Integrating the Length Along a Cosine Wave using Elliptic Integrals

Kurt Nalty

May 20, 2014

Length Along a Cosine Wave

I wanted to measure the length travelled along a sinusoidal path. The integral is easy to write.

$$\begin{aligned} y &= \cos(x) \\ dy &= -\sin(x)dx \\ ds^2 &= dx^2 + dy^2 \\ s(x) &= \int_0^x \sqrt{1 + \sin^2(\phi)}d\phi \end{aligned}$$

I knew from prior experience that this should be representable as an elliptic integral. As an aside, this integral is in Gradshteyn, p173, 2.597 #2.

The elliptic integral of the second kind is commonly written using a modulus k and argument (also called amplitude) ϕ, as

$$E(\phi, k) = \int_0^\phi \sqrt{1 - k^2 \sin^2\theta}d\theta$$

If we use an imaginary modulus $k = i$, we have our integral. This same integral is also written with a different term, the parameter $m = k^2$. With m negative, we again have our answer.

$$E(\phi, m) = \int_0^\phi \sqrt{1 - m\sin^2\theta}d\theta$$

So, we have

$$s(x) = \int_0^x \sqrt{1+\sin^2(\phi)}d\phi = E(\phi, m=-1) = E(\phi, k=i)$$

I have two packages for evaluating elliptic integrals that I commonly use. For completely open source, free to copy work, I prefer to use the Gnu Scientific Libray. For open source, elegant, but copyright protected work (by RLH), I prefer to use the MyMathLib.com routines.

The MyMathLib.com routines are ready to work, out of the box, as they support the m format. RLH's routine is only defined over $-\pi/2 < \phi < \pi/2$, so a small wrapper is needed for extended argument range, which uses the relationship $E(N\pi \pm \phi, k) = 2NE(\pi/2, k) \pm E(\phi, k)$.

GSL does not support m or ik directly. Instead, we use a formula from `http://dlmf.nist.gov/19.7` to calculate $E(\phi, ik)$ using an imaginary modulus.

Sample verification code comparing numerical integration with these two other approaches show good agreement. Figure 1 compares numerical integration with the MyMathLib formula.

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include <gsl/gsl_math.h>
#include <gsl/gsl_precision.h>
#include <gsl/gsl_sf_ellint.h>
#include <gsl/gsl_sf_elljac.h>
#include "legendre_elliptic_integral_second_kind.c"
#define npts 120

// This program is checking two formulas for negative m
// from MyMathLib.com and NIST, p. 492 using gsl routines

// gcc Check_E_using_gsl.c -lgsl  -lgslcblas -lm

// double gsl_sf_ellint_E (double phi, double k, gsl_mode_t mode);

int main(void)
{
```

```
int i,j,err_code,zone;
double a,b,c,phi,theta,k,chi,chi_prime,E,m;
double x,y,z,dx,dy,dz,s,ds,pi,s_m,s1,s2,s3,s4;
double E_theta_chi, alpha, beta, u, sn, cd, cn, dn;
FILE* Output;
double x_loc,sint,cost,kappa;

pi = M_PI;

Output = fopen("Plot.txt","w");
s = 0.0;
dx = 0.02;
for (x = 0.0*pi; x<3.5*pi; x+=dx)
{

// begin by integrating /int_0^x /sqrt{1 + sin^x} dx =
//           E(x,k=i) = E(x,m=-1), alpha = sin x
    k = 1.0;

    y = cos(x);
    dy = -sin(x)*dx;
    ds = sqrt(dx*dx + dy*dy);
    s1 = s;
    s += ds;

// next use the MyMathLib routine for m

    if (x >= 0.0) {
        zone = (1 + x*2/M_PI)/2;
    }
    else {
        zone = -(1 - x/M_PI_2)/2;
    }
    x_loc = x - zone*M_PI;
    s2 = Legendre_Elliptic_Integral_Second_Kind( x_loc, 'm', -k*k );
    s2 += 2.0*zone*Legendre_Elliptic_Integral_Second_Kind(M_PI_2,'m',-k*k );

// now calculate using NIST 19.7.5  p492
```

```
        x_loc = x - zone*M_PI;

        kappa = k/sqrt(1.0+k*k);
        sint = sin(x_loc)*sqrt(1.0+k*k)/sqrt(1.0+k*k*sin(x_loc)*sin(x_loc));
        cost = cos(x_loc)/sqrt(1.0+k*k*sin(x_loc)*sin(x_loc));
        theta = asin(sint);                              // theta = pi/2

        s3 = sqrt(1+k*k) *(gsl_sf_ellint_E(theta, kappa, GSL_PREC_DOUBLE)
                   - kappa*kappa*sint*cost/sqrt(1 - kappa*kappa*sint*sint) );
  s3 += 2.0*sqrt(1+k*k)*zone*(gsl_sf_ellint_E(M_PI_2, kappa, GSL_PREC_DOUBLE));

    // now we use a cleaned up version of the above formula

        s4 = sqrt(1.0 + k*k)*gsl_sf_ellint_E(theta, kappa, GSL_PREC_DOUBLE)
            - k*k*sin(x)*cos(x)/sqrt(1.0 + k*k*sin(x)*sin(x));
  s4 += 2.0*sqrt(1+k*k)*zone*(gsl_sf_ellint_E(M_PI_2, kappa, GSL_PREC_DOUBLE));

        fprintf(Output,"%f  %f  %f  %f  %f \n",x,s1,s2,s3,s4);
    }
    fclose(Output);
}
```

More General Case

Treating our curve as a scope trace of a sine wave, we can change the amplitude and frequency, and see the effect on our integral.

$$
\begin{aligned}
y &= A\cos(2\pi f x) \\
dy &= -2\pi A f \sin(2\pi f x)dx \\
s &= \int_0^x \sqrt{1 + (2\pi A f)^2 \sin^2 \phi d\phi} \\
&= E(x, m = -2\pi A f)/(2\pi f)
\end{aligned}
$$

The magnitude of the modulus can easily exceed 1. MyMathLib has no problem with this at all. Figure 2 is an illustration with $A = 10$ and $f = 1.5$. Sample code follows.

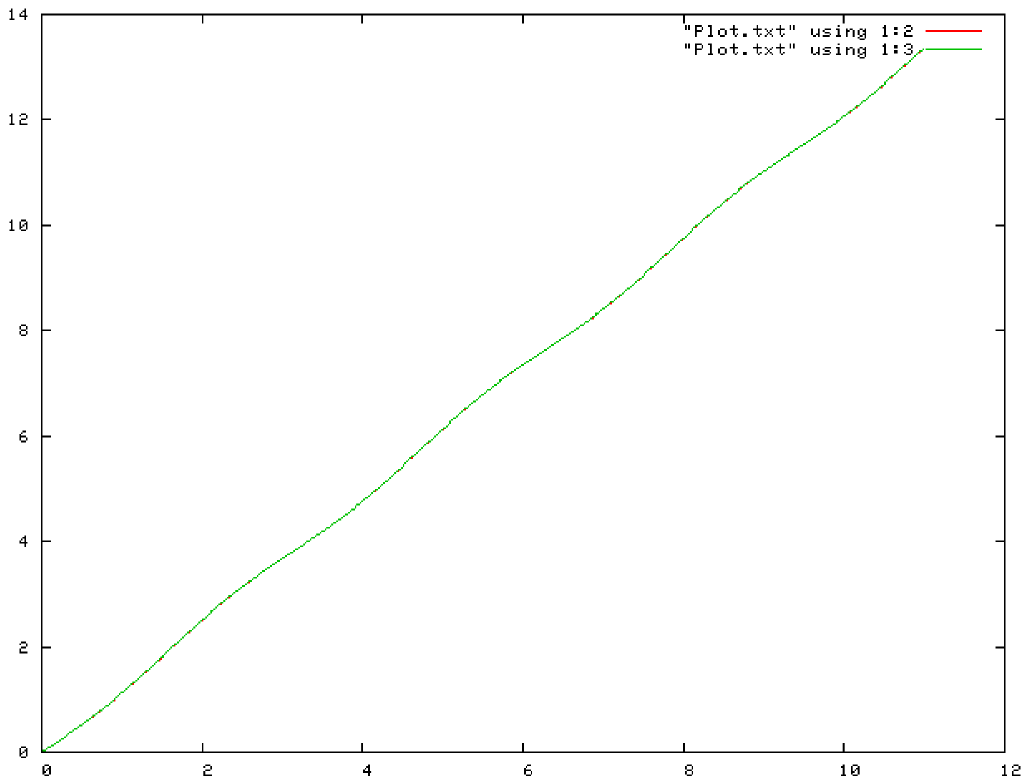

Figure 1: Comparision of Numerical versus Elliptical Solutions

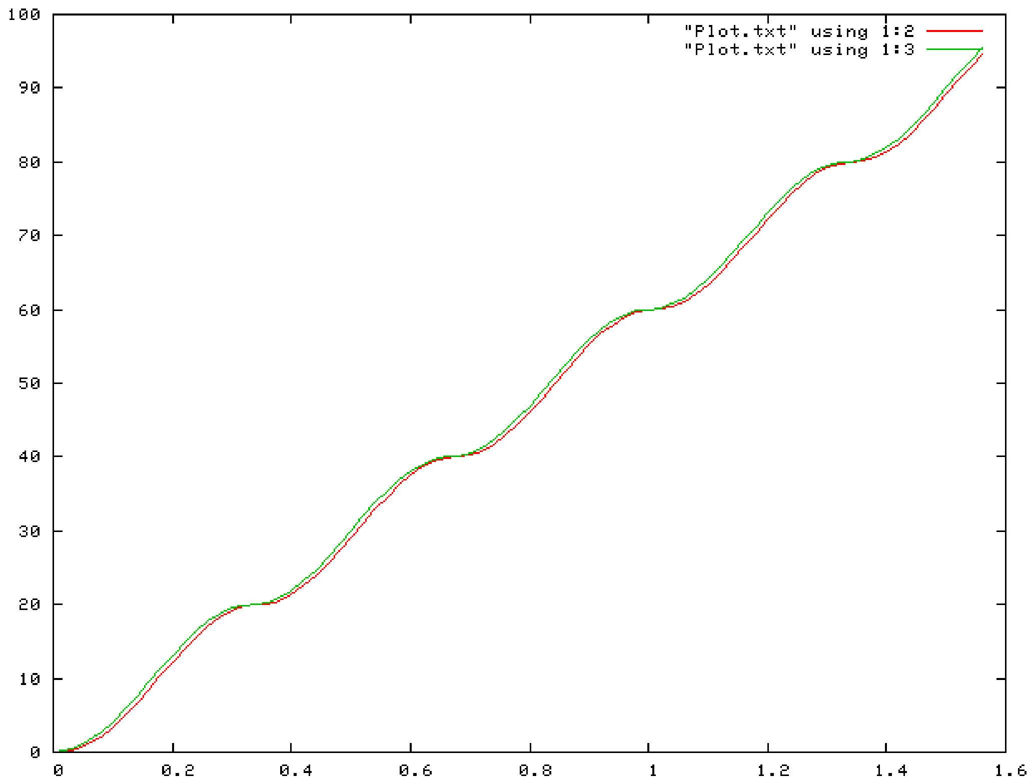

Figure 2: Comparison with $A = 10$, $f = 1.5$

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include "legendre_elliptic_integral_second_kind.c"
#define npts 120

// This program is checking two formulas for negative m

// gcc Check_E_using_gsl.c -lgsl -lm
// gcc Check_E_using_gsl.c -lgsl  -lgslcblas -lm

// double gsl_sf_ellint_E (double phi, double k, gsl_mode_t mode);

int main(void)
{
    int i,j,err_code,zone;
    double a,b,c,phi,theta,k,chi,chi_prime,E,m;
    double x,y,z,dx,dy,dz,s,ds,pi,s_m,s1,s2,s3,s4;
    double E_theta_chi, alpha, beta, u, sn, cd, cn, dn;
    FILE* Output;
    double x_loc,sint,cost,kappa,A,f,X;

    pi = PI;    // PI defined in legendre_elliptic_integral_second_kind.c

    Output = fopen("Plot.txt","w");
    s = 0.0;
    dx = 0.02;
    A = 10.0;
    f = 1.5;
    for (x = 0.0*pi; x<0.5*pi; x+=dx)
    {

    // begin by simple numerical integration
        k = 2.0*pi*A*f;

        y = A*cos(2.0*pi*f*x);
        dy = -2.0*pi*A*f*sin(2.0*pi*f*x)*dx;
        ds = sqrt(dx*dx + dy*dy);
```

```
        s1 = s;
        s += ds;

    // next use the MyMathLib routine for m

        X = 2.0*pi*f*x;

        if (X >= 0.0) {
            zone = (1 + X/PI_2)/2;
        }
        else {
            zone = -(1 - X/PI_2)/2;
        }
        x_loc = X - zone*PI;
        s2 = Legendre_Elliptic_Integral_Second_Kind( x_loc, 'm', -k*k );
   s2 += 2.0*zone*Legendre_Elliptic_Integral_Second_Kind(PI_2,'m',-k*k );
        s2 /= 2.0*pi*f;
        fprintf(Output,"%f  %f  %f \n",x,s1,s2);
    }
    fclose(Output);
}
```

Comments on Two Solutions for this Integral

The NIST transformation matches the Gradshteyn 2.597 #2 solution.

$$
\begin{aligned}
\phi &= \sin^{-1}\left(\frac{\sqrt{1+k^2}\sin(x)}{\sqrt{1+k^2\sin^2 x}}\right) \\
\int \sqrt{1+k^2\sin^2 x}dx &= \sqrt{1+k^2}E\left(\phi, \frac{k}{\sqrt{1+k^2}}\right) - k^2\frac{\sin(x)\cos(x)}{\sqrt{1+k^2\sin^2(x)}}
\end{aligned}
$$

By contrast, RLH uses a different, simpler, relationship for negative values of m, which works well.

$$
\int \sqrt{1+k^2\sin^2 x}dx = \sqrt{1+k^2}\left(E\left(\frac{\pi}{2}, \frac{k}{\sqrt{1+k^2}}\right) + E\left(x-\frac{\pi}{2}, \frac{k}{\sqrt{1+k^2}}\right)\right)
$$

It is easy to verify the correctness of the RLH formula. Begin with the definition of the elliptical function $E(\phi, k)$, and it's derivative with respect tp ϕ.

$$
\begin{aligned}
E(\phi, k) &= \int_0^{\phi} \sqrt{1 - k^2 \sin^2(x)}dx \\
\frac{\partial E(\phi, k)}{\partial \phi} &= \sqrt{1 - k^2 \sin^2(\phi)}
\end{aligned}
$$

We want to verify that

$$
\int_0^{\phi} \sqrt{1 + k^2 \sin^2 x}dx = \sqrt{1+k^2}\left(E\left(\frac{\pi}{2}, \frac{k}{\sqrt{1+k^2}}\right) + E\left(\phi - \frac{\pi}{2}, \frac{k}{\sqrt{1+k^2}}\right)\right)
$$

We begin by examining the constant of integration. When ϕ is 0, due to the odd nature of E with respect to ϕ, we get the desired value of 0. Now, we take the derivative of the right hand side with respect to ϕ.

$$
\begin{aligned}
f(\phi, k) &= \sqrt{1+k^2}\left(E\left(\frac{\pi}{2}, \frac{k}{\sqrt{1+k^2}}\right) + E\left(\phi - \frac{\pi}{2}, \frac{k}{\sqrt{1+k^2}}\right)\right) \\
\frac{\partial f}{\partial \phi} &= \frac{\partial}{\partial \phi}\left[\sqrt{1+k^2}\left(E\left(\frac{\pi}{2}, \frac{k}{\sqrt{1+k^2}}\right) + E\left(\phi - \frac{\pi}{2}, \frac{k}{\sqrt{1+k^2}}\right)\right)\right] \\
&= \sqrt{1+k^2}\left(\sqrt{1 - \left(\frac{k^2}{1+k^2}\right)\sin^2\left(\phi - \frac{\pi}{2}\right)}\right) \\
&= \sqrt{1+k^2}\left(\sqrt{\frac{1 + k^2 - k^2 \sin^2\left(\phi - \frac{\pi}{2}\right)}{1+k^2}}\right) \\
&= \sqrt{1 + k^2 \cos^2\left(\phi - \frac{\pi}{2}\right)} \\
&= \sqrt{1 + k^2 \sin^2 \phi}
\end{aligned}
$$

This is, of course, the desired result.