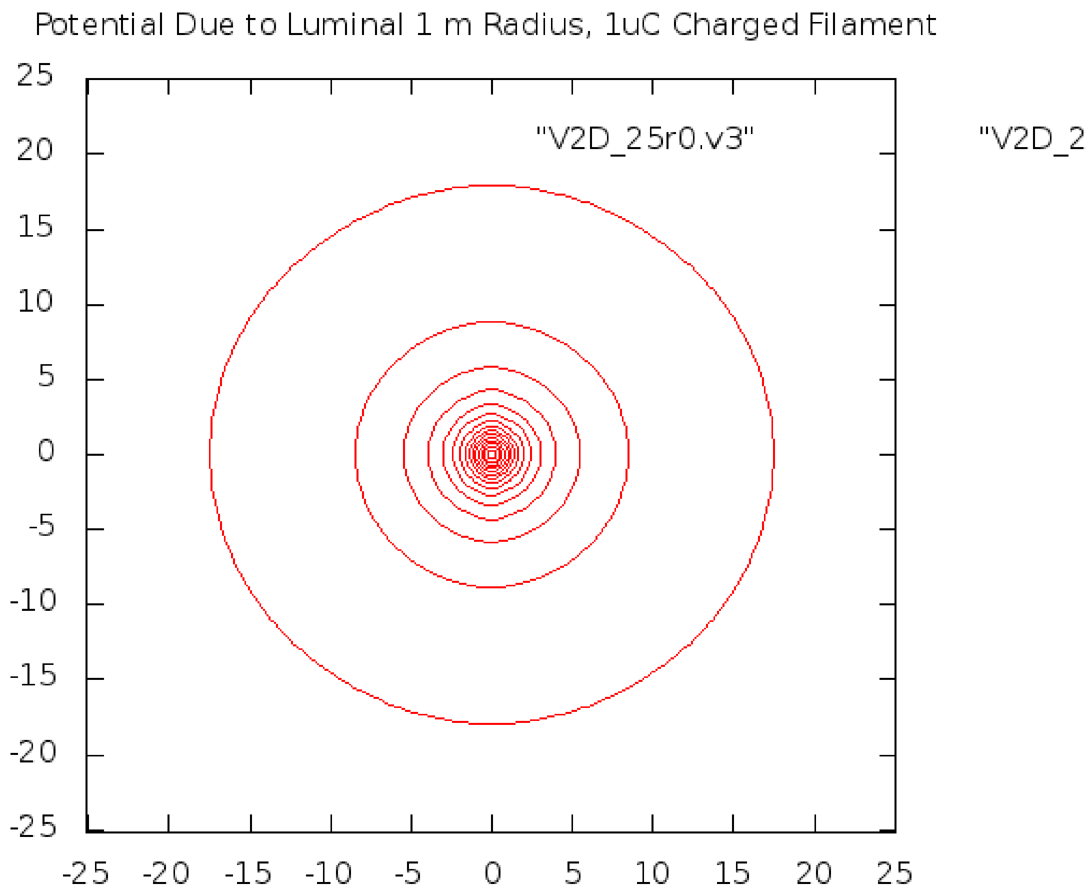

Figure 12: Combined Potential Contour to 25 Radii

We are working in cylindrical coordinates. The gradient and curl components are

$$\begin{aligned}
\vec{\nabla} V &= \vec{a}_\rho \frac{\partial V}{\partial \rho} + \vec{a}_\phi \frac{1}{\rho}\frac{\partial V}{\partial \phi} + \vec{a}_z \frac{\partial V}{\partial z} \\
\vec{\nabla} \times \vec{A} &= \vec{a}_\rho \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) + \vec{a}_\phi \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) + \vec{a}_z \frac{1}{\rho} \left(\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi} \right)
\end{aligned}$$

Repeating our potentials, we see V is independent of ϕ, and $A_\rho = A_z = 0$.

$$\begin{aligned}
k &= \sqrt{\frac{4R\rho}{z^2 + (R+\rho)^2}} \\
V(\rho, \phi, z) &= \frac{q}{2\pi^2 \epsilon} \frac{1}{\sqrt{(R+\rho)^2 + z^2}} K(k) \\
A_\phi(\rho, \phi, z) &= \frac{\mu I}{2\pi} \frac{\sqrt{z^2 + (\rho+R)^2}}{\rho} \left[\left(\frac{z^2 + \rho^2 + R^2}{z^2 + (\rho+R)^2} \right) K(k) - E(k) \right]
\end{aligned}$$

Consequently, our $\vec{E}$ and $\vec{B}$ fields simplify to

$$\begin{aligned}
\vec{E} &= -\frac{\partial V}{\partial \rho} \vec{a}_\rho - \frac{\partial V}{\partial z} \vec{a}_z \\
\vec{B} &= -\frac{\partial A_\phi}{\partial z} \vec{a}_\rho + \frac{1}{\rho} \frac{\partial(\rho A_\phi)}{\partial \rho} \vec{a}_z
\end{aligned}$$

Our derivatives for the complete elliptic functions are

$$\begin{aligned}
\frac{K(k)}{dk} &= \frac{E(k) - (1-k^2)K(k)}{k(1-k^2)} \\
\frac{E(k)}{dk} &= \frac{E(k) - K(k)}{k}
\end{aligned}$$

Carefully keeping track of our terms, we find

$$\begin{aligned}
B_r &= -\frac{\mu I z}{2\pi\rho\sqrt{(R+\rho)^2+z^2}}\left(K(k)-E(k)\frac{R^2+\rho^2+z^2}{(R-\rho)^2+z^2}\right)\\
B_\phi &= 0\\
B_z &= \frac{\mu I}{2\pi\sqrt{(R+\rho)^2+z^2}}\left(K(k)+E(k)\frac{R^2-\rho^2-z^2}{(R-\rho)^2+z^2}\right)\\
E_\rho &= \frac{q}{4\pi\epsilon\rho\sqrt{(R+\rho)^2+z^2}}\left(\frac{K(k)}{\pi}-\left[\frac{E(k)}{\pi}\frac{z^2+R^2-\rho^2}{(R-\rho)^2+z^2}\right]\right)\\
E_\phi &= 0\\
E_z &= \frac{q}{4\pi\epsilon\rho\sqrt{(\rho+R)^2+z^2}}\left(\frac{E(k)}{\pi}\frac{2\rho z}{(R-\rho)^2+z^2}\right)
\end{aligned}$$

Individually, each of these terms goes infinite as we approach the ring at $(R,\phi,0)$. Let's now look at the combined forces from $\vec{F}=q(\vec{E}+\vec{v}\times\vec{B})$.

Radial Combined Force Component F_ρ

We start with the radial component.

$$\begin{aligned}
\frac{F_\rho}{q} &= E_\rho + cB_z\\
&= \frac{q}{4\pi\epsilon\rho\sqrt{(R+\rho)^2+z^2}}\left(\frac{K(k)}{\pi}-\left[\frac{E(k)}{\pi}\frac{z^2+R^2-\rho^2}{(R-\rho)^2+z^2}\right]\right)\\
&\quad+\frac{c\mu I}{2\pi\sqrt{(R+\rho)^2+z^2}}\left(K(k)+E(k)\frac{R^2-\rho^2-z^2}{(R-\rho)^2+z^2}\right)
\end{aligned}$$

Substituting for I, we have

$$\begin{aligned}
I &= \frac{qc}{2\pi R}\\
\frac{F_\rho}{q} &= \frac{q}{4\pi\epsilon\rho\sqrt{(R+\rho)^2+z^2}}\left(\frac{K(k)}{\pi}-\left[\frac{E(k)}{\pi}\frac{z^2+R^2-\rho^2}{(R-\rho)^2+z^2}\right]\right)\\
&\quad+\frac{c^2\mu q}{4\pi^2R\sqrt{(R+\rho)^2+z^2}}\left(K(k)+E(k)\frac{R^2-\rho^2-z^2}{(R-\rho)^2+z^2}\right)
\end{aligned}$$

Now, $c^2 = 1/(\mu\epsilon)$, so

$$
\begin{aligned}
\frac{F_\rho}{q} &= \frac{q}{4\pi^2\epsilon\rho\sqrt{(R+\rho)^2+z^2}}\left(K(k) - E(k)\frac{z^2+R^2-\rho^2}{(R-\rho)^2+z^2}\right) \\
&\quad + \frac{q}{4\pi^2\epsilon R\sqrt{(R+\rho)^2+z^2}}\left(K(k) + E(k)\frac{R^2-\rho^2-z^2}{(R-\rho)^2+z^2}\right) \\
&= \frac{q}{4\pi^2\epsilon\sqrt{(R+\rho)^2+z^2}}G \\
G &= \left(\frac{1}{\rho}K(k) + \frac{1}{R}K(k)\right) + \left(\frac{-E(k)}{\rho}\frac{R^2-\rho^2+z^2}{(R-\rho)^2+z^2} + \frac{E(k)}{R}\frac{R^2-\rho^2-z^2}{(R-\rho)^2+z^2}\right) \\
&= \frac{\rho+R}{\rho R}K(k) - \frac{\rho+R}{\rho R}E(k)
\end{aligned}
$$

$$
\begin{aligned}
\frac{F_\rho}{q} &= \frac{q}{4\pi^2\epsilon\sqrt{(R+\rho)^2+z^2}}\left[\frac{\rho+R}{\rho R}K(k) - \frac{\rho+R}{\rho R}E(k)\right] \\
&= \frac{q}{4\pi\epsilon\rho\sqrt{(R+\rho)^2+z^2}}\left[\frac{\rho+R}{R}\left(\frac{K(k)-E(k)}{\pi}\right)\right]
\end{aligned}
$$

What we see here is very interesting. This formula still has an infinity on the ring due to $K(k=1)$. To me, this implies that $F = q(\vec{E} + \vec{v} \times B)$ with the static formulas for $\vec{E}$ and $\vec{B}$ is incomplete.

Axial Combined Force Component F_z

$$
\begin{aligned}
\frac{F_z}{q} &= E_z - cB_r \\
&= \frac{q}{4\pi\epsilon\rho\sqrt{(\rho+R)^2+z^2}}\left(\frac{E(k)}{\pi}\frac{2\rho z}{(R-\rho)^2+z^2}\right) \\
&\quad + \frac{c\mu I z}{2\pi\rho\sqrt{(R+\rho)^2+z^2}}\left(K(k) - E(k)\frac{R^2+\rho^2+z^2}{(R-\rho)^2+z^2}\right)
\end{aligned}
$$

Now

$$I = \frac{qc}{2\pi R} \quad \text{so,}$$

$$
\begin{aligned}
\frac{F_z}{q} &= \frac{q}{4\pi\epsilon\rho\sqrt{(\rho+R)^2+z^2}}\left(\frac{E(k)}{\pi}\frac{2\rho z}{(R-\rho)^2+z^2}\right) \\
&\quad + \frac{c^2\mu q z}{4\pi^2 R\rho\sqrt{(R+\rho)^2+z^2}}\left(K(k)-E(k)\frac{R^2+\rho^2+z^2}{(R-\rho)^2+z^2}\right) \\
&= \frac{q}{4\pi\epsilon\rho\sqrt{(\rho+R)^2+z^2}}\left(\frac{E(k)}{\pi}\frac{2\rho z}{(R-\rho)^2+z^2}\right) \\
&\quad + \frac{qz}{4\pi^2\epsilon\rho R\sqrt{(R+\rho)^2+z^2}}\left(K(k)-E(k)\frac{R^2+\rho^2+z^2}{(R-\rho)^2+z^2}\right) \\
&= \frac{qz}{4\pi^2\epsilon\rho R\sqrt{(\rho+R)^2+z^2}}\left(E(k)\frac{2\rho R}{(R-\rho)^2+z^2}\right) \\
&\quad + \frac{qz}{4\pi^2\epsilon\rho R\sqrt{(R+\rho)^2+z^2}}\left(K(k)-E(k)\frac{R^2+\rho^2+z^2}{(R-\rho)^2+z^2}\right) \\
&= \frac{qz}{4\pi^2\epsilon\rho R\sqrt{(R+\rho)^2+z^2}}\left(K(k)-E(k)\frac{R^2+\rho^2+z^2-2\rho R}{(R-\rho)^2+z^2}\right) \\
&= \frac{qz}{4\pi^2\epsilon\rho R\sqrt{(R+\rho)^2+z^2}}\left(K(k)-E(k)\right) \\
&= \frac{q}{4\pi\epsilon\rho\sqrt{(R+\rho)^2+z^2}}\frac{z}{R}\left(\frac{K(k)-E(k)}{\pi}\right)
\end{aligned}
$$

We see that as we take $z = 0$, this term goes to zero.

Summary of Combined Forces

$$
\begin{aligned}
\frac{F_\rho}{q} &= \frac{q}{4\pi\epsilon\rho\sqrt{(R+\rho)^2+z^2}}\left[\frac{\rho+R}{R}\left(\frac{K(k)-E(k)}{\pi}\right)\right] \\
\frac{F_z}{q} &= \frac{q}{4\pi\epsilon\rho\sqrt{(R+\rho)^2+z^2}}\left[\frac{z}{R}\left(\frac{K(k)-E(k)}{\pi}\right)\right]
\end{aligned}
$$

We have the interesting result that the composite force is always radial.

$$\begin{aligned}
\frac{\vec{F}}{q} &= \frac{q}{4\pi\epsilon\rho\sqrt{(R+\rho)^2+z^2}}\left[\frac{\rho+R}{R}\left(\frac{K(k)-E(k)}{\pi}\right)\right]\vec{a}_\rho \\
&\quad + \frac{q}{4\pi\epsilon\rho\sqrt{(R+\rho)^2+z^2}}\left[\frac{z}{R}\left(\frac{K(k)-E(k)}{\pi}\right)\right]\vec{a}_z \\
&= \frac{q}{4\pi\epsilon\rho R}\left(\frac{K(k)-E(k)}{\pi}\right)\vec{a}_r
\end{aligned}$$

A simple numerical check, available at `http://www.kurtnalty.com/Verify_Force_Far_Field.c`, verifies that

$$\frac{1}{\rho R}\frac{K(k)-E(k)}{\pi} \to \frac{1}{\rho^2+z^2}$$

leading to the conventional force law a few diameters from the ring.

Removing the Infinity by Including the Convective Derivative

Much as I like the correspondance of the previous section with far field EM, I recognise that the infinity associated with $K(k=1)$ at the ring cannot occur given the smooth potential created by the assumption of the luminal ring. The missing terms which cancel this infinity at the ring, turn out to be the convective derivative terms of $\vec{A}$, which are traditionally part of the $\vec{E}$ field.

The correct starting point should have been

$$\begin{aligned}
\vec{E} &= -\vec{\nabla}V + (\vec{v}\cdot\vec{\nabla})\vec{A} \\
\vec{B} &= \vec{\nabla}\times\vec{A}
\end{aligned}$$

At the ring itself, $\vec{v}$ is both source and sensor. In conventional electrodynamics, the convective derivative term is bundled in the time partial derivative of $\vec{A}$ term.

For this ring, we have (in cylindrical coordinates)

$$
\begin{aligned}
\vec{v} &= c\vec{a}_\phi \\
\vec{\nabla} &= \vec{a}_\rho \frac{\partial}{\partial \rho} + \vec{a}_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi} + \vec{a}_z \frac{\partial}{\partial z} \\
\vec{v}\cdot\vec{\nabla} &= \frac{c}{\rho}\frac{\partial}{\partial \phi} \\
(\vec{v}\cdot\nabla)\vec{A} &= \frac{c}{\rho}\frac{\partial}{\partial \phi}\left(A_\phi \vec{a}_\phi\right) \\
&= \frac{c}{\rho}\left[\vec{a}_\phi \frac{\partial A_\phi}{\partial \phi} + A_\phi \frac{\partial \vec{a}_\phi}{\partial \phi}\right] \\
&= \frac{c}{\rho}\left[A_\phi \frac{\partial \vec{a}_\phi}{\partial \phi}\right] \\
&= -\frac{cA_\phi}{\rho}\vec{a}_\rho
\end{aligned}
$$

Let's put this term into a more standard form.

$$
\begin{aligned}
A_\phi(\rho,\phi,z) &= \frac{\mu I}{2\pi}\frac{\sqrt{z^2+(\rho+R)^2}}{\rho}\left[\left(\frac{z^2+\rho^2+R^2}{z^2+(\rho+R)^2}\right)K(k) - E(k)\right] \\
&= \frac{\mu q c}{4\pi^2 R}\frac{\sqrt{z^2+(\rho+R)^2}}{\rho}\left[\left(\frac{z^2+\rho^2+R^2}{z^2+(\rho+R)^2}\right)K(k) - E(k)\right] \\
\frac{cA_\phi}{\rho} &= \frac{\mu q c^2}{4\pi^2 \rho R}\frac{\sqrt{z^2+(\rho+R)^2}}{\rho}\left[\left(\frac{z^2+\rho^2+R^2}{z^2+(\rho+R)^2}\right)K(k) - E(k)\right] \\
&= \frac{q}{4\pi\epsilon\rho R}\frac{\sqrt{z^2+(\rho+R)^2}}{\rho}\left[\left(\frac{z^2+\rho^2+R^2}{z^2+(\rho+R)^2}\right)\frac{K(k)}{\pi} - \frac{E(k)}{\pi}\right] \\
&= \frac{q}{4\pi\epsilon\rho\sqrt{(\rho+R)^2+z^2}}\left[\left(\frac{z^2+\rho^2+R^2}{\rho R}\right)\frac{K(k)}{\pi} - \frac{(\rho+R)^2+z^2}{\rho R}\frac{E(k)}{\pi}\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{cA_\phi}{\rho} &= \frac{q}{4\pi\epsilon\rho\sqrt{(\rho+R)^2+z^2}}\left[\left(-2+\frac{4}{k^2}\right)\frac{K(k)}{\pi} - \frac{4}{k^2}\frac{E(k)}{\pi}\right] \\
&= \frac{q}{4\pi\epsilon\rho\sqrt{(\rho+R)^2+z^2}}\left[\frac{4}{\pi}\frac{K(k)-E(k)}{k^2} - \frac{2}{\pi}K(k)\right]
\end{aligned}
$$

This term strongly approaches zero as we move away from the ring. As $k \to 0$, $(K(k) - E(k))/k^2 \to \pi/4$. Likewise, $K(0) = \pi/2$, with the result that the term in brackets disappears.

Our composite force now becomes

$$\begin{aligned}
\frac{F_\rho}{q} &= \frac{q}{4\pi\epsilon\rho\sqrt{(R+\rho)^2+z^2}}\left[\frac{\rho+R}{R}\left(\frac{K(k)-E(k)}{\pi}\right)\right] - \frac{cA_\phi}{\rho} \\
\frac{F_z}{q} &= \frac{q}{4\pi\epsilon\rho\sqrt{(\rho+R)^2+z^2}}\left[\frac{z}{R}\left(\frac{K(k)-E(k)}{\pi}\right)\right]
\end{aligned}$$

Our axial term is well behaved at the loop due to the $z = 0$ factor. We now look at the radial term.

$$\begin{aligned}
\frac{F_\rho}{q} &= \frac{q}{4\pi\epsilon\rho\sqrt{(\rho+R)^2+z^2}}\left[\frac{\rho+R}{R}\left(\frac{K(k)-E(k)}{\pi}\right)\right] - \frac{cA_\phi}{\rho} \\
&= \frac{q}{4\pi\epsilon\rho\sqrt{(\rho+R)^2+z^2}}\left[\frac{K(k)}{\pi}\left(\frac{\rho+R}{R}+2-\frac{4}{k^2}\right)+\frac{E(k)}{\pi}\left(\frac{4}{k^2}-\frac{\rho+R}{R}\right)\right]
\end{aligned}$$

This expression is very satisfactory. At the ring itself, $z = 0, \rho = R$ and $k = 1$. We see the infinity of $K(k = 1)$ is cancelled by the factor of zero. Note $E(1) = 1$. The finite force on the ring is

$$\begin{aligned}
\frac{F_\rho}{q} &= \frac{q}{4\pi\epsilon\rho\sqrt{(\rho+R)^2+z^2}}\left[\frac{2}{\pi}E(k)\right] \\
F_\rho &= \frac{q^2}{4\pi\epsilon R2R}\frac{2}{\pi}E(k) \\
&= \frac{q^2E(k)}{4\pi^2\epsilon RR} \\
&= \frac{q^2}{4\pi^2\epsilon R^2}
\end{aligned}$$

Plots of Normalized Force Along X and Z Axis

It is instructive to look at the normalized force, where I take the ratio of the complete force above divided by the point source force law. Near, and in the ring, the forces are suppressed. As we get out several ring diameter in any direction, we begin to approach the classical forces. These normalized forces are illustrated in Figure 13.

Discussion

The Parson Ring model traditionally began with the assumption of the charge ring rotating at c. This derivation provides a strong justification as to *why* the ring rotates at c.

In the expressions above, we get no clue as to what R *should* be. We will back this number out from the known energy of the electron.

Before we start the calculation, I want to write down a number of constants for the electron and electromagnetism.

$$
\begin{aligned}
\mu_0 &= 4\pi 10^{-7}\ \text{H/m} \\
\epsilon_0 &= 8.854 \cdot 10^{-12}\ \text{C/m} \\
c &= 2.998 \cdot 10^{8}\ \text{m/s} \\
E_e &= 511\ \text{keV} \\
m_e &= 9.1 \cdot 10^{-31}\ \text{kg} \\
q_e &= 1.6 \cdot 10^{-19}\ \text{C} \\
h &= 6.626 \cdot 10^{-34}\ \text{J s} \\
\hbar &= 1.0546 \cdot 10^{-34}\ \text{J s} \\
\alpha &= \frac{q^2}{(4\pi\epsilon)\hbar c} = 1/137.04 \\
r_{\text{compton}} &= \frac{h}{m_e c} = 2.4263 \cdot 10^{-12}\ \text{m} \\
r_{\text{classical}} &= \frac{\alpha\hbar}{m_e c} = \frac{q^2}{4\pi\epsilon}\frac{1}{m_e c^2} = 2.8179 \cdot 10^{-15}\ \text{m}
\end{aligned}
$$

Let's assume that the residual energy from the $E(k)$ provides for the rest mass of the electron. The energy content of the electron is 511 keV, and since the charge of the electron is $e = q$, we have the ring self-potential must

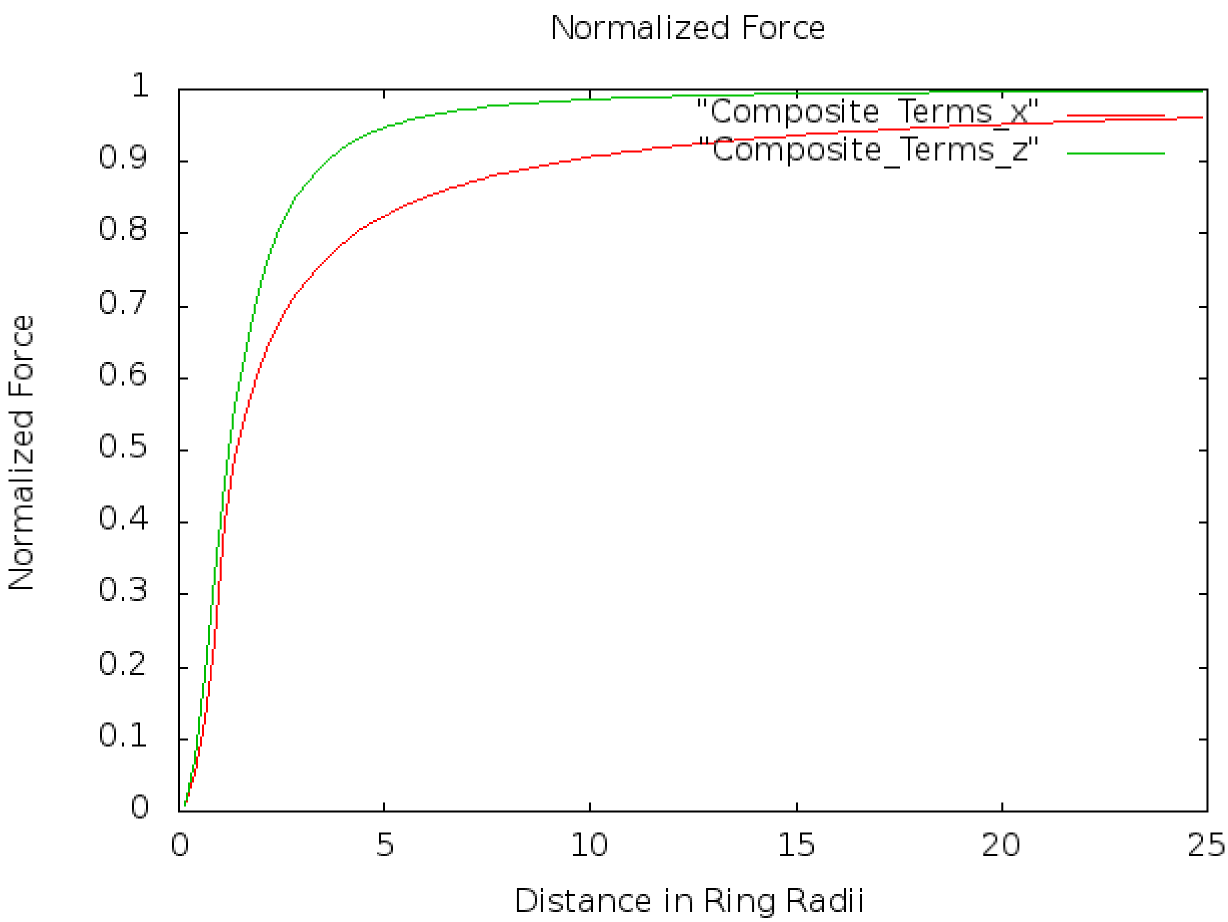

Figure 13: Ratio of Parson Ring Force over Classical Point Source Force.

be 511 kV. We have the combined potential residue as

$$
\begin{aligned}
V(r,\phi,z) &= K(k)\left[\frac{q}{4R\pi^2\epsilon} - \frac{\mu v I}{2\pi}\right] - E(k)\left[\frac{\mu v I}{\pi}\right] \\
V(R,\phi,0) &= -E(k)\left[\frac{\mu v I}{\pi}\right] \text{ given } v = c \\
&= (-1)\frac{\mu c}{\pi}\frac{qc}{2\pi R} \\
&= \frac{-\mu c^2 q}{2\pi^2 R} = 511kV = \frac{m_e c^2}{q} \\
R &= \frac{-\mu c^2 q}{2\pi^2 511\cdot 10^3} = \left(\frac{2}{\pi}\right)\left(\frac{q^2}{4\pi\epsilon}\right)\frac{1}{m_e c^2} \\
&= 1.79\cdot 10^{-15}\text{ m}
\end{aligned}
$$

This radius is $2/\pi$ times the classical electron radius ($2.8179\cdot 10^{-15}$ m). The classical electron radius was calculated assuming a spherical charge distribution. This mismatch in value does not cause me concern, as our geometries are different.

Now, let's examine the force on the ring. From previously, on the ring itself, we have

$$
\begin{aligned}
F_\rho &= \frac{q^2}{4\pi^2\epsilon R^2} \\
\text{assuming } R &= 1.79\cdot 10^{-15}\text{ m} \\
F_\rho &= 22.85\text{ N}
\end{aligned}
$$

We now compare this to the electron mass times kinematic acceleration.

$$
\begin{aligned}
v &= c = R\omega \\
a &= \omega^2 R = \frac{c^2}{R} \\
ma &= \frac{mc^2}{R} \\
&= 45.75N
\end{aligned}
$$

It would be nice to have these items line up. The factor of 2 needs some investigation. One thought is to split the observable mass into ρ and z components. The ρ component then matches the half of the kinematic term, and the z component accounts for the other half. Meanwhile, I am happy to have an absense of infinities in this work.

Here are a few observations.

A constant speed ring implies a constant internal acceleration.

A constant ring radius implies constant curvature.

A constant speed ring has constant angular momentum.

However, nothing in this work has locked in the mass, radius, or the angular momentum. My expectation is that application of the Weber, Ampere or Whittaker elemental potentials will remove the extra freedom we have here.

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