Ramanujan's Challenge

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Abstract

Around 1910, Ramanujan, devotee of Mahalakshmi of Namakkal, posed a number of puzzles and challenges in the *Journal of the Indian Mathematical Society*, one of which was to evaluate

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{\ldots}}}}}$$

This note walks through one solution of this puzzle, hopefully expressing my deep appreciation of this perceptive and inspired genius.

Puzzle Walkthrough

After six months, with no answers from the audience, Ramanujan provided the answer to the puzzle above, as

$$3 \;=\; \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{\ldots}}}}}$$

The answer, by itself, totally misses the beauty of the recursive formula found in the above. This, I believe, is part of the appeal of Ramanujan's work. He provides a puzzle and answer, and leaves the detection of the scaffolding and development of the generating tools as the exercise to the interested reader.

We are told the number is 3, but let's pretend we are starting from scratch. My first step is to calculate partial expressions, looking for convergence numerically.

```
#include <stdio.h>
#include <math.h>

int main(void)
{
    int i,j,k;
    double a,b,c,d,s,t[20];

    t[0]  = sqrt(1.0);
    t[1]  = sqrt(1.0+2.0);
    t[2]  = sqrt(1.0+2.0*sqrt(1.0+3.0));
    t[3]  = sqrt(1.0+2.0*sqrt(1.0+3.0*sqrt(1.0+4.0)));
    t[4]  = sqrt(1.0+2.0*sqrt(1.0+3.0*sqrt(1.0+4.0*sqrt(1.0+5.0))));
    t[5]  = sqrt(1.0+2.0*sqrt(1.0+3.0*sqrt(1.0+4.0*sqrt(1.0+5.0*
              sqrt(1.0+6.0)))));
    t[6]  = sqrt(1.0+2.0*sqrt(1.0+3.0*sqrt(1.0+4.0*sqrt(1.0+5.0*
              sqrt(1.0+6.0*sqrt(1.0+7.0))))));
    t[7]  = sqrt(1.0+2.0*sqrt(1.0+3.0*sqrt(1.0+4.0*sqrt(1.0+5.0*
              sqrt(1.0+6.0*sqrt(1.0+7.0*sqrt(1.0+8.0)))))));
    t[8]  = sqrt(1.0+2.0*sqrt(1.0+3.0*sqrt(1.0+4.0*sqrt(1.0+5.0*
              sqrt(1.0+6.0*sqrt(1.0+7.0*sqrt(1.0+8.0*sqrt(1.0+9.0))))))));
    t[9]  = sqrt(1.0+2.0*sqrt(1.0+3.0*sqrt(1.0+4.0*sqrt(1.0+5.0*
              sqrt(1.0+6.0*sqrt(1.0+7.0*sqrt(1.0+8.0*sqrt(1.0+9.0*
                 sqrt(1.0+10.0)))))))));
    t[10] = sqrt(1.0+2.0*sqrt(1.0+3.0*sqrt(1.0+4.0*sqrt(1.0+5.0*
              sqrt(1.0+6.0*sqrt(1.0+7.0*sqrt(1.0+8.0*sqrt(1.0+9.0*
                sqrt(1.0+10.0*sqrt(1.0+11.0))))))))));
    t[11] = sqrt(1.0+2.0*sqrt(1.0+3.0*sqrt(1.0+4.0*sqrt(1.0+5.0*
              sqrt(1.0+6.0*sqrt(1.0+7.0*sqrt(1.0+8.0*sqrt(1.0+9.0*
                 sqrt(1.0+10.0*sqrt(1.0+11.0*sqrt(1.0+12.0)))))))))));

    for (i=0;i<12;i++) printf("t[%d] = %f  \n",i,t[i]);

    return (0);



}
```

The results from above are

```
t[0]  = 1.000000
t[1]  = 1.732051
t[2]  = 2.236068
t[3]  = 2.559830
t[4]  = 2.755053
t[5]  = 2.867103
t[6]  = 2.929173
t[7]  = 2.962723
t[8]  = 2.980554
t[9]  = 2.989920
t[10] = 2.994800
t[11] = 2.997327
```

which indeed clearly indicate 3 as the result.

So now, we start trying to detect the method by which this result was achieved. We assume the answer of 3 is correct, and see what patterns we can find from this. We do a few numerical consistency checks, this time with pen and paper. We are looking at the underlined pattern, and square, subtract and divide to investigate this further.

$$
\begin{aligned}
\underline{3} &= \sqrt{\underline{1} + \underline{2}\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{\ldots}}}}} \\
9 - 1 &= 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{\ldots}}}} \\
\underline{4} &= \sqrt{\underline{1} + \underline{3}\sqrt{1 + 4\sqrt{1 + 5\sqrt{\ldots}}}} \\
16 - 1 &= 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + 6\sqrt{\ldots}}}} \\
\underline{5} &= \sqrt{\underline{1} + \underline{4}\sqrt{1 + 5\sqrt{1 + 6\sqrt{\ldots}}}}
\end{aligned}
$$

So we have a clear enough pattern. We hypothesize two forms for a $f(x)$:

$$\begin{aligned} f(x) &= x+1 = \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{\ldots}}}} \\ f(x+1) &= (x+1)+1 = \sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{1+(x+3)\sqrt{\ldots}}}} \\ f(x) &= x+1 = \sqrt{1+xf(x+1)} = \sqrt{1+x(x+2)} = \sqrt{x^2+2x+1} \end{aligned}$$

We see this is consistent. We also see clues as to how he developed this formula.

Start with the square identity, then define $f(x)$ by taking square roots of both sides of the square identity.

$$\begin{aligned} (x+1)^2 &= 1+x(x+2) \\ f(x)=(x+1) &= \sqrt{1+x(x+2)} \end{aligned}$$

Notice that

$$f(x+1)=(x+2) = \sqrt{1+(x+1)(x+3)}$$

Substitute in the above to get

$$\begin{aligned} f(x)=(x+1) &= \sqrt{1+x(x+2)} \\ f(x+1)=(x+2) &= \sqrt{1+(x+1)(x+3)} \\ f(x)=(x+1) &= \sqrt{1+x\sqrt{1+(x+1)(x+3)}} \\ f(x+2)=(x+3) &= \sqrt{1+(x+2)(x+4)} \\ f(x)=(x+1) &= \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)(x+4)}}} \\ f(x+3)=(x+4) &= \sqrt{1+(x+3)(x+5)} \\ f(x)=(x+1) &= \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{1+(x+3)(x+5)}}}} \\ f(x)=(x+1) &= \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{\ldots}}}} \quad \text{and so on} \ldots \end{aligned}$$

For the specific case of $x = 2$, we pickup the challenge problem, with the answer of 3.

More General Formula, Notebook 1, Page 105

We now tackle a more general formula from Ramanujan's Notebook 1, page 105. Ramanujan uses n in his formula, which can confuse due to the common usage of n as an index. I will use b in place of n here.

Ramanujan's formula is

$$x + (a + b) = \sqrt{ax + (a + b)^2 + x\sqrt{a(x + b) + (a + b)^2 + (x + b)\sqrt{\ldots}}}$$

My initial attack begins with

$$f(x, a, b) = x + (a + b) = \sqrt{(a + b)^2 + x\,(a + b) + x\,(x + (a + b))}$$

We spot our opportunity for a recursive definition.

$$\begin{aligned}
\underline{x + (a + b)} &= \sqrt{(a + b)^2 + x\,(a + b) + x\left(\underline{x + (a + b)}\right)} \\
x + (a + b) &= \sqrt{(a + b)^2 + x\,(a + b) + x\sqrt{(a + b)^2 + x\,(a + b) + x\,(x + (a + b))}} \\
&= \sqrt{(a + b)^2 + x\,(a + b) + x\sqrt{(a + b)^2 + x\,(a + b) + x\sqrt{\ldots})}}
\end{aligned}$$

Unfortunately, this is not a match. (It is also too easy to spot the recursive formula.) So we begin our second attack, where we shift one of the bx terms in the initial identity.

$$f(x, a, b) = x + (a + b) = \sqrt{(a + b)^2 + ax + x\,((x + b) + (a + b))}$$

Using this definition, we note

$$\begin{aligned}
f((x + b), a, b) &= \sqrt{(a + b)^2 + a(x + b) + (x + b)\,((x + 2b) + (a + b))} \\
f((x + 2b), a, b) &= \sqrt{(a + b)^2 + a(x + 2b) + (x + 2b)\,((x + 3b) + (a + b))} \\
f((x + 3b), a, b) &= \sqrt{(a + b)^2 + a(x + 3b) + (x + 3b)\,((x + 4b) + (a + b))}
\end{aligned}$$

We now start our substitutions

$$
\begin{aligned}
x+(a+b) &= \sqrt{(a+b)^2+ax+x\left(\underline{(x+b)+(a+b)}\right)} \\
&= \sqrt{(a+b)^2+ax+x\sqrt{(a+b)^2+a(x+b)+(x+b)\underline{((x+2b)+(a+b))}}} \\
&= \sqrt{(a+b)^2+ax+x\sqrt{(a+b)^2+a(x+b)+(x+b)\sqrt{\ldots}}} \\
\sqrt{\ldots} &= \sqrt{(a+b)^2+a(x+2b)+(x+2b)\underline{((x+3b)+(a+b))}}
\end{aligned}
$$

which matches and slightly extends Ramanujan's formula.

Using this formula, with $x=2$, $a=0$, $b=1$ presents the earlier result of

$$
\begin{aligned}
x+(a+b) &= \sqrt{(a+b)^2+ax+x\sqrt{(a+b)^2+a(x+b)+(x+b)\sqrt{\ldots}}} \\
3=2+(0+1) &= \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{\ldots}}}}
\end{aligned}
$$

A Simpler Challenge

To test our understanding of the recursive root definitions, let's study the expression

$$
f(1) = \sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4+\sqrt{5+\ldots}}}}}
$$

As long as the terms under the square root grow at a rate less than quadratic, I expect convergence.

We define a candidate function, and see the relationships

$$
\begin{aligned}
f(x) &= \sqrt{x+\sqrt{(x+1)+\sqrt{(x+2)+\ldots}}} \\
f(x+1) &= f^2(x)-x \\
f(x-1) &= \sqrt{(x-1)+f(x)}
\end{aligned}
$$

This function hits zero between $-2 < x < -1$. The function goes seriously vertical after $x > 9$, as each unit step right squares the new value compared to the old. Unlike the tangent function, even though we are radically increasing on the right travel, we never reset, such as at $\tan(\pi/2)$. This curve is continous in the positive y zone, and has the appearance to me, of a horse riding boot, with the toe left. Figure 1 provides a coarse plot for this function.

A simple C program finds $f(1) = 1.757971$.

```
f[-1] = 0.570864
f[0] = 1.325885
f[1] = 1.757971
f[2] = 2.090464
f[3] = 2.370038
f[4] = 2.617080
f[5] = 2.849105
f[6] = 3.117400
f[7] = 3.718184
f[8] = 6.824892
f[9] = 38.579152
f[10] = 1479.350935
```

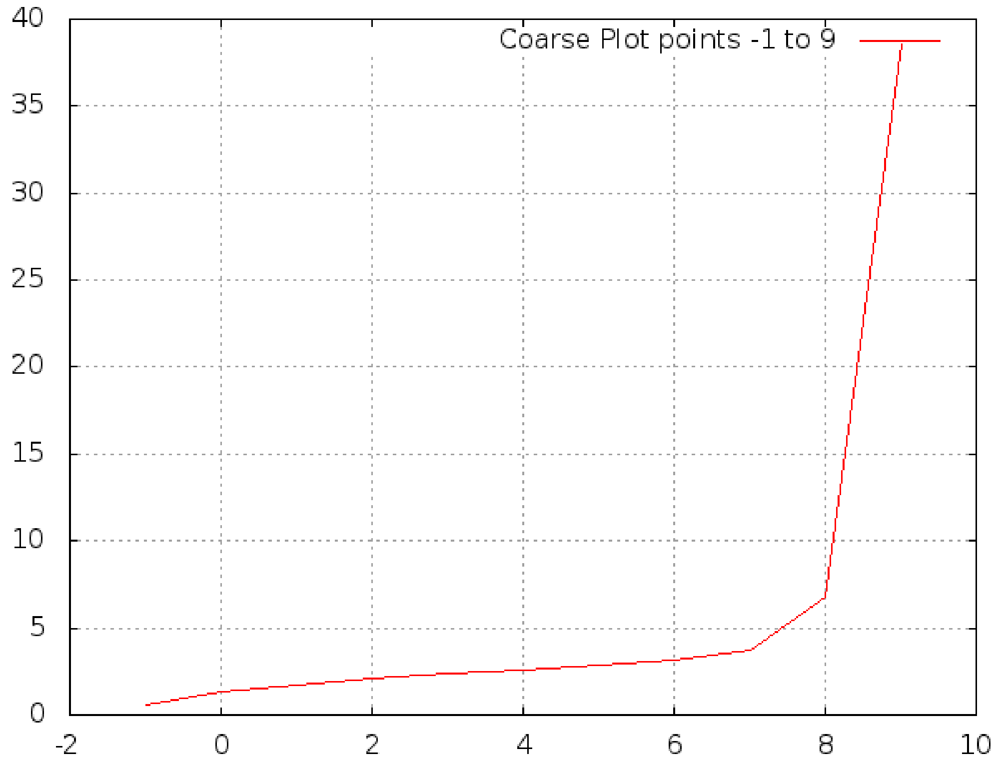

Figure 1: Course Plot for F(x)