Regressive Versus Antiwedge Products

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Abstract

The Hestenes' (1986) regressive product differs from the Lengyel antiwedge product.

Minkowski Geometric Algebra

Minkowski geometric algebra is a spacetime algebra using east coast metric $(+,+,+,-)$. Our components are one scalar, e_q, three space directions e_x, e_y, and e_z, a time dimension e_t, and higher order products of the previous, yielding three pure space bivectors e_xe_y, e_xe_z, and e_ye_z, three spacetime bivectors e_xe_t, e_ye_t, and e_ze_t, four trivectors $e_xe_ye_z, e_xe_ye_t, e_xe_ze_t$, and $e_ye_ze_t$, and a quadvector $e_xe_ye_ze_t$.

A generic multivector is the scaled sum of the previous elements, which I usually write as

$$\begin{aligned}
&Ae_q + \\
&Be_x + Ce_y + De_z + Ee_t + \\
&Fe_{xy} + Ge_{xz} + He_{yz} + Je_{xt} + Ke_{yt} + Le_{zt} + \\
&Me_{xyz} + Ne_{xyt} + Pe_{xzt} + Re_{yzt} + \\
&Se_{xyzt}
\end{aligned}$$

We can define this algebra by a multiplication table. In condensed format, to fit on one page, we have Table 1.

*	q	x	y	z	t	xy	xz	yz	xt	yt	zt	xyz	xyt	xzt	yzt	xyzt
q	q	x	y	z	t	xy	xz	yz	xt	yt	zt	xyz	xyt	xzt	yzt	xyzt
x	x	q	xy	xz	xt	y	z	xyz	t	xyt	xzt	yz	yt	zt	xyzt	yzt
y	y	-xy	q	yz	yt	-x	-xyz	z	-xyt	t	yzt	-xz	-xt	-xyzt	zt	-xzt
z	z	-xz	-yz	q	zt	xyz	-x	-y	-xzt	-yzt	t	xy	xyzt	-xt	-yt	xyt
t	t	-xt	-yt	-zt	-q	xyt	xzt	yzt	x	y	z	-xyzt	-xy	-xz	-yz	xyz
xy	xy	-y	x	xyz	xyt	-q	-yz	xz	-yt	xt	xyzt	-z	-t	-yzt	xzt	-zt
xz	xz	-z	-xyz	x	xzt	yz	-q	-xy	-zt	-xyzt	xt	y	yzt	-t	-xyt	yt
yz	yz	xyz	-z	y	yzt	-xz	xy	-q	xyzt	-zt	yt	-x	-xzt	xyt	-t	-xt
xt	xt	-t	-xyt	-xzt	-x	yt	zt	xyzt	q	xy	xz	-yzt	-y	-z	-xyz	yz
yt	yt	xyt	-t	-yzt	-y	-xt	-xyzt	zt	-xy	q	yz	xzt	x	xyz	-z	-xz
zt	zt	xzt	yzt	-t	-z	xyzt	-xt	-yt	-xz	-yz	q	-xyt	-xyz	x	y	xy
xyz	xyz	yz	-xz	xy	xyzt	-z	y	-x	yzt	-xzt	xyt	-q	-zt	yt	-xt	-t
xyt	xyt	yt	-xt	-xyzt	-xy	-t	-yzt	xzt	-y	x	xyz	zt	q	yz	-xz	-z
xzt	xzt	zt	xyzt	-xt	-xz	yzt	-t	-xyt	-z	-xyz	x	-yt	-yz	q	xy	y
yzt	yzt	-xyzt	zt	-yt	-yz	-xzt	xyt	-t	xyz	-z	y	xt	xz	-xy	q	-x
xyzt	xyzt	-yzt	xzt	-xyt	-xyz	-zt	yt	-xt	yz	-xz	xy	t	z	-y	x	-q

Table 1: Minkowski Geometric Algebra Multiplication Table

Minkowski Wedge Product

The wedge product is independent of the metric, and is the same for four dimensional spacetime as well as Minkowski spacetime.

We can define this algebra by a multiplication table. In condensed format, to fit on one page, we have Table 2.

Hestenes' Regressive Product

The regressive product is independent of the metric, and is the same for four dimensional spacetime as well as Minkowski spacetime. We can define this algebra by a multiplication table. In condensed format, to fit on one page, we have Table 3.

This product can also be defined by several equivalent formulae, where $I = e_{xyzt}$ is the pseudoscalar for Minkowski spacetime, $I^2 = -1$, and $I^{-1} = -I$.

$$\begin{aligned} (A \vee B) * I^{-1} &= \left(\left(A * I^{-1}\right) \wedge \left(B * I^{-1}\right)\right) \\ A \vee B &= \left(\left(A * I^{-1}\right) \wedge \left(B * I^{-1}\right)\right) * I \\ &= ((A * I) \wedge (B * I)) * I \end{aligned}$$

From the first definition, we can define a dual monadic operator, equal to postmultiplication by I^{-1}, and then define

$$\operatorname{dual}(A \vee B) = \operatorname{dual}(A) \wedge \operatorname{dual}(B)$$

Lengyel's Antiwedge Product

The antiwedge product is independent of the metric, and is the same for four dimensional spacetime as well as Minkowski spacetime.

We can define this algebra by a multiplication table. In condensed format, to fit on one page, we have Table 4.

Conclusion

We see that when both factors have odd grade, the product signs are different.

^	q	x	y	z	t	xy	xz	yz	xt	yt	zt	xyz	xyt	xzt	yzt	xyzt
q	q	x	y	z	t	xy	xz	yz	xt	yt	zt	xyz	xyt	xzt	yzt	xyzt
x	x	0	xy	xz	xt	0	0	xyz	0	xyt	xzt	0	0	0	xyzt	0
y	y	-xy	0	yz	yt	0	-xyz	0	-xyt	0	yzt	0	0	-xyzt	0	0
z	z	-xz	-yz	0	zt	xyz	0	0	-xzt	-yzt	0	0	xyzt	0	0	0
t	t	-xt	-yt	-zt	0	xyt	xzt	yzt	0	0	0	-xyzt	0	0	0	0
xy	xy	0	0	xyz	xyt	0	0	0	0	0	xyzt	0	0	0	0	0
xz	xz	0	-xyz	0	xzt	0	0	0	0	-xyzt	0	0	0	0	0	0
yz	yz	xyz	0	0	yzt	0	0	0	xyzt	0	0	0	0	0	0	0
xt	xt	0	-xyt	-xzt	0	0	0	xyzt	0	0	0	0	0	0	0	0
yt	yt	xyt	0	-yzt	0	0	-xyzt	0	0	0	0	0	0	0	0	0
zt	zt	xzt	yzt	0	0	xyzt	0	0	0	0	0	0	0	0	0	0
xyz	xyz	0	0	0	xyzt	0	0	0	0	0	0	0	0	0	0	0
xyt	xyt	0	0	-xyzt	0	0	0	0	0	0	0	0	0	0	0	0
xzt	xzt	0	xyzt	0	0	0	0	0	0	0	0	0	0	0	0	0
yzt	yzt	-xyzt	0	0	0	0	0	0	0	0	0	0	0	0	0	0
xyzt	xyzt	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 2: Minkowski Wedge Product Multiplication Table

V	q	x	y	z	t	xy	xz	yz	xt	yt	zt	xyz	xyt	xzt	yzt	xyzt
q	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	q
x	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-q	x
y	0	0	0	0	0	0	0	0	0	0	0	0	0	q	0	y
z	0	0	0	0	0	0	0	0	0	0	0	0	-q	0	0	z
t	0	0	0	0	0	0	0	0	0	0	0	q	0	0	0	t
xy	0	0	0	0	0	0	0	0	0	0	q	0	0	x	y	xy
xz	0	0	0	0	0	0	0	0	0	-q	0	0	-x	0	z	xz
yz	0	0	0	0	0	0	0	0	q	0	0	0	-y	-z	0	yz
xt	0	0	0	0	0	0	0	q	0	0	0	x	0	0	t	xt
yt	0	0	0	0	0	0	-q	0	0	0	0	y	0	-t	0	yt
zt	0	0	0	0	0	q	0	0	0	0	0	z	t	0	0	zt
xyz	0	0	0	0	-q	0	0	0	x	y	z	0	-xy	-xz	-yz	xyz
xyt	0	0	0	q	0	0	-x	-y	0	0	t	xy	0	-xt	-yt	xyt
xzt	0	0	-q	0	0	x	0	-z	0	-t	0	xz	xt	0	-zt	xzt
yzt	0	q	0	0	0	y	z	0	t	0	0	yz	yt	zt	0	yzt
xyzt	q	x	y	z	t	xy	xz	yz	xt	yt	zt	xyz	xyt	xzt	yzt	xyzt

Table 3: Hestenes' Regressive Product Multiplication Table

V	q	x	y	z	t	xy	xz	yz	xt	yt	zt	xyz	xyt	xzt	yzt	xyzt
q	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	q
x	0	0	0	0	0	0	0	0	0	0	0	0	0	0	q	x
y	0	0	0	0	0	0	0	0	0	0	0	0	0	-q	0	y
z	0	0	0	0	0	0	0	0	0	0	0	0	q	0	0	z
t	0	0	0	0	0	0	0	0	0	0	0	-q	0	0	0	t
xy	0	0	0	0	0	0	0	0	0	0	q	0	0	x	y	xy
xz	0	0	0	0	0	0	0	0	0	-q	0	0	-x	0	z	xz
yz	0	0	0	0	0	0	0	0	q	0	0	0	-y	-z	0	yz
xt	0	0	0	0	0	0	0	q	0	0	0	x	0	0	t	xt
yt	0	0	0	0	0	0	-q	0	0	0	0	y	0	-t	0	yt
zt	0	0	0	0	0	q	0	0	0	0	0	z	t	0	0	zt
xyz	0	0	0	0	q	0	0	0	x	y	z	0	xy	xz	yz	xyz
xyt	0	0	0	-q	0	0	-x	-y	0	0	t	-xy	0	xt	yt	xyt
xzt	0	0	q	0	0	x	0	-z	0	-t	0	-xz	-xt	0	zt	xzt
yzt	0	-q	0	0	0	y	z	0	t	0	0	-yz	-yt	-zt	0	yzt
xyzt	q	x	y	z	t	xy	xz	yz	xt	yt	zt	xyz	xyt	xzt	yzt	xyzt

Table 4: Lengyel's Antiwedge Product Multiplication Table

References

[1] Leo Dorst, Daniel Fontune and Stephen Mann, *Geometric Algebra for Computer Science* Morgan Kaufmann Publishers, ISBN 978-0-12-374942-0