Symmetries In Minkowski Geometric Algebra

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Abstract

This note documents a nice symmetry in Minkowski (+,+,+,-) signature geometric algebra. These symmetries allow twelve different mappings of a common spanning set of 4x4 matrices to Minkowski geometric algebra. These symmetries also imply eleven more Lorentz contraction relationships exist in multivector form in Minkowski geometric algebra.

Minkowski Geometric Algebra

Minkowski geometric algebra is a four dimensional Clifford algebra with signature (+,+,+,-). This is different from, and complementary to the Space Time Algebra (STA) of Hestenes [3], Doran [1] and others.

Elements of Minkowski Geometric Algebra

In Minkoswki geometric algebra, we have one scalar which commutes with all other elements, four vectors which anti-commute among themselves, six bivectors with a split between three pure spatial bivectors versus three time-space bivectors, four trivectors, and one quadvector. All multiplication is associative.

In basis notation form, we can write $e_q = 1$ for our scalar, which commutes with everything. We have four basis vectors, e_x, e_y, e_z, and e_t. The square of these basis vectors is according to the Minkowski signature.

$$\begin{aligned} e_x^2 &= e_y^2 = e_z^2 = 1 \\ e_t^2 &= -1 \end{aligned}$$

These basis vectors anti-commute, and these pairwise products define our six bivectors.

$$\begin{aligned}
e_{xy} &= e_x e_y = -e_y e_x \\
e_{yz} &= e_y e_z = -e_z e_y \\
e_{zx} &= e_z e_x = -e_x e_z \\
e_{tx} &= e_t e_x = -e_x e_t \\
e_{ty} &= e_t e_y = -e_y e_t \\
e_{tz} &= e_t e_z = -e_z e_t
\end{aligned}$$

These bivectors have a space=time split in behavior. The purely spatial bivectors square to negative one, while the space-time bivectors square to positive one (due to $e_t^2 = -1$).

$$\begin{aligned}
e_{xy}^2 &= e_x e_y e_x e_y = -e_x(e_x e_y)e_y = -e_x e_x = -1 \\
e_{yz}^2 &= -1 \\
e_{zx}^2 &= -1 \\
e_{tx}^2 &= e_t e_x e_t e_x = -e_t(e_x e_x)e_t = -e_t e_t = 1 \\
e_{ty}^2 &= 1 \\
e_{tz}^2 &= 1
\end{aligned}$$

Pairs of bivectors whose product is also a bivector anticommute. For example, $e_{xy}e_{zx} = -e_{zx}e_{xy}$. However, products of bivectors which result in the quadvector commute. For example, $e_{xy}e_{zt} = e_{zt}e_{xy}$.

We have four trivectors, with a space-time split on the squares. The trivectors anti-commute among themselves.

$$\begin{aligned}
e_{xyz} &= e_x e_y e_z \\
e_{xyt} &= e_x e_y e_t \\
e_{xzt} &= e_x e_z e_t \\
e_{yzt} &= e_y e_z e_t
\end{aligned}$$

$$\begin{aligned}
e_{xyz}^2 &= -1 \\
e_{xyt}^2 &= 1 \\
e_{xzt}^2 &= 1 \\
e_{yzt}^2 &= 1
\end{aligned}$$

e_x	e_y	e_z	e_{xyz}
e_{yzt}	e_{xzt}	e_{xyt}	e_t
e_{xt}	e_{yt}	e_{zt}	e_{xyzt}
e_{yz}	e_{xz}	e_{xy}	

Table 1: Sets of Products Across and Down

Scalar	X	Y	Z	XY	XZ	YZ	XYZ	Comment
1	e_x	e_y	e_z	e_{xy}	e_{xz}	e_{yz}	e_{xyz}	Vector
1	e_{yzt}	e_{xzt}	e_{xyt}	e_{yx}	e_{xz}	e_{zy}	e_t	Anti-vector
1	e_{xt}	e_{yt}	e_{zt}	e_{xy}	e_{xz}	e_{yz}	e_{xyzt}	Even Rank Subspace
1	e_x	e_{yzt}	e_{xt}	e_{xyzt}	e_t	e_{xyz}	e_{yz}	Isn't
1	e_y	e_{xzt}	e_{yt}	e_{yxzt}	e_t	e_{xzy}	e_{xz}	This
1	e_z	e_{xyt}	e_{zt}	e_{xyzt}	e_t	e_{xyz}	e_{xy}	Neat!

Table 2: Six Euclidean Subspaces

Six Euclidean Subspaces

Having established a representation for Minkowski geometric algebra, it is interesting to create a 3x3 grid for the elements which square to positive one, where the product across each row, and product down each column provide terms which squares to negative one. Such a table is shown in table 1. As an aside, the diagonal triple products in this table are ± 1.

Each of these triads mathematically mimics a set of basis for Euclidean three-space. Table 2 explicitly lists these six subspaces of Minkowski space which map to three dimensional Euclidean space. As each of these subspaces as the same mathematical structure as three space, we can use our standard tools, such as translations, scaling, rotations, cylindrical coordinates, spherical coordinates, and so forth in these subspaces.

The first three subspaces are known to me, as trivial for the top line, dual space as presented in Langyel [5] for the second line, and even rank subspace as presented in Hestenes [2] and Doran [1]. The bottom three lines, I believe, are new to me.

The next page presents a specific implementation of a Minkowsky space, mapped to real 4x4 matrices.

Implementation #1

	(P1)	(+a)	(+b)	(+c)	(+E)	(+A)	(-B)	(+C)	(-g)	(-h)	(-i)	(+D)	(-f)	(+e)	(-d)	(-F)
(P1)	(P1)	(+a)	(+b)	(+c)	(+E)	(+A)	(-B)	(+C)	(-g)	(-h)	(-i)	(+D)	(-f)	(+e)	(-d)	(-F)
(+a)	(+a)	(P1)	(+A)	(-B)	(-g)	(+b)	(+c)	(+D)	(+E)	(-f)	(+e)	(+C)	(-h)	(-i)	(-F)	(-d)
(+b)	(+b)	(-A)	(P1)	(+C)	(-h)	(-a)	(-D)	(+c)	(+f)	(+E)	(-d)	(+B)	(+g)	(+F)	(-i)	(-e)
(+c)	(+c)	(+B)	(-C)	(P1)	(-i)	(+D)	(-a)	(-b)	(-e)	(+d)	(+E)	(+A)	(-F)	(+g)	(+h)	(-f)
(+E)	(+E)	(+g)	(+h)	(+i)	(M1)	(-f)	(+e)	(-d)	(+a)	(+b)	(+c)	(+F)	(-A)	(+B)	(-C)	(+D)
(+A)	(+A)	(-b)	(+a)	(+D)	(-f)	(M1)	(-C)	(-B)	(+h)	(-g)	(-F)	(-c)	(-E)	(+d)	(+e)	(+i)
(-B)	(-B)	(-c)	(-D)	(+a)	(+e)	(+C)	(M1)	(-A)	(+i)	(+F)	(-g)	(+b)	(-d)	(-E)	(+f)	(-h)
(+C)	(+C)	(+D)	(-c)	(+b)	(-d)	(+B)	(+A)	(M1)	(-F)	(+i)	(-h)	(-a)	(-e)	(-f)	(-E)	(+g)
(-g)	(-g)	(-E)	(+f)	(-e)	(-a)	(-h)	(-i)	(-F)	(P1)	(+A)	(-B)	(+d)	(-b)	(-c)	(-D)	(+C)
(-h)	(-h)	(-f)	(-E)	(+d)	(-b)	(+g)	(+F)	(-i)	(-A)	(P1)	(+C)	(+e)	(+a)	(+D)	(-c)	(+B)
(-i)	(-i)	(+e)	(-d)	(-E)	(-c)	(-F)	(+g)	(+h)	(+B)	(-C)	(P1)	(+f)	(-D)	(+a)	(+b)	(+A)
(+D)	(+D)	(+C)	(+B)	(+A)	(-F)	(-c)	(+b)	(-a)	(-d)	(-e)	(-f)	(M1)	(+i)	(-h)	(+g)	(-E)
(-f)	(-f)	(-h)	(+g)	(+F)	(-A)	(-E)	(+d)	(+e)	(-b)	(+a)	(+D)	(-i)	(P1)	(+C)	(+B)	(-c)
(+e)	(+e)	(-i)	(-F)	(+g)	(+B)	(-d)	(-E)	(+f)	(-c)	(-D)	(+a)	(+h)	(-C)	(P1)	(+A)	(+b)
(-d)	(-d)	(+F)	(-i)	(+h)	(-C)	(-e)	(-f)	(-E)	(+D)	(-c)	(+b)	(-g)	(-B)	(-A)	(P1)	(-a)
(-F)	(-F)	(+d)	(+e)	(+f)	(-D)	(+i)	(-h)	(+g)	(+C)	(+B)	(+A)	(+E)	(+c)	(-b)	(+a)	(M1)

```
    Unity                xyzt
[ 1  0  0  0]        [ 0  1  0  0]
[ 0  1  0  0]        [-1  0  0  0]
[ 0  0  1  0]        [ 0  0  0  1]
[ 0  0  0  1]        [ 0  0 -1  0]

      x             y             z             t
[ 0  1  0  0] [-1  0  0  0] [ 0  0  0  1] [ 0  0  0 -1]
[ 1  0  0  0] [ 0  1  0  0] [ 0  0  1  0] [ 0  0 -1  0]
[ 0  0  0 -1] [ 0  0 -1  0] [ 0  1  0  0] [ 0  1  0  0]
[ 0  0 -1  0] [ 0  0  0  1] [ 1  0  0  0] [ 1  0  0  0]

     xy            xz            yz            xt            yt            zt
[ 0  1  0  0] [ 0  0  1  0] [ 0  0  0 -1] [ 0  0 -1  0] [ 0  0  0  1] [ 1  0  0  0]
[-1  0  0  0] [ 0  0  0  1] [ 0  0  1  0] [ 0  0  0 -1] [ 0  0 -1  0] [ 0  1  0  0]
[ 0  0  0 -1] [-1  0  0  0] [ 0 -1  0  0] [-1  0  0  0] [ 0 -1  0  0] [ 0  0 -1  0]
[ 0  0  1  0] [ 0 -1  0  0] [ 1  0  0  0] [ 0 -1  0  0] [ 1  0  0  0] [ 0  0  0 -1]

    xyz           xyt           xzt           yzt
[ 0  0  1  0] [ 0  0 -1  0] [ 0  1  0  0] [-1  0  0  0]
[ 0  0  0 -1] [ 0  0  0  1] [ 1  0  0  0] [ 0  1  0  0]
[-1  0  0  0] [-1  0  0  0] [ 0  0  0  1] [ 0  0  1  0]
[ 0  1  0  0] [ 0  1  0  0] [ 0  0  1  0] [ 0  0  0 -1]
```

X	Y	Z	T	Comment
e_x	e_y	e_z	e_t	Initial Map
e_x	e_y	e_z	e_{xyzt}	Other Choice
e_{yzt}	e_{xzt}	e_{xyt}	e_{xyz}	Anti-vector
e_{yzt}	e_{xzt}	e_{xyt}	e_{xyzt}	
e_{xt}	e_{yt}	e_{zt}	e_{xyz}	Even Rank Subspace
e_{xt}	e_{yt}	e_{zt}	e_t	
e_x	e_{yzt}	e_{xt}	e_{xz}	
e_x	e_{yzt}	e_{xt}	e_{xy}	
e_y	e_{xzt}	e_{yt}	e_{yz}	
e_y	e_{xzt}	e_{yt}	e_{xy}	
e_z	e_{xyt}	e_{zt}	e_{yz}	
e_z	e_{xyt}	e_{zt}	e_{xz}	

Table 3: Twelve Minkowski Remappings

Twelve Remappings Inside Minkowski Space

Having seen six mappings of subspaces of Minkowski space to Euclidean three space, inquiring minds want to know if similar such remappings can be done within Minkowski space to Minkowski space itself (outside trivial relabellings such as interchanging x and y axis).

In a fashion similar to the Euclidean subspaces, we can use the six groups of Euclidean spaces as a starting point for Minkowski remappings. The novelty, or twist, in this case, is to take the other trivectors in the 3x3 array from above as the replacement for the timelike basis.

Table 3 presents these 12 remappings, Minkowski to Minkowski. The full table is too wide to fit my standard page size, so I'm only providing the basii in this table. I plan to revise this note later with a full, sidewise table.

References

[1] Chris Doran and Anthony Lasenby, *Geometric Algebra for Physicists* Cambridge University Press, ISBN 978-0-521-71595-9

[2] David Hestenes, *New Foundations for Classical Mechanics* Kluwer Academic Publishers, ISBN 0-7923-5514-8

[3] David Hestenes and Garret Sobczyk, *Clifford Algebra to Geometric Calculus* D. Reidal Publishing Company, ISBN 978-90-277-2581-5

[4] David Hestenes, *Universal Geometric Algebra*, Quarterly Journal of Pure and Applied Mathematics, Volume 62 (1988)

[5] Eric Langyel, Game Developers Conference 2012 Slideset `http://www.terathon.com/lengyel/`