Project Aldehyde - $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$

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Abstract

This article is my study of the divisional algebra approach to the standard model, whose main contributors are John Baez, Geoffrey Dixon, Cohl Furey, Murat Günaydin, Feza Gürsey, and John Huerta. I am focusing mainly on Cohl Furey's thesis, and will recast her work in terms of geometric algebra, where applicable.

Preliminary Mathematics

In this section, I provide a succint description of the algebra (or ring), multiplication table and component implementation formulas.

Divisional Algebra

The four divisional algebras define a magnitude as a square root of the sum of the squares of the components. These magnitudes compound under the defined multiplication. No nilpotents are present other than the trivial zero case. If the magnitude is non-zero, a reciprocal is well defined.

Real Numbers $\mathbb{R}$

Real numbers are sortable, associative and commutative, and trivial.

Complex Numbers $\mathbb{C}$

Complex numbers have two independent components, and thus lose the unique sortability function of real numbers. Complex numbers are commutative and associative. My usual format for complex numbers is $Z = (a + IA)$, where $I = \sqrt{-1}$, lowercase variable for real component, and uppercase variable for imaginary components. Python, C and C++ have support for complex numbers using the I format above.

Using '.r' for the real field, and '.i' for the imaginary field, complex multiplication can be defined by

```
typedef struct
{
    double r;    double i;
} Complex;

Complex Product_Complex( Complex a, Complex b){
    Complex w;
    w.r = a.r*b.r - a.i*b.i;
    w.i = a.r*b.i + a.i*b.r;
    return w;
}
```

Complex numbers mapped nicely into $C\ell_2$, $C\ell_3$ and $C\ell_{4,1}$.

$$
\begin{aligned}
a + IA &\rightarrow a + Ae_{xy} \quad C\ell_2 \\
a + IA &\rightarrow a + Ae_{xyz} \quad C\ell_3 \\
a + IA &\rightarrow a + Ae_{wxyzt} \quad C\ell_{4,1}
\end{aligned}
$$

The determinant of this multivector for $C\ell_2$ is $a^2 + A^2$, which matches the standard complex norm.

Quaternions $\mathbb{H}$

Quaternions have four independent components; a scalar q, whose basis squares to one, and three imaginary units (i, j, and k) which anti-commute among themselves and square to negative one. In general, quaternions are non-commutative, although associative.

```
typedef struct
{
    double q;    double i;    double j;    double k;
} Quaternion;
```

```
 * |  q  i  j  k
-----------------
 q |  q  i  j  k
 i |  i -q  k -j
 j |  j -k -q  i
 k |  k  j -i -q
```

```
Quaternion Product_Quaternion(Quaternion u, Quaternion v) {
    Quaternion w;
    w.q = (+u.q*v.q-u.i*v.i-u.j*v.j-u.k*v.k);
    w.i = (+u.q*v.i+u.i*v.q+u.j*v.k-u.k*v.j);
    w.j = (+u.q*v.j-u.i*v.k+u.j*v.q+u.k*v.i);
    w.k = (+u.q*v.k+u.i*v.j-u.j*v.i+u.k*v.q);
    return w;
}
```

Quaternions map into $C\ell_3$ as

$$a + bi + cj + dk \to a + be_{xy} + ce_{yz} + de_{xz}$$

with determinant $a^2 + b^2 + c^2 + d^2$.

Octonions $\mathbb{O}$

Octonions have eight independent components; a scalar component q, which squares to one, and seven imaginary units (i, j, k, E, I, J, and K) which square to negative one. Octonions are neither commutative nor associative, but are alternative and flexible, meaning $(aa)b = a(ab)$, $(ab)a = a(ba)$, and $(ba)a = b(aa)$. Many variations on octonions exist. The table below, which I prefer, is the Cayley format, clearly showing quaternions as a subset of octonions. My software also supports John Baez's preferred format.

```
typedef struct
{
    double q;    double i;    double j;    double k;
    double E;    double I;    double J;    double K;
} Octonion;
```

```
 * |  q  i  j  k  E  I  J  K    Cayley's Octonion Table
-----------------------------   Showing Quaternion Inheritance
 q |  q  i  j  k  E  I  J  K
 i |  i -q  k -j  I -E -K  J
 j |  j -k -q  i  J  K -E -I
 k |  k  j -i -q  K -J  I -E
 E |  E -I -J -K -q  i  j  k
 I |  I  E -K  J -i -q -k  j
 J |  J  K  E -I -j  k -q -i
 K |  K -J  I  E -k -j  i -q
```

```
Octonion Product_Octonion(const Octonion u, const Octonion v) {
    Octonion w;
    double a,b,c,d,e,f,g,h,A,B,C,D,E,F,G,H;
        a = u.q;    b = u.i;    c = u.j;    d = u.k;
        e = u.E;    f = u.I;    g = u.J;    h = u.K;
        A = v.q;    B = v.i;    C = v.j;    D = v.k;
        E = v.E;    F = v.I;    G = v.J;    H = v.K;

    if(Baez) {
        w.q = (a*A - b*B - c*C - d*D - e*E - f*F - g*G - h*H) ; // Baez
        w.i = (a*B + b*A + c*E + d*H - e*C + f*G - g*F - h*D) ;
        w.j = (a*C - b*E + c*A + d*F + e*B - f*D + g*H - h*G) ;
        w.k = (a*D - b*H - c*F + d*A + e*G + f*C - g*E + h*B) ;
        w.E = (a*E + b*C - c*B - d*G + e*A + f*H + g*D - h*F) ;
        w.I = (a*F - b*G + c*D - d*C - e*H + f*A + g*B + h*E) ;
        w.J = (a*G + b*F - c*H + d*E - e*D - f*B + g*A + h*C) ;
        w.K = (a*H + b*D + c*G - d*B + e*F - f*E - g*C + h*A) ;
    } else {
        w.q = (a*A - b*B - c*C - d*D - e*E - f*F - g*G - h*H) ; // Cayley
        w.i = (a*B + b*A + c*D - d*C + e*F - f*E - g*H + h*G) ;
        w.j = (a*C - b*D + c*A + d*B + e*G + f*H - g*E - h*F) ;
        w.k = (a*D + b*C - c*B + d*A + e*H - f*G + g*F - h*E) ;
        w.E = (a*E - b*F - c*G - d*H + e*A + f*B + g*C + h*D) ;
        w.I = (a*F + b*E - c*H + d*G - e*B + f*A - g*D + h*C) ;
        w.J = (a*G + b*H + c*E - d*F - e*C + f*D + g*A - h*B) ;
        w.K = (a*H - b*G + c*F + d*E - e*D - f*C + g*B + h*A) ;
    }
    return w;
}
```

Geometric Algebra

Geometric algebras are associative at all dimensions, in contrast to octonions. The magnitude of a multivector is best defined as the appropriate root of a determinant associated with the multivector. Zero determinants occur for nilpotents, idempotents, and eigenpotents, meaning that these expressions do not have a well defined reciprocal. However, when the determinant is non-zero (which is most of the time), a unique reciprocal does exist.

GA2E

Two dimensional geometric algebra has a scalar basis e_q, which squares to one and commutes with everything, two vector basis e_x and e_y, which square to one and anti-commute with each other, and a single areal bivector, e_{xy}, which squares to negative one and anti-commutes with the basis vectors.

```
typedef struct
{
    double q;
    double x, y;
    double xy;
} GA2E;
```

```
 *    | q     x     y     xy
------------------------------
 q    | q     x     y     xy
 x    | x     q     xy    y
 y    | y    -xy    q    -x
 xy   | xy   -y     x    -q
```

```
c.q   =  + a.q*b.q   + a.x*b.x   + a.y*b.y   - a.xy*b.xy  ;
c.x   =  + a.q*b.x   + a.x*b.q   - a.y*b.xy  + a.xy*b.y   ;
c.y   =  + a.q*b.y   + a.x*b.xy  + a.y*b.q   - a.xy*b.x   ;
c.xy  =  + a.q*b.xy  + a.x*b.y   - a.y*b.x   + a.xy*b.q   ;
```

This algebra can be implemented using real 2x2 matrices.

$$e_q = 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$e_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$e_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$e_x e_y = e_{xy} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

A general 2D multivector, $q + ae_x + be_y + ce_{xy}$ in matrix format becomes

$$\begin{bmatrix} q+a & b-c \\ b+c & q-a \end{bmatrix}$$

This matrix has trace $2q$ and determinant $q^2 - a^2 - b^2 + c^2$.

Complex numbers map to the even grade elements of $C\ell_2$.

$$a + IA \to a + Ae_{xy}$$

The determinant of this multivector is $a^2 + A^2$, which matches the standard complex norm.

GA3E

Three dimensional Euclidean geometric algebra has a scalar basis e_q, which squares to one and commutes with everything, three vector basis e_x, e_y and e_z, which square to one and anti-commute among themselves, three areal bivectors e_{xy}, e_{xz} and e_{yz} which square to negative one, and one volume trivector e_{xyz}, which squares to negative one, commute with everything, and thus mimics $\sqrt{-1}$. The three vector and bivector terms have been a source of historical confusion, such as axial versus pseudovectors. Quaternions can be mapped to the even grade subset $(q, e_{xy}, e_{xz}, e_{yz})$. In current literature, about half the authors use e_{xz}, while the others use e_{zx}.

```
typedef struct
{
    double q;
    double x,  y,  z;
    double xy, xz, yz;
    double xyz;
} GA3E;
```

*	q	x	y	z	xy	xz	yz	xyz
q	q	x	y	z	xy	xz	yz	xyz
x	x	q	xy	xz	y	z	xyz	yz
y	y	-xy	q	yz	-x	-xyz	z	-xz
z	z	-xz	-yz	q	xyz	-x	-y	xy
xy	xy	-y	x	xyz	-q	-yz	xz	-z
xz	xz	-z	-xyz	x	yz	-q	-xy	y
yz	yz	xyz	-z	y	-xz	xy	-q	-x
xyz	xyz	yz	-xz	xy	-z	y	-x	-q

```
c.q   =  + a.q*b.q   + a.x*b.x   + a.y*b.y   + a.z*b.z
         - a.xy*b.xy  - a.xz*b.xz  - a.yz*b.yz  - a.xyz*b.xyz ;
c.x   =  + a.q*b.x   + a.x*b.q   - a.y*b.xy  - a.z*b.xz
         + a.xy*b.y   + a.xz*b.z   - a.yz*b.xyz - a.xyz*b.yz  ;
c.y   =  + a.q*b.y   + a.x*b.xy  + a.y*b.q   - a.z*b.yz
         - a.xy*b.x   + a.xz*b.xyz + a.yz*b.z   + a.xyz*b.xz  ;
c.z   =  + a.q*b.z   + a.x*b.xz  + a.y*b.yz  + a.z*b.q
         - a.xy*b.xyz - a.xz*b.x   - a.yz*b.y   - a.xyz*b.xy  ;
c.xy  =  + a.q*b.xy  + a.x*b.y   - a.y*b.x   + a.z*b.xyz
         + a.xy*b.q   - a.xz*b.yz  + a.yz*b.xz  + a.xyz*b.z   ;
c.xz  =  + a.q*b.xz  + a.x*b.z   - a.y*b.xyz - a.z*b.x
         + a.xy*b.yz  + a.xz*b.q   - a.yz*b.xy  - a.xyz*b.y   ;
```

```
c.yz  =  + a.q*b.yz  + a.x*b.xyz + a.y*b.z   - a.z*b.y
         - a.xy*b.xz  + a.xz*b.xy  + a.yz*b.q   + a.xyz*b.x   ;
c.xyz =  + a.q*b.xyz + a.x*b.yz  - a.y*b.xz  + a.z*b.xy
         + a.xy*b.z   - a.xz*b.y   + a.yz*b.x   + a.xyz*b.q   ;
```

We can implement three dimensional Euclidean geometric algebra as a complexified two dimensional matrix algebra. One implementation of matrices (notice the e_{zx} convention) is

```
 Unity       e_x        e_y        e_xy
[ 1  0]    [ 0  1]    [ 0 -1]    [ 1  0]
[ 0  1]    [ 1  0]    [ 1  0]    [ 0 -1]

 e_xyz       e_yz       e_zx       e_z
[ I  0]    [ 0  I]    [ 0 -I]    [ I  0]
[ 0  I]    [ I  0]    [ I  0]    [ 0 -I]
```

For the three dimensional multivector

$$A + Be_x + Ce_y + De_z + Ee_{xy} + Fe_{zx} + Ge_{yz} + He_{xyz}$$

we can write the equivalent matrix and determinant

```
ThreeD = [[ + A + E + I*D + I*H, + B - C - I*F + I*G],
           [ + B + C + I*F + I*G, + A - E - I*D + I*H]]
```

$$\det_{\text{3D}} = (A + I * H)^2 - (B + I * G)^2 - (C + I * F)^2 - (D + I * E)^2$$

Complex numbers map into $C\ell_3$ as

$$a + IA \to a + Ae_{xyz}$$

with determinant $a^2 + A^2$. Quaternions map into $C\ell_3$ as

$$a + bi + cj + dk \to a + be_{xy} + ce_{yz} + de_{xz}$$

with determinant $a^2 + b^2 + c^2 + d^2$.

Minkowski SpaceTime ($C\ell_{3,1}$)

Minkowski spacetime has one scalar, which squares to one, three space vector basis, which square to one, and a time basis, which squares to negative one. We now have six bivectors, four trivectors, and a quadvector. None of these elements faithfully mimics $\sqrt{-1}$.

*	q	x	y	z	t	xy	xz	yz	xt	yt	zt	xyz	xyt	xzt	yzt	xyzt
q	q	x	y	z	t	xy	xz	yz	xt	yt	zt	xyz	xyt	xzt	yzt	xyzt
x	x	q	xy	xz	xt	y	z	xyz	t	xyt	xzt	yz	yt	zt	xyzt	yzt
y	y	-xy	q	yz	yt	-x	-xyz	z	-xyt	t	yzt	-xz	-xt	-xyzt	zt	-xzt
z	z	-xz	-yz	q	zt	xyz	-x	-y	-xzt	-yzt	t	xy	xyzt	-xt	-yt	xyt
t	t	-xt	-yt	-zt	-q	xyt	xzt	yzt	x	y	z	-xyzt	-xy	-xz	-yz	xyz
xy	xy	-y	x	xyz	xyt	-q	-yz	xz	-yt	xt	xyzt	-z	-t	-yzt	xzt	-zt
xz	xz	-z	-xyz	x	xzt	yz	-q	-xy	-zt	-xyzt	xt	y	yzt	-t	-xyt	yt
yz	yz	xyz	-z	y	yzt	-xz	xy	-q	xyzt	-zt	yt	-x	-xzt	xyt	-t	-xt
xt	xt	-t	-xyt	-xzt	-x	yt	zt	xyzt	q	xy	xz	-yzt	-y	-z	-xyz	yz
yt	yt	xyt	-t	-yzt	-y	-xt	-xyzt	zt	-xy	q	yz	xzt	x	xyz	-z	-xz
zt	zt	xzt	yzt	-t	-z	xyzt	-xt	-yt	-xz	-yz	q	-xyt	-xyz	x	y	xy
xyz	xyz	yz	-xz	xy	xyzt	-z	y	-x	yzt	-xzt	xyt	-q	-zt	yt	-xt	-t
xyt	xyt	yt	-xt	-xyzt	-xy	-t	-yzt	xzt	-y	x	xyz	zt	q	yz	-xz	-z
xzt	xzt	zt	xyzt	-xt	-xz	yzt	-t	-xyt	-z	-xyz	x	-yt	-yz	q	xy	y
yzt	yzt	-xyzt	zt	-yt	-yz	-xzt	xyt	-t	xyz	-z	y	xt	xz	-xy	q	-x
xyzt	xyzt	-yzt	xzt	-xyt	-xyz	-zt	yt	-xt	yz	-xz	xy	t	z	-y	x	-q

Table 1: Minkowski Geometric Algebra Multiplication Table

```
w.q    = + u.q*v.q + u.x*v.x + u.y*v.y + u.z*v.z - u.t*v.t - u.xy*v.xy - u.xz*v.xz - u.yz*v.yz
         + u.xt*v.xt + u.yt*v.yt + u.zt*v.zt - u.xyz*v.xyz + u.xyt*v.xyt + u.xzt*v.xzt + u.yzt*v.yzt - u.xyzt*v.xyzt;
w.x    = + u.q*v.x + u.x*v.q - u.y*v.xy - u.z*v.xz + u.t*v.xt + u.xy*v.y + u.xz*v.z - u.yz*v.xyz
         - u.xt*v.t + u.yt*v.xyt + u.zt*v.xzt - u.xyz*v.yz + u.xyt*v.yt + u.xzt*v.zt - u.yzt*v.xyzt + u.xyzt*v.yzt ;
w.y    = + u.q*v.y + u.x*v.xy + u.y*v.q - u.z*v.yz + u.t*v.yt - u.xy*v.x + u.xz*v.xyz + u.yz*v.z
         - u.xt*v.xyt - u.yt*v.t + u.zt*v.yzt + u.xyz*v.xz - u.xyt*v.xt + u.xzt*v.xyzt + u.yzt*v.zt - u.xyzt*v.xzt ;
w.z    = + u.q*v.z + u.x*v.xz + u.y*v.yz + u.z*v.q + u.t*v.zt - u.xy*v.xyz - u.xz*v.x - u.yz*v.y
         - u.xt*v.xzt - u.yt*v.yzt - u.zt*v.t - u.xyz*v.xy - u.xyt*v.xyzt - u.xzt*v.xt - u.yzt*v.yt + u.xyzt*v.xyt ;
w.t    = + u.q*v.t + u.x*v.xt + u.y*v.yt + u.z*v.zt + u.t*v.q - u.xy*v.xyt - u.xz*v.xzt - u.yz*v.yzt
         - u.xt*v.x - u.yt*v.y - u.zt*v.z - u.xyz*v.xyzt - u.xyt*v.xy - u.xzt*v.xz - u.yzt*v.yz + u.xyzt*v.xyz ;
w.xy   = + u.q*v.xy + u.x*v.y - u.y*v.x + u.z*v.xyz - u.t*v.xyt + u.xy*v.q - u.xz*v.yz + u.yz*v.xz
         + u.xt*v.yt - u.yt*v.xt + u.zt*v.xyzt + u.xyz*v.z - u.xyt*v.t + u.xzt*v.yzt - u.yzt*v.xzt + u.xyzt*v.zt ;
w.xz   = + u.q*v.xz + u.x*v.z - u.y*v.xyz - u.z*v.x - u.t*v.xzt + u.xy*v.yz + u.xz*v.q - u.yz*v.xy
         + u.xt*v.zt - u.yt*v.xyzt - u.zt*v.xt - u.xyz*v.y - u.xyt*v.yzt - u.xzt*v.t + u.yzt*v.xyt - u.xyzt*v.yt ;
w.yz   = + u.q*v.yz + u.x*v.xyz + u.y*v.z - u.z*v.y - u.t*v.yzt - u.xy*v.xz + u.xz*v.xy + u.yz*v.q
         + u.xt*v.xyzt + u.yt*v.zt - u.zt*v.yt + u.xyz*v.x + u.xyt*v.xzt - u.xzt*v.xyt - u.yzt*v.t + u.xyzt*v.xt ;
w.xt   = + u.q*v.xt + u.x*v.t - u.y*v.xyt - u.z*v.xzt - u.t*v.x + u.xy*v.yt + u.xz*v.zt - u.yz*v.xyzt
         + u.xt*v.q - u.yt*v.xy - u.zt*v.xz - u.xyz*v.yzt - u.xyt*v.y - u.xzt*v.z + u.yzt*v.xyz - u.xyzt*v.yz ;
w.yt   = + u.q*v.yt + u.x*v.xyt + u.y*v.t - u.z*v.yzt - u.t*v.y - u.xy*v.xt + u.xz*v.xyzt + u.yz*v.zt
         + u.xt*v.xy + u.yt*v.q - u.zt*v.yz + u.xyz*v.xzt + u.xyt*v.x - u.xzt*v.xyz - u.yzt*v.z + u.xyzt*v.xz ;
w.zt   = + u.q*v.zt + u.x*v.xzt + u.y*v.yzt + u.z*v.t - u.t*v.z - u.xy*v.xyzt - u.xz*v.xt - u.yz*v.yt
         + u.xt*v.xz + u.yt*v.yz + u.zt*v.q - u.xyz*v.xyt + u.xyt*v.xyz + u.xzt*v.x + u.yzt*v.y - u.xyzt*v.xy ;
w.xyz  = + u.q*v.xyz + u.x*v.yz - u.y*v.xz + u.z*v.xy + u.t*v.xyzt + u.xy*v.z - u.xz*v.y + u.yz*v.x
         - u.xt*v.yzt + u.yt*v.xzt - u.zt*v.xyt + u.xyz*v.q + u.xyt*v.zt - u.xzt*v.yt + u.yzt*v.xt - u.xyzt*v.t ;
w.xyt  = + u.q*v.xyt + u.x*v.yt - u.y*v.xt + u.z*v.xyzt + u.t*v.xy + u.xy*v.t - u.xz*v.yzt + u.yz*v.xzt
         - u.xt*v.y + u.yt*v.x - u.zt*v.xyz + u.xyz*v.zt + u.xyt*v.q - u.xzt*v.yz + u.yzt*v.xz - u.xyzt*v.z ;
w.xzt  = + u.q*v.xzt + u.x*v.zt - u.y*v.xyzt - u.z*v.xt + u.t*v.xz + u.xy*v.yzt + u.xz*v.t - u.yz*v.xyt
         - u.xt*v.z + u.yt*v.xyz + u.zt*v.x - u.xyz*v.yt + u.xyt*v.yz + u.xzt*v.q - u.yzt*v.xy + u.xyzt*v.y ;
w.yzt  = + u.q*v.yzt + u.x*v.xyzt + u.y*v.zt - u.z*v.yt + u.t*v.yz - u.xy*v.xzt + u.xz*v.xyt + u.yz*v.t
         - u.xt*v.xyz - u.yt*v.z + u.zt*v.y + u.xyz*v.xt - u.xyt*v.xz + u.xzt*v.xy + u.yzt*v.q - u.xyzt*v.x ;
w.xyzt = + u.q*v.xyzt + u.x*v.yzt - u.y*v.xzt + u.z*v.xyt - u.t*v.xyz + u.xy*v.zt - u.xz*v.yt + u.yz*v.xt
         + u.xt*v.yz - u.yt*v.xz + u.zt*v.xy + u.xyz*v.t - u.xyt*v.z + u.xzt*v.y - u.yzt*v.x + u.xyzt*v.q ;
```

Table 2: Minkowski Geometric Algebra Multiplication Equations

The multiplication table among Minkowski geometric elements is shown in sidewise Table 1, while sidewise Table 2 provides the component level equations.

The Minkowski geometric algebra can be mapped to the real Dirac 4x4 matrices, which are direct product of real 2x2 matrices representing two dimensional Euclidean geometric algebra. This means $C\ell_{3,1} = C\ell_2 \otimes C\ell_2$.

Minkowski FiveSpaceTime $(C\ell_{4,1})$

Minkowski FiveSpaceTime geometric algebra has four space axis, e_w, e_x, e_y, and e_z, which square to one, and our friendly time axis e_t, which squares to negative one. We have ten bivectors, ten trivectors, five quadvectors and one pentavector e_{wxyzt}, which mimic $\sqrt{-1}$. The 32 complex Dirac 4x4 matrices directly map to this space.

Direct Product Investigation

I first look at matrix implementations of 2D and Minkwski geometric algebras, and then study symbolic $C\ell_2 \otimes C\ell_2$ directly.

Minkowski and $C\ell_2$ Basis Matrices

Sixteen 4x4 matrices provide a basis for geometric algebra in a Minkowski space with metric signature of (+,+,+,-).

A preformatted list of the these matrices is shown below.

On inspection, these matrices are seen to be the direct or Kronecker product of four different 2x2 matrices which define geometric algebra in the plane. I will use a_x to define the 2x2 matrix set, and e_x for the 4x4 matrix.

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad a_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad a_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad a_{xy} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Illustrating the direct product by example,

$$a \otimes b = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$$

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     Real Matrix Implementation of Minkowski Geometric Algebra

     Unity              xyzt
[  1  0  0  0]     [  0  0  0  1]
[  0  1  0  0]     [  0  0  1  0]
[  0  0  1  0]     [  0 -1  0  0]
[  0  0  0  1]     [ -1  0  0  0]

        x              y              z              t
[  0  0  1  0] [  1  0  0  0] [  0  0  0  1] [  0  0 -1  0]
[  0  0  0  1] [  0  1  0  0] [  0  0 -1  0] [  0  0  0  1]
[  1  0  0  0] [  0  0 -1  0] [  0 -1  0  0] [  1  0  0  0]
[  0  1  0  0] [  0  0  0 -1] [  1  0  0  0] [  0 -1  0  0]

       xy             xz             yz             xt             yt             zt
[  0  0 -1  0] [  0 -1  0  0] [  0  0  0  1] [  1  0  0  0] [  0  0 -1  0] [  0 -1  0  0]
[  0  0  0 -1] [  1  0  0  0] [  0  0 -1  0] [  0 -1  0  0] [  0  0  0  1] [ -1  0  0  0]
[  1  0  0  0] [  0  0  0  1] [  0  1  0  0] [  0  0 -1  0] [ -1  0  0  0] [  0  0  0 -1]
[  0  1  0  0] [  0  0 -1  0] [ -1  0  0  0] [  0  0  0  1] [  0  1  0  0] [  0  0 -1  0]

       xyz            xyt            xzt            yzt
[  0  1  0  0] [ -1  0  0  0] [  0  0  0 -1] [  0 -1  0  0]
[ -1  0  0  0] [  0  1  0  0] [  0  0 -1  0] [ -1  0  0  0]
[  0  0  0  1] [  0  0 -1  0] [  0 -1  0  0] [  0  0  0  1]
[  0  0 -1  0] [  0  0  0  1] [ -1  0  0  0] [  0  0  1  0]
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Table 3 generates a set of 16 basis 4x4 matrices from the direct product of the 2x2 geometric algebra basis. As an example of the process, here are two sample calculations used in the table.

$$a_1 \otimes a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$a_1 \otimes a_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Table 4 replaces the 4x4 matrices with the Minkowski basis symbols, and the 2x2 matrices with the $C\ell_2$ basis symbols.

Implementation of $C\ell_2 \otimes C\ell_2$

My elements of two dimensional geometric algebra are (a_1, a_x, a_y, a_{xy}).

$\otimes$	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}$	$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}$
$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix}$	$\begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{matrix}$
$\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{matrix}$
$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{matrix}$
$\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}$

Table 3: Minkowski Basis Via Direct Product

$\otimes$	a_1	a_x	a_y	a_{xy}
a_1	e_q	$-e_{xyt}$	$-e_{zt}$	e_{xyz}
a_x	e_y	e_{xt}	$-e_{yzt}$	$-e_{xz}$
a_y	e_x	$-e_{yt}$	$-e_{xzt}$	e_{yz}
a_{xy}	$-e_{xy}$	$-e_t$	e_{xyzt}	e_z

Table 4: Minkowski Basis Via Direct Product Symbolic Format

My sixteen elements of $C\ell_2 \otimes C\ell_2$ are

$$\begin{array}{r}(a_1 \otimes a_1), (a_1 \otimes a_x), (a_1 \otimes a_y), (a_1 \otimes a_{xy}), \\ (a_x \otimes a_1), (a_x \otimes a_x), (a_x \otimes a_y), (a_x \otimes a_{xy}), \\ (a_y \otimes a_1), (a_y \otimes a_x), (a_y \otimes a_y), (a_y \otimes a_{xy}), \\ (a_{xy} \otimes a_1), (a_{xy} \otimes a_x), (a_{xy} \otimes a_y), (a_{xy} \otimes a_{xy})\end{array}$$

Our next step is to combine our products, using

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

The next step is to map these element to Minkowski geometric algebra elements. Multiple implementations exist. This is the one I currently like best.

$$\begin{aligned}
e_1 &= (a_1 \otimes a_1) \\
\\
e_x &= (a_x \otimes a_1) \\
e_y &= (a_y \otimes a_1) \\
e_z &= (a_{xy} \otimes a_{xy}) \\
e_t &= (a_{xy} \otimes a_y) \\
\\
e_{xy} &= (a_{xy} \otimes a_1) \\
e_{xz} &= (a_y \otimes a_{xy}) \\
e_{yz} &= -(a_x \otimes a_{xy}) \\
e_{xt} &= (a_y \otimes a_y) \\
e_{yt} &= -(a_x \otimes a_y) \\
e_{zt} &= -(a_1 \otimes a_x) \\
\\
e_{xyz} &= -(a_1 \otimes a_{xy}) \\
e_{xyt} &= -(a_1 \otimes a_y) \\
e_{xzt} &= -(a_x \otimes a_x) \\
e_{yzt} &= -(a_y \otimes a_x) \\
\\
e_{xyzt} &= -(a_{xy} \otimes a_x)
\end{aligned}$$

$\otimes$	$(a_1 \otimes a_1)$	$(a_1 \otimes a_x)$	$(a_1 \otimes a_y)$	$(a_1 \otimes a_{xy})$
$(a_1 \otimes a_1)$	$(a_1 \otimes a_1)(a_1 \otimes a_1)$	$(a_1 \otimes a_1)(a_1 \otimes a_x)$	$(a_1 \otimes a_1)(a_1 \otimes a_y)$	$(a_1 \otimes a_1)(a_1 \otimes a_{xy})$
$(a_1 \otimes a_x)$	$(a_1 \otimes a_x)(a_1 \otimes a_1)$	$(a_1 \otimes a_x)(a_1 \otimes a_x)$	$(a_1 \otimes a_x)(a_1 \otimes a_y)$	$(a_1 \otimes a_x)(a_1 \otimes a_{xy})$
$(a_1 \otimes a_y)$	$(a_1 \otimes a_y)(a_1 \otimes a_1)$	$(a_1 \otimes a_y)(a_1 \otimes a_x)$	$(a_1 \otimes a_y)(a_1 \otimes a_y)$	$(a_1 \otimes a_y)(a_1 \otimes a_{xy})$
$(a_1 \otimes a_{xy})$	$(a_1 \otimes a_{xy})(a_1 \otimes a_1)$	$(a_1 \otimes a_{xy})(a_1 \otimes a_x)$	$(a_1 \otimes a_{xy})(a_1 \otimes a_y)$	$(a_1 \otimes a_{xy})(a_1 \otimes a_{xy})$
$(a_x \otimes a_1)$	$(a_x \otimes a_1)(a_1 \otimes a_1)$	$(a_x \otimes a_1)(a_1 \otimes a_x)$	$(a_x \otimes a_1)(a_1 \otimes a_y)$	$(a_x \otimes a_1)(a_1 \otimes a_{xy})$
$(a_x \otimes a_x)$	$(a_x \otimes a_x)(a_1 \otimes a_1)$	$(a_x \otimes a_x)(a_1 \otimes a_x)$	$(a_x \otimes a_x)(a_1 \otimes a_y)$	$(a_x \otimes a_x)(a_1 \otimes a_{xy})$
$(a_x \otimes a_y)$	$(a_x \otimes a_y)(a_1 \otimes a_1)$	$(a_x \otimes a_y)(a_1 \otimes a_x)$	$(a_x \otimes a_y)(a_1 \otimes a_y)$	$(a_x \otimes a_y)(a_1 \otimes a_{xy})$
$(a_x \otimes a_{xy})$	$(a_x \otimes a_{xy})(a_1 \otimes a_1)$	$(a_x \otimes a_{xy})(a_1 \otimes a_x)$	$(a_x \otimes a_{xy})(a_1 \otimes a_y)$	$(a_x \otimes a_{xy})(a_1 \otimes a_{xy})$
$(a_y \otimes a_1)$	$(a_y \otimes a_1)(a_1 \otimes a_1)$	$(a_y \otimes a_1)(a_1 \otimes a_x)$	$(a_y \otimes a_1)(a_1 \otimes a_y)$	$(a_y \otimes a_1)(a_1 \otimes a_{xy})$
$(a_y \otimes a_x)$	$(a_y \otimes a_x)(a_1 \otimes a_1)$	$(a_y \otimes a_x)(a_1 \otimes a_x)$	$(a_y \otimes a_x)(a_1 \otimes a_y)$	$(a_y \otimes a_x)(a_1 \otimes a_{xy})$
$(a_y \otimes a_y)$	$(a_y \otimes a_y)(a_1 \otimes a_1)$	$(a_y \otimes a_y)(a_1 \otimes a_x)$	$(a_y \otimes a_y)(a_1 \otimes a_y)$	$(a_y \otimes a_y)(a_1 \otimes a_{xy})$
$(a_y \otimes a_{xy})$	$(a_y \otimes a_{xy})(a_1 \otimes a_1)$	$(a_y \otimes a_{xy})(a_1 \otimes a_x)$	$(a_y \otimes a_{xy})(a_1 \otimes a_y)$	$(a_y \otimes a_{xy})(a_1 \otimes a_{xy})$
$(a_{xy} \otimes a_1)$	$(a_{xy} \otimes a_1)(a_1 \otimes a_1)$	$(a_{xy} \otimes a_1)(a_1 \otimes a_x)$	$(a_{xy} \otimes a_1)(a_1 \otimes a_y)$	$(a_{xy} \otimes a_1)(a_1 \otimes a_{xy})$
$(a_{xy} \otimes a_x)$	$(a_{xy} \otimes a_x)(a_1 \otimes a_1)$	$(a_{xy} \otimes a_x)(a_1 \otimes a_x)$	$(a_{xy} \otimes a_x)(a_1 \otimes a_y)$	$(a_{xy} \otimes a_x)(a_1 \otimes a_{xy})$
$(a_{xy} \otimes a_y)$	$(a_{xy} \otimes a_y)(a_1 \otimes a_1)$	$(a_{xy} \otimes a_y)(a_1 \otimes a_x)$	$(a_{xy} \otimes a_y)(a_1 \otimes a_y)$	$(a_{xy} \otimes a_y)(a_1 \otimes a_{xy})$
$(a_{xy} \otimes a_{xy})$	$(a_{xy} \otimes a_{xy})(a_1 \otimes a_1)$	$(a_{xy} \otimes a_{xy})(a_1 \otimes a_x)$	$(a_{xy} \otimes a_{xy})(a_1 \otimes a_y)$	$(a_{xy} \otimes a_{xy})(a_1 \otimes a_{xy})$

Table 5: Direct Product, 1 of 4

$\otimes$	$(a_x \otimes a_1)$	$(a_x \otimes a_x)$	$(a_x \otimes a_y)$	$(a_x \otimes a_{xy})$
$(a_1 \otimes a_1)$	$(a_1 \otimes a_1)(a_x \otimes a_1)$	$(a_1 \otimes a_1)(a_x \otimes a_x)$	$(a_1 \otimes a_1)(a_x \otimes a_y)$	$(a_1 \otimes a_1)(a_x \otimes a_{xy})$
$(a_1 \otimes a_x)$	$(a_1 \otimes a_x)(a_x \otimes a_1)$	$(a_1 \otimes a_x)(a_x \otimes a_x)$	$(a_1 \otimes a_x)(a_x \otimes a_y)$	$(a_1 \otimes a_x)(a_x \otimes a_{xy})$
$(a_1 \otimes a_y)$	$(a_1 \otimes a_y)(a_x \otimes a_1)$	$(a_1 \otimes a_y)(a_x \otimes a_x)$	$(a_1 \otimes a_y)(a_x \otimes a_y)$	$(a_1 \otimes a_y)(a_x \otimes a_{xy})$
$(a_1 \otimes a_{xy})$	$(a_1 \otimes a_{xy})(a_x \otimes a_1)$	$(a_1 \otimes a_{xy})(a_x \otimes a_x)$	$(a_1 \otimes a_{xy})(a_x \otimes a_y)$	$(a_1 \otimes a_{xy})(a_x \otimes a_{xy})$
$(a_x \otimes a_1)$	$(a_x \otimes a_1)(a_x \otimes a_1)$	$(a_x \otimes a_1)(a_x \otimes a_x)$	$(a_x \otimes a_1)(a_x \otimes a_y)$	$(a_x \otimes a_1)(a_x \otimes a_{xy})$
$(a_x \otimes a_x)$	$(a_x \otimes a_x)(a_x \otimes a_1)$	$(a_x \otimes a_x)(a_x \otimes a_x)$	$(a_x \otimes a_x)(a_x \otimes a_y)$	$(a_x \otimes a_x)(a_x \otimes a_{xy})$
$(a_x \otimes a_y)$	$(a_x \otimes a_y)(a_x \otimes a_1)$	$(a_x \otimes a_y)(a_x \otimes a_x)$	$(a_x \otimes a_y)(a_x \otimes a_y)$	$(a_x \otimes a_y)(a_x \otimes a_{xy})$
$(a_x \otimes a_{xy})$	$(a_x \otimes a_{xy})(a_x \otimes a_1)$	$(a_x \otimes a_{xy})(a_x \otimes a_x)$	$(a_x \otimes a_{xy})(a_x \otimes a_y)$	$(a_x \otimes a_{xy})(a_x \otimes a_{xy})$
$(a_y \otimes a_1)$	$(a_y \otimes a_1)(a_x \otimes a_1)$	$(a_y \otimes a_1)(a_x \otimes a_x)$	$(a_y \otimes a_1)(a_x \otimes a_y)$	$(a_y \otimes a_1)(a_x \otimes a_{xy})$
$(a_y \otimes a_x)$	$(a_y \otimes a_x)(a_x \otimes a_1)$	$(a_y \otimes a_x)(a_x \otimes a_x)$	$(a_y \otimes a_x)(a_x \otimes a_y)$	$(a_y \otimes a_x)(a_x \otimes a_{xy})$
$(a_y \otimes a_y)$	$(a_y \otimes a_y)(a_x \otimes a_1)$	$(a_y \otimes a_y)(a_x \otimes a_x)$	$(a_y \otimes a_y)(a_x \otimes a_y)$	$(a_y \otimes a_y)(a_x \otimes a_{xy})$
$(a_y \otimes a_{xy})$	$(a_y \otimes a_{xy})(a_x \otimes a_1)$	$(a_y \otimes a_{xy})(a_x \otimes a_x)$	$(a_y \otimes a_{xy})(a_x \otimes a_y)$	$(a_y \otimes a_{xy})(a_x \otimes a_{xy})$
$(a_{xy} \otimes a_1)$	$(a_{xy} \otimes a_1)(a_x \otimes a_1)$	$(a_{xy} \otimes a_1)(a_x \otimes a_x)$	$(a_{xy} \otimes a_1)(a_x \otimes a_y)$	$(a_{xy} \otimes a_1)(a_x \otimes a_{xy})$
$(a_{xy} \otimes a_x)$	$(a_{xy} \otimes a_x)(a_x \otimes a_1)$	$(a_{xy} \otimes a_x)(a_x \otimes a_x)$	$(a_{xy} \otimes a_x)(a_x \otimes a_y)$	$(a_{xy} \otimes a_x)(a_x \otimes a_{xy})$
$(a_{xy} \otimes a_y)$	$(a_{xy} \otimes a_y)(a_x \otimes a_1)$	$(a_{xy} \otimes a_y)(a_x \otimes a_x)$	$(a_{xy} \otimes a_y)(a_x \otimes a_y)$	$(a_{xy} \otimes a_y)(a_x \otimes a_{xy})$
$(a_{xy} \otimes a_{xy})$	$(a_{xy} \otimes a_{xy})(a_x \otimes a_1)$	$(a_{xy} \otimes a_{xy})(a_x \otimes a_x)$	$(a_{xy} \otimes a_{xy})(a_x \otimes a_y)$	$(a_{xy} \otimes a_{xy})(a_x \otimes a_{xy})$

Table 6: Direct Product, 2 of 4

$\otimes$	$(a_y \otimes a_1)$	$(a_y \otimes a_x)$	$(a_y \otimes a_y)$	$(a_y \otimes a_{xy})$
$(a_1 \otimes a_1)$	$(a_1 \otimes a_1)(a_y \otimes a_1)$	$(a_1 \otimes a_1)(a_y \otimes a_x)$	$(a_1 \otimes a_1)(a_y \otimes a_y)$	$(a_1 \otimes a_1)(a_y \otimes a_{xy})$
$(a_1 \otimes a_x)$	$(a_1 \otimes a_x)(a_y \otimes a_1)$	$(a_1 \otimes a_x)(a_y \otimes a_x)$	$(a_1 \otimes a_x)(a_y \otimes a_y)$	$(a_1 \otimes a_x)(a_y \otimes a_{xy})$
$(a_1 \otimes a_y)$	$(a_1 \otimes a_y)(a_y \otimes a_1)$	$(a_1 \otimes a_y)(a_y \otimes a_x)$	$(a_1 \otimes a_y)(a_y \otimes a_y)$	$(a_1 \otimes a_y)(a_y \otimes a_{xy})$
$(a_1 \otimes a_{xy})$	$(a_1 \otimes a_{xy})(a_y \otimes a_1)$	$(a_1 \otimes a_{xy})(a_y \otimes a_x)$	$(a_1 \otimes a_{xy})(a_y \otimes a_y)$	$(a_1 \otimes a_{xy})(a_y \otimes a_{xy})$
$(a_x \otimes a_1)$	$(a_x \otimes a_1)(a_y \otimes a_1)$	$(a_x \otimes a_1)(a_y \otimes a_x)$	$(a_x \otimes a_1)(a_y \otimes a_y)$	$(a_x \otimes a_1)(a_y \otimes a_{xy})$
$(a_x \otimes a_x)$	$(a_x \otimes a_x)(a_y \otimes a_1)$	$(a_x \otimes a_x)(a_y \otimes a_x)$	$(a_x \otimes a_x)(a_y \otimes a_y)$	$(a_x \otimes a_x)(a_y \otimes a_{xy})$
$(a_x \otimes a_y)$	$(a_x \otimes a_y)(a_y \otimes a_1)$	$(a_x \otimes a_y)(a_y \otimes a_x)$	$(a_x \otimes a_y)(a_y \otimes a_y)$	$(a_x \otimes a_y)(a_y \otimes a_{xy})$
$(a_x \otimes a_{xy})$	$(a_x \otimes a_{xy})(a_y \otimes a_1)$	$(a_x \otimes a_{xy})(a_y \otimes a_x)$	$(a_x \otimes a_{xy})(a_y \otimes a_y)$	$(a_x \otimes a_{xy})(a_y \otimes a_{xy})$
$(a_y \otimes a_1)$	$(a_y \otimes a_1)(a_y \otimes a_1)$	$(a_y \otimes a_1)(a_y \otimes a_x)$	$(a_y \otimes a_1)(a_y \otimes a_y)$	$(a_y \otimes a_1)(a_y \otimes a_{xy})$
$(a_y \otimes a_x)$	$(a_y \otimes a_x)(a_y \otimes a_1)$	$(a_y \otimes a_x)(a_y \otimes a_x)$	$(a_y \otimes a_x)(a_y \otimes a_y)$	$(a_y \otimes a_x)(a_y \otimes a_{xy})$
$(a_y \otimes a_y)$	$(a_y \otimes a_y)(a_y \otimes a_1)$	$(a_y \otimes a_y)(a_y \otimes a_x)$	$(a_y \otimes a_y)(a_y \otimes a_y)$	$(a_y \otimes a_y)(a_y \otimes a_{xy})$
$(a_y \otimes a_{xy})$	$(a_y \otimes a_{xy})(a_y \otimes a_1)$	$(a_y \otimes a_{xy})(a_y \otimes a_x)$	$(a_y \otimes a_{xy})(a_y \otimes a_y)$	$(a_y \otimes a_{xy})(a_y \otimes a_{xy})$
$(a_{xy} \otimes a_1)$	$(a_{xy} \otimes a_1)(a_y \otimes a_1)$	$(a_{xy} \otimes a_1)(a_y \otimes a_x)$	$(a_{xy} \otimes a_1)(a_y \otimes a_y)$	$(a_{xy} \otimes a_1)(a_y \otimes a_{xy})$
$(a_{xy} \otimes a_x)$	$(a_{xy} \otimes a_x)(a_y \otimes a_1)$	$(a_{xy} \otimes a_x)(a_y \otimes a_x)$	$(a_{xy} \otimes a_x)(a_y \otimes a_y)$	$(a_{xy} \otimes a_x)(a_y \otimes a_{xy})$
$(a_{xy} \otimes a_y)$	$(a_{xy} \otimes a_y)(a_y \otimes a_1)$	$(a_{xy} \otimes a_y)(a_y \otimes a_x)$	$(a_{xy} \otimes a_y)(a_y \otimes a_y)$	$(a_{xy} \otimes a_y)(a_y \otimes a_{xy})$
$(a_{xy} \otimes a_{xy})$	$(a_{xy} \otimes a_{xy})(a_y \otimes a_1)$	$(a_{xy} \otimes a_{xy})(a_y \otimes a_x)$	$(a_{xy} \otimes a_{xy})(a_y \otimes a_y)$	$(a_{xy} \otimes a_{xy})(a_y \otimes a_{xy})$

Table 7: Direct Product, 3 of 4

$\otimes$	$(a_{xy} \otimes a_1)$	$(a_{xy} \otimes a_x)$	$(a_{xy} \otimes a_y)$	$(a_{xy} \otimes a_{xy})$
$(a_1 \otimes a_1)$	$(a_1 \otimes a_1)(a_{xy} \otimes a_1)$	$(a_1 \otimes a_1)(a_{xy} \otimes a_x)$	$(a_1 \otimes a_1)(a_{xy} \otimes a_y)$	$(a_1 \otimes a_1)(a_{xy} \otimes a_{xy})$
$(a_1 \otimes a_x)$	$(a_1 \otimes a_x)(a_{xy} \otimes a_1)$	$(a_1 \otimes a_x)(a_{xy} \otimes a_x)$	$(a_1 \otimes a_x)(a_{xy} \otimes a_y)$	$(a_1 \otimes a_x)(a_{xy} \otimes a_{xy})$
$(a_1 \otimes a_y)$	$(a_1 \otimes a_y)(a_{xy} \otimes a_1)$	$(a_1 \otimes a_y)(a_{xy} \otimes a_x)$	$(a_1 \otimes a_y)(a_{xy} \otimes a_y)$	$(a_1 \otimes a_y)(a_{xy} \otimes a_{xy})$
$(a_1 \otimes a_{xy})$	$(a_1 \otimes a_{xy})(a_{xy} \otimes a_1)$	$(a_1 \otimes a_{xy})(a_{xy} \otimes a_x)$	$(a_1 \otimes a_{xy})(a_{xy} \otimes a_y)$	$(a_1 \otimes a_{xy})(a_{xy} \otimes a_{xy})$
$(a_x \otimes a_1)$	$(a_x \otimes a_1)(a_{xy} \otimes a_1)$	$(a_x \otimes a_1)(a_{xy} \otimes a_x)$	$(a_x \otimes a_1)(a_{xy} \otimes a_y)$	$(a_x \otimes a_1)(a_{xy} \otimes a_{xy})$
$(a_x \otimes a_x)$	$(a_x \otimes a_x)(a_{xy} \otimes a_1)$	$(a_x \otimes a_x)(a_{xy} \otimes a_x)$	$(a_x \otimes a_x)(a_{xy} \otimes a_y)$	$(a_x \otimes a_x)(a_{xy} \otimes a_{xy})$
$(a_x \otimes a_y)$	$(a_x \otimes a_y)(a_{xy} \otimes a_1)$	$(a_x \otimes a_y)(a_{xy} \otimes a_x)$	$(a_x \otimes a_y)(a_{xy} \otimes a_y)$	$(a_x \otimes a_y)(a_{xy} \otimes a_{xy})$
$(a_x \otimes a_{xy})$	$(a_x \otimes a_{xy})(a_{xy} \otimes a_1)$	$(a_x \otimes a_{xy})(a_{xy} \otimes a_x)$	$(a_x \otimes a_{xy})(a_{xy} \otimes a_y)$	$(a_x \otimes a_{xy})(a_{xy} \otimes a_{xy})$
$(a_y \otimes a_1)$	$(a_y \otimes a_1)(a_{xy} \otimes a_1)$	$(a_y \otimes a_1)(a_{xy} \otimes a_x)$	$(a_y \otimes a_1)(a_{xy} \otimes a_y)$	$(a_y \otimes a_1)(a_{xy} \otimes a_{xy})$
$(a_y \otimes a_x)$	$(a_y \otimes a_x)(a_{xy} \otimes a_1)$	$(a_y \otimes a_x)(a_{xy} \otimes a_x)$	$(a_y \otimes a_x)(a_{xy} \otimes a_y)$	$(a_y \otimes a_x)(a_{xy} \otimes a_{xy})$
$(a_y \otimes a_y)$	$(a_y \otimes a_y)(a_{xy} \otimes a_1)$	$(a_y \otimes a_y)(a_{xy} \otimes a_x)$	$(a_y \otimes a_y)(a_{xy} \otimes a_y)$	$(a_y \otimes a_y)(a_{xy} \otimes a_{xy})$
$(a_y \otimes a_{xy})$	$(a_y \otimes a_{xy})(a_{xy} \otimes a_1)$	$(a_y \otimes a_{xy})(a_{xy} \otimes a_x)$	$(a_y \otimes a_{xy})(a_{xy} \otimes a_y)$	$(a_y \otimes a_{xy})(a_{xy} \otimes a_{xy})$
$(a_{xy} \otimes a_1)$	$(a_{xy} \otimes a_1)(a_{xy} \otimes a_1)$	$(a_{xy} \otimes a_1)(a_{xy} \otimes a_x)$	$(a_{xy} \otimes a_1)(a_{xy} \otimes a_y)$	$(a_{xy} \otimes a_1)(a_{xy} \otimes a_{xy})$
$(a_{xy} \otimes a_x)$	$(a_{xy} \otimes a_x)(a_{xy} \otimes a_1)$	$(a_{xy} \otimes a_x)(a_{xy} \otimes a_x)$	$(a_{xy} \otimes a_x)(a_{xy} \otimes a_y)$	$(a_{xy} \otimes a_x)(a_{xy} \otimes a_{xy})$
$(a_{xy} \otimes a_y)$	$(a_{xy} \otimes a_y)(a_{xy} \otimes a_1)$	$(a_{xy} \otimes a_y)(a_{xy} \otimes a_x)$	$(a_{xy} \otimes a_y)(a_{xy} \otimes a_y)$	$(a_{xy} \otimes a_y)(a_{xy} \otimes a_{xy})$
$(a_{xy} \otimes a_{xy})$	$(a_{xy} \otimes a_{xy})(a_{xy} \otimes a_1)$	$(a_{xy} \otimes a_{xy})(a_{xy} \otimes a_x)$	$(a_{xy} \otimes a_{xy})(a_{xy} \otimes a_y)$	$(a_{xy} \otimes a_{xy})(a_{xy} \otimes a_{xy})$

Table 8: Direct Product, 4 of 4

$\otimes$	$(a_1 \otimes a_1)$	$(a_1 \otimes a_x)$	$(a_1 \otimes a_y)$	$(a_1 \otimes a_{xy})$
$(a_1 \otimes a_1)$	$(a_1 \otimes a_1)$	$(a_1 \otimes a_x)$	$(a_1 \otimes a_y)$	$(a_1 \otimes a_{xy})$
$(a_1 \otimes a_x)$	$(a_1 \otimes a_x)$	$(a_1 \otimes a_1)$	$(a_1 \otimes a_{xy})$	$(a_1 \otimes a_y)$
$(a_1 \otimes a_y)$	$(a_1 \otimes a_y)$	-$(a_1 \otimes a_{xy})$	$(a_1 \otimes a_1)$	-$(a_1 \otimes a_x)$
$(a_1 \otimes a_{xy})$	$(a_1 \otimes a_{xy})$	-$(a_1 \otimes a_y)$	$(a_1 \otimes a_x)$	-$(a_1 \otimes a_1)$
$(a_x \otimes a_1)$	$(a_x \otimes a_1)$	$(a_x \otimes a_x)$	$(a_x \otimes a_y)$	$(a_x \otimes a_{xy})$
$(a_x \otimes a_x)$	$(a_x \otimes a_x)$	$(a_x \otimes a_1)$	$(a_x \otimes a_{xy})$	$(a_x \otimes a_y)$
$(a_x \otimes a_y)$	$(a_x \otimes a_y)$	-$(a_x \otimes a_{xy})$	$(a_x \otimes a_1)$	-$(a_x \otimes a_x)$
$(a_x \otimes a_{xy})$	$(a_x \otimes a_{xy})$	-$(a_x \otimes a_y)$	$(a_x \otimes a_x)$	-$(a_x \otimes a_1)$
$(a_y \otimes a_1)$	$(a_y \otimes a_1)$	$(a_y \otimes a_x)$	$(a_y \otimes a_y)$	$(a_y \otimes a_{xy})$
$(a_y \otimes a_x)$	$(a_y \otimes a_x)$	$(a_y \otimes a_1)$	$(a_y \otimes a_{xy})$	$(a_y \otimes a_y)$
$(a_y \otimes a_y)$	$(a_y \otimes a_y)$	-$(a_y \otimes a_{xy})$	$(a_y \otimes a_1)$	-$(a_y \otimes a_x)$
$(a_y \otimes a_{xy})$	$(a_y \otimes a_{xy})$	-$(a_y \otimes a_y)$	$(a_y \otimes a_x)$	-$(a_y \otimes a_1)$
$(a_{xy} \otimes a_1)$	$(a_{xy} \otimes a_1)$	$(a_{xy} \otimes a_x)$	$(a_{xy} \otimes a_y)$	$(a_{xy} \otimes a_{xy})$
$(a_{xy} \otimes a_x)$	$(a_{xy} \otimes a_x)$	$(a_{xy} \otimes a_1)$	$(a_{xy} \otimes a_{xy})$	$(a_{xy} \otimes a_y)$
$(a_{xy} \otimes a_y)$	$(a_{xy} \otimes a_y)$	-$(a_{xy} \otimes a_{xy})$	$(a_{xy} \otimes a_1)$	-$(a_{xy} \otimes a_x)$
$(a_{xy} \otimes a_{xy})$	$(a_{xy} \otimes a_{xy})$	-$(a_{xy} \otimes a_y)$	$(a_{xy} \otimes a_x)$	-$(a_{xy} \otimes a_1)$

Table 9: Condensed Direct Product, 1 of 4

$\otimes$	$(a_x \otimes a_1)$	$(a_x \otimes a_x)$	$(a_x \otimes a_y)$	$(a_x \otimes a_{xy})$
$(a_1 \otimes a_1)$	$(a_x \otimes a_1)$	$(a_x \otimes a_x)$	$(a_x \otimes a_y)$	$(a_x \otimes a_{xy})$
$(a_1 \otimes a_x)$	$(a_x \otimes a_x)$	$(a_x \otimes a_1)$	$(a_x \otimes a_{xy})$	$(a_x \otimes a_y)$
$(a_1 \otimes a_y)$	$(a_x \otimes a_y)$	-$(a_x \otimes a_{xy})$	$(a_x \otimes a_1)$	-$(a_x \otimes a_x)$
$(a_1 \otimes a_{xy})$	$(a_x \otimes a_{xy})$	-$(a_x \otimes a_y)$	$(a_x \otimes a_x)$	-$(a_x \otimes a_1)$
$(a_x \otimes a_1)$	$(a_1 \otimes a_1)$	$(a_1 \otimes a_x)$	$(a_1 \otimes a_y)$	$(a_1 \otimes a_{xy})$
$(a_x \otimes a_x)$	$(a_1 \otimes a_x)$	$(a_1 \otimes a_1)$	$(a_1 \otimes a_{xy})$	$(a_1 \otimes a_y)$
$(a_x \otimes a_y)$	$(a_1 \otimes a_y)$	-$(a_1 \otimes a_{xy})$	$(a_1 \otimes a_1)$	-$(a_1 \otimes a_x)$
$(a_x \otimes a_{xy})$	$(a_1 \otimes a_{xy})$	-$(a_1 \otimes a_y)$	$(a_1 \otimes a_x)$	-$(a_1 \otimes a_1)$
$(a_y \otimes a_1)$	-$(a_{xy} \otimes a_1)$	-$(a_{xy} \otimes a_x)$	-$(a_{xy} \otimes a_y)$	-$(a_{xy} \otimes a_{xy})$
$(a_y \otimes a_x)$	-$(a_{xy} \otimes a_x)$	-$(a_{xy} \otimes a_1)$	-$(a_{xy} \otimes a_{xy})$	-$(a_{xy} \otimes a_y)$
$(a_y \otimes a_y)$	-$(a_{xy} \otimes a_y)$	$(a_{xy} \otimes a_{xy})$	-$(a_{xy} \otimes a_1)$	$(a_{xy} \otimes a_x)$
$(a_y \otimes a_{xy})$	-$(a_{xy} \otimes a_{xy})$	$(a_{xy} \otimes a_y)$	-$(a_{xy} \otimes a_x)$	$(a_{xy} \otimes a_1)$
$(a_{xy} \otimes a_1)$	-$(a_y \otimes a_1)$	-$(a_y \otimes a_x)$	-$(a_y \otimes a_y)$	-$(a_y \otimes a_{xy})$
$(a_{xy} \otimes a_x)$	-$(a_y \otimes a_x)$	-$(a_y \otimes a_1)$	-$(a_y \otimes a_{xy})$	-$(a_y \otimes a_y)$
$(a_{xy} \otimes a_y)$	-$(a_y \otimes a_y)$	$(a_y \otimes a_{xy})$	-$(a_y \otimes a_1)$	$(a_y \otimes a_x)$
$(a_{xy} \otimes a_{xy})$	-$(a_y \otimes a_{xy})$	$(a_y \otimes a_y)$	-$(a_y \otimes a_x)$	$(a_y \otimes a_1)$

Table 10: Condensed Direct Product, 2 of 4

$\otimes$	$(a_y \otimes a_1)$	$(a_y \otimes a_x)$	$(a_y \otimes a_y)$	$(a_y \otimes a_{xy})$
$(a_1 \otimes a_1)$	$(a_y \otimes a_1)$	$(a_y \otimes a_x)$	$(a_y \otimes a_y)$	$(a_y \otimes a_{xy})$
$(a_1 \otimes a_x)$	$(a_y \otimes a_x)$	$(a_y \otimes a_1)$	$(a_y \otimes a_{xy})$	$(a_y \otimes a_y)$
$(a_1 \otimes a_y)$	$(a_y \otimes a_y)$	-$(a_y \otimes a_{xy})$	$(a_y \otimes a_1)$	-$(a_y \otimes a_x)$
$(a_1 \otimes a_{xy})$	$(a_y \otimes a_{xy})$	-$(a_y \otimes a_y)$	$(a_y \otimes a_x)$	-$(a_y \otimes a_1)$
$(a_x \otimes a_1)$	$(a_{xy} \otimes a_1)$	$(a_{xy} \otimes a_x)$	$(a_{xy} \otimes a_y)$	$(a_{xy} \otimes a_{xy})$
$(a_x \otimes a_x)$	$(a_{xy} \otimes a_x)$	$(a_{xy} \otimes a_1)$	$(a_{xy} \otimes a_{xy})$	$(a_{xy} \otimes a_y)$
$(a_x \otimes a_y)$	$(a_{xy} \otimes a_y)$	-$(a_{xy} \otimes a_{xy})$	$(a_{xy} \otimes a_1)$	-$(a_{xy} \otimes a_x)$
$(a_x \otimes a_{xy})$	$(a_{xy} \otimes a_{xy})$	-$(a_{xy} \otimes a_y)$	$(a_{xy} \otimes a_x)$	-$(a_{xy} \otimes a_1)$
$(a_y \otimes a_1)$	$(a_1 \otimes a_1)$	$(a_1 \otimes a_x)$	$(a_1 \otimes a_y)$	$(a_1 \otimes a_{xy})$
$(a_y \otimes a_x)$	$(a_1 \otimes a_x)$	$(a_1 \otimes a_1)$	$(a_1 \otimes a_{xy})$	$(a_1 \otimes a_y)$
$(a_y \otimes a_y)$	$(a_1 \otimes a_y)$	-$(a_1 \otimes a_{xy})$	$(a_1 \otimes a_1)$	-$(a_1 \otimes a_x)$
$(a_y \otimes a_{xy})$	$(a_1 \otimes a_{xy})$	-$(a_1 \otimes a_y)$	$(a_1 \otimes a_x)$	-$(a_1 \otimes a_1)$
$(a_{xy} \otimes a_1)$	$(a_x \otimes a_1)$	$(a_x \otimes a_x)$	$(a_x \otimes a_y)$	$(a_x \otimes a_{xy})$
$(a_{xy} \otimes a_x)$	$(a_x \otimes a_x)$	$(a_x \otimes a_1)$	$(a_x \otimes a_{xy})$	$(a_x \otimes a_y)$
$(a_{xy} \otimes a_y)$	$(a_x \otimes a_y)$	-$(a_x \otimes a_{xy})$	$(a_x \otimes a_1)$	-$(a_x \otimes a_x)$
$(a_{xy} \otimes a_{xy})$	$(a_x \otimes a_{xy})$	-$(a_x \otimes a_y)$	$(a_x \otimes a_x)$	-$(a_x \otimes a_1)$

Table 11: Condensed Direct Product, 3 of 4

$\otimes$	$(a_{xy} \otimes a_1)$	$(a_{xy} \otimes a_x)$	$(a_{xy} \otimes a_y)$	$(a_{xy} \otimes a_{xy})$
$(a_1 \otimes a_1)$	$(a_{xy} \otimes a_1)$	$(a_{xy} \otimes a_x)$	$(a_{xy} \otimes a_y)$	$(a_{xy} \otimes a_{xy})$
$(a_1 \otimes a_x)$	$(a_{xy} \otimes a_x)$	$(a_{xy} \otimes a_1)$	$(a_{xy} \otimes a_{xy})$	$(a_{xy} \otimes a_y)$
$(a_1 \otimes a_y)$	$(a_{xy} \otimes a_y)$	-$(a_{xy} \otimes a_{xy})$	$(a_{xy} \otimes a_1)$	-$(a_{xy} \otimes a_x)$
$(a_1 \otimes a_{xy})$	$(a_{xy} \otimes a_{xy})$	-$(a_{xy} \otimes a_y)$	$(a_{xy} \otimes a_x)$	-$(a_{xy} \otimes a_1)$
$(a_x \otimes a_1)$	$(a_y \otimes a_1)$	$(a_y \otimes a_x)$	$(a_y \otimes a_y)$	$(a_y \otimes a_{xy})$
$(a_x \otimes a_x)$	$(a_y \otimes a_x)$	$(a_y \otimes a_1)$	$(a_y \otimes a_{xy})$	$(a_y \otimes a_y)$
$(a_x \otimes a_y)$	$(a_y \otimes a_y)$	-$(a_y \otimes a_{xy})$	$(a_y \otimes a_1)$	-$(a_y \otimes a_x)$
$(a_x \otimes a_{xy})$	$(a_y \otimes a_{xy})$	-$(a_y \otimes a_y)$	$(a_y \otimes a_x)$	-$(a_y \otimes a_1)$
$(a_y \otimes a_1)$	-$(a_x \otimes a_1)$	-$(a_x \otimes a_x)$	-$(a_x \otimes a_y)$	-$(a_x \otimes a_{xy})$
$(a_y \otimes a_x)$	-$(a_x \otimes a_x)$	-$(a_x \otimes a_1)$	-$(a_x \otimes a_{xy})$	-$(a_x \otimes a_y)$
$(a_y \otimes a_y)$	-$(a_x \otimes a_y)$	$(a_x \otimes a_{xy})$	-$(a_x \otimes a_1)$	$(a_x \otimes a_x)$
$(a_y \otimes a_{xy})$	-$(a_x \otimes a_{xy})$	$(a_x \otimes a_y)$	-$(a_x \otimes a_x)$	$(a_x \otimes a_1)$
$(a_{xy} \otimes a_1)$	-$(a_1 \otimes a_1)$	-$(a_1 \otimes a_x)$	-$(a_1 \otimes a_y)$	-$(a_1 \otimes a_{xy})$
$(a_{xy} \otimes a_x)$	-$(a_1 \otimes a_x)$	-$(a_1 \otimes a_1)$	-$(a_1 \otimes a_{xy})$	-$(a_1 \otimes a_y)$
$(a_{xy} \otimes a_y)$	-$(a_1 \otimes a_y)$	$(a_1 \otimes a_{xy})$	-$(a_1 \otimes a_1)$	$(a_1 \otimes a_x)$
$(a_{xy} \otimes a_{xy})$	-$(a_1 \otimes a_{xy})$	$(a_1 \otimes a_y)$	-$(a_1 \otimes a_x)$	$(a_1 \otimes a_1)$

Table 12: Condensed Direct Product, 4 of 4

Substituting symbols, re-ordering the rows and columns, and cleaning up the signs recovers the Minkowski Table 1. We have now verified $C\ell_2 \otimes C\ell_2 = C_{3,1}$.

It is important to note that the formal procedure above never defined the $\otimes$ operation involved in expressions like $(a_x \otimes a_y)$. We know it is non-commutative, and we saw a previous implementation using square matrices, but for the implementation just above, we have made a group table for Minkowski spacetime, and made symbol assignments independent of the implementation details.

Verify $\mathbb{C} \otimes \mathbb{H} = C\ell_3$

Our next task to verify $\mathbb{C} \otimes \mathbb{H} = C\ell_3$ (and $sl(2, \mathbb{C})$).

Cohl Furey and others jump directly to complexified quaternions. Starting from complexified quaternions, it is easy to to show $\mathbb{C} \otimes \mathbb{H} = C\ell_3$.

For her complex quaternions, Cohl Furey uses the representation

$$c_0 + c_1\, i\epsilon_x + c_2\, i\epsilon_y + c_3\, i\epsilon_z$$

The c_i are generic complex numbers, $i = \sqrt{-1}$, and ${\epsilon_x}^2 = {\epsilon_y}^2 = {\epsilon_z}^2 = \epsilon_x\epsilon_y\epsilon_z = -1$

I will typically use notation

$$(a + IA) + (b + IB)i + (c + IC)j + (d + ID)k$$

My convention has I $= \sqrt{-1}$, the lowercase letters are the real portion, the uppercase letters are the complex portion, and the quaternion basis vectors are the classic i, j, and k, with $i^2 = j^2 = k^2 = ijk = -1$.

Correlating our notations,

```
a =  real_part(c_0);     A = imag_part(c_0);
b = -imag_part(c_1);     B = real_part(c_1);
c = -imag_part(c_2);     C = real_part(c_2);
d = -imag_part(c_3);     D = real_part(c_3);
```

Using my notation, we map quaternions and complex numbers to $C\ell_3$ using

$$\begin{aligned} I &= e_{xyz} \\ i &= e_{xy} \\ j &= e_{yz} \\ k &= e_{xz} \end{aligned}$$

My notational form becomes

$$\begin{aligned}
& (a+IA)+(b+IB)i+(c+IC)j+(d+ID)k \\
= \;& (a+Ae_{xyz})+(b+Be_{xyz})e_{xy}+(c+Ce_{xyz})e_{yz}+(d+De_{xyz})e_{xz} \\
= \;& a+Ae_{xyz}+be_{xy}-Be_z+ce_{yz}-Ce_x+de_{xz}+De_y \\
= \;& a-Ce_x+De_y-Be_z+be_{xy}+de_{xz}+ce_{yz}+Ae_{xyz}
\end{aligned}$$

This trivially completes the exercise. The next task is to repeat the symbolic exercise akin to the previous section.

Symbolically Verify $\mathbb{C} \otimes \mathbb{H} = C\ell_3$

Represent our complex number by $aa_1 + ba_I$, and our quaternion by $ce_1 + de_i + ee_j + fe_k$. Our pairwise products are

$a_1 \otimes e_1 \quad a_1 \otimes e_i \quad a_1 \otimes e_j \quad a_1 \otimes e_k$

$a_I \otimes e_1 \quad a_I \otimes e_i \quad a_I \otimes e_j \quad a_I \otimes e_k$

We build our sidewise Tables 13 and 14 showing the direct product.

We now substitute our symbols for $C\ell_3$, and generate Table 16, "Translated $\mathbb{C} \otimes \mathbb{H}$ Direct Product".

$$\begin{aligned}
e_q &= (a_1 \otimes e_1) \\
e_x &= -(a_I \otimes e_j) \\
e_y &= (a_I \otimes e_k) \\
e_z &= -(a_I \otimes e_i) \\
e_{xy} &= (a_1 \otimes e_i) \\
e_{yz} &= (a_1 \otimes e_j) \\
e_{xz} &= (a_1 \otimes e_k) \\
e_{xyz} &= (a_I \otimes e_1)
\end{aligned}$$

Absorb minus signs in headers, reorder rows and columns, and the proof is complete. $C\ell_3 = \mathbb{C} \otimes \mathbb{H}$.

Evaluation of $\mathbb{H} \otimes \mathbb{C} = C\ell_3$

As my next exercise to gain experience, I did the symbolic tables for quaternified complex numbers, $\mathbb{H} \otimes \mathbb{C}$. This group also proved to be congruent with

$\otimes$	$(a_1 \otimes e_1)$	$(a_1 \otimes e_i)$	$(a_1 \otimes e_j)$	$(a_1 \otimes e_k)$
$(a_1 \otimes e_1)$	$(a_1 \otimes e_1)(a_1 \otimes e_1)$	$(a_1 \otimes e_1)(a_1 \otimes e_i)$	$(a_1 \otimes e_1)(a_1 \otimes e_j)$	$(a_1 \otimes e_1)(a_1 \otimes e_k)$
$(a_1 \otimes e_i)$	$(a_1 \otimes e_i)(a_1 \otimes e_1)$	$(a_1 \otimes e_i)(a_1 \otimes e_i)$	$(a_1 \otimes e_i)(a_1 \otimes e_j)$	$(a_1 \otimes e_i)(a_1 \otimes e_k)$
$(a_1 \otimes e_j)$	$(a_1 \otimes e_j)(a_1 \otimes e_1)$	$(a_1 \otimes e_j)(a_1 \otimes e_i)$	$(a_1 \otimes e_j)(a_1 \otimes e_j)$	$(a_1 \otimes e_j)(a_1 \otimes e_k)$
$(a_1 \otimes e_k)$	$(a_1 \otimes e_k)(a_1 \otimes e_1)$	$(a_1 \otimes e_k)(a_1 \otimes e_i)$	$(a_1 \otimes e_k)(a_1 \otimes e_j)$	$(a_1 \otimes e_k)(a_1 \otimes e_k)$
$(a_I \otimes e_1)$	$(a_I \otimes e_1)(a_1 \otimes e_1)$	$(a_I \otimes e_1)(a_1 \otimes e_i)$	$(a_I \otimes e_1)(a_1 \otimes e_j)$	$(a_I \otimes e_1)(a_1 \otimes e_k)$
$(a_I \otimes e_i)$	$(a_I \otimes e_i)(a_1 \otimes e_1)$	$(a_I \otimes e_i)(a_1 \otimes e_i)$	$(a_I \otimes e_i)(a_1 \otimes e_j)$	$(a_I \otimes e_i)(a_1 \otimes e_k)$
$(a_I \otimes e_j)$	$(a_I \otimes e_j)(a_1 \otimes e_1)$	$(a_I \otimes e_j)(a_1 \otimes e_i)$	$(a_I \otimes e_j)(a_1 \otimes e_j)$	$(a_I \otimes e_j)(a_1 \otimes e_k)$
$(a_I \otimes e_k)$	$(a_I \otimes e_k)(a_1 \otimes e_1)$	$(a_I \otimes e_k)(a_1 \otimes e_i)$	$(a_I \otimes e_k)(a_1 \otimes e_j)$	$(a_I \otimes e_k)(a_1 \otimes e_k)$

Table 13: $\mathbb{C} \otimes \mathbb{H}$ Direct Product, Part 1 of 2

$\otimes$	$(a_I \otimes e_1)$	$(a_I \otimes e_i)$	$(a_I \otimes e_j)$	$(a_I \otimes e_k)$
$(a_1 \otimes e_1)$	$(a_1 \otimes e_1)(a_I \otimes e_1)$	$(a_1 \otimes e_1)(a_I \otimes e_i)$	$(a_1 \otimes e_1)(a_I \otimes e_j)$	$(a_1 \otimes e_1)(a_I \otimes e_k)$
$(a_1 \otimes e_i)$	$(a_1 \otimes e_i)(a_I \otimes e_1)$	$(a_1 \otimes e_i)(a_I \otimes e_i)$	$(a_1 \otimes e_i)(a_I \otimes e_j)$	$(a_1 \otimes e_i)(a_I \otimes e_k)$
$(a_1 \otimes e_j)$	$(a_1 \otimes e_j)(a_I \otimes e_1)$	$(a_1 \otimes e_j)(a_I \otimes e_i)$	$(a_1 \otimes e_j)(a_I \otimes e_j)$	$(a_1 \otimes e_j)(a_I \otimes e_k)$
$(a_1 \otimes e_k)$	$(a_1 \otimes e_k)(a_I \otimes e_1)$	$(a_1 \otimes e_k)(a_I \otimes e_i)$	$(a_1 \otimes e_k)(a_I \otimes e_j)$	$(a_1 \otimes e_k)(a_I \otimes e_k)$
$(a_I \otimes e_1)$	$(a_I \otimes e_1)(a_I \otimes e_1)$	$(a_I \otimes e_1)(a_I \otimes e_i)$	$(a_I \otimes e_1)(a_I \otimes e_j)$	$(a_I \otimes e_1)(a_I \otimes e_k)$
$(a_I \otimes e_i)$	$(a_I \otimes e_i)(a_I \otimes e_1)$	$(a_I \otimes e_i)(a_I \otimes e_i)$	$(a_I \otimes e_i)(a_I \otimes e_j)$	$(a_I \otimes e_i)(a_I \otimes e_k)$
$(a_I \otimes e_j)$	$(a_I \otimes e_j)(a_I \otimes e_1)$	$(a_I \otimes e_j)(a_I \otimes e_i)$	$(a_I \otimes e_j)(a_I \otimes e_j)$	$(a_I \otimes e_j)(a_I \otimes e_k)$
$(a_I \otimes e_k)$	$(a_I \otimes e_k)(a_I \otimes e_1)$	$(a_I \otimes e_k)(a_I \otimes e_i)$	$(a_I \otimes e_k)(a_I \otimes e_j)$	$(a_I \otimes e_k)(a_I \otimes e_k)$

Table 14: $\mathbb{C} \otimes \mathbb{H}$ Direct Product, Part 2 of 2

$\otimes$	$(a_1 \otimes e_1)$	$(a_1 \otimes e_i)$	$(a_1 \otimes e_j)$	$(a_1 \otimes e_k)$	$(a_I \otimes e_1)$	$(a_I \otimes e_i)$	$(a_I \otimes e_j)$	$(a_I \otimes e_k)$
$(a_1 \otimes e_1)$	$(a_1 \otimes e_1)$	$(a_1 \otimes e_i)$	$(a_1 \otimes e_j)$	$(a_1 \otimes e_k)$	$(a_I \otimes e_1)$	$(a_I \otimes e_i)$	$(a_I \otimes e_j)$	$(a_I \otimes e_k)$
$(a_1 \otimes e_i)$	$(a_1 \otimes e_i)$	$-(a_1 \otimes e_1)$	$(a_1 \otimes e_k)$	$-(a_1 \otimes e_j)$	$(a_I \otimes e_i)$	$-(a_I \otimes e_1)$	$(a_I \otimes e_k)$	$-(a_I \otimes e_j)$
$(a_1 \otimes e_j)$	$(a_1 \otimes e_j)$	$-(a_1 \otimes e_k)$	$-(a_1 \otimes e_1)$	$(a_1 \otimes e_i)$	$(a_I \otimes e_j)$	$-(a_I \otimes e_k)$	$-(a_I \otimes e_1)$	$(a_I \otimes e_i)$
$(a_1 \otimes e_k)$	$(a_1 \otimes e_k)$	$(a_1 \otimes e_j)$	$-(a_1 \otimes e_i)$	$(-a_1 \otimes e_1)$	$(a_I \otimes e_k)$	$(a_I \otimes e_j)$	$-(a_I \otimes e_i)$	$-(a_I \otimes e_1)$
$(a_I \otimes e_1)$	$(a_I \otimes e_1)$	$(a_I \otimes e_i)$	$(a_I \otimes e_j)$	$(a_I \otimes e_k)$	$-(a_1 \otimes e_1)$	$-(a_1 \otimes e_i)$	$-(a_1 \otimes e_j)$	$-(a_1 \otimes e_k)$
$(a_I \otimes e_i)$	$(a_I \otimes e_i)$	$-(a_I \otimes e_1)$	$(a_I \otimes e_k)$	$-(a_I \otimes e_j)$	$-(a_1 \otimes e_i)$	$(a_1 \otimes e_1)$	$-(a_1 \otimes e_k)$	$(a_1 \otimes e_j)$
$(a_I \otimes e_j)$	$(a_I \otimes e_j)$	$-(a_I \otimes e_k)$	$-(a_I \otimes e_1)$	$(a_I \otimes e_i)$	$-(a_1 \otimes e_j)$	$(a_1 \otimes e_k)$	$(a_1 \otimes e_1)$	$-(a_1 \otimes e_i)$
$(a_I \otimes e_k)$	$(a_I \otimes e_k)$	$(a_I \otimes e_j)$	$-(a_I \otimes e_i)$	$-(a_I \otimes e_1)$	$-(a_1 \otimes e_k)$	$-(a_1 \otimes e_j)$	$(a_1 \otimes e_i)$	$(a_1 \otimes e_1)$

Table 15: Condensed $\mathbb{C} \otimes \mathbb{H}$ Direct Product

$\otimes$	e_q	e_{xy}	e_{yz}	e_{xz}	e_{xyz}	-e_z	-e_x	e_y
e_q	e_q	e_{xy}	e_{yz}	e_{xz}	e_{xyz}	-e_z	-e_x	e_y
e_{xy}	e_{xy}	-e_q	e_{xz}	-e_{yz}	-e_z	-e_{xyz}	e_y	e_x
e_{yz}	e_{yz}	-e_{xz}	-e_q	e_{xy}	-e_x	-e_y	-e_{xyz}	-e_z
e_{xz}	e_{xz}	e_{yz}	-e_{xy}	-e_q	e_y	-e_x	e_z	-e_{xyz}
e_{xyz}	e_{xyz}	-e_z	-e_x	e_y	-e_q	-e_{xy}	-e_{yz}	-e_{xz}
-e_z	-e_z	-e_{xyz}	e_y	e_x	-e_{xy}	e_q	-e_{xz}	e_{yz}
-e_x	-e_x	-e_y	-e_{xyz}	-e_z	-e_{yz}	e_{xz}	e_q	-e_{xy}
e_y	e_y	-e_x	e_z	-e_{xyz}	-e_{xz}	-e_{yz}	e_{xy}	e_q

Table 16: Translated $\mathbb{C} \otimes \mathbb{H}$ Direct Product

$\otimes$	e_q	e_x	e_y	e_z	e_{xy}	e_{xz}	e_{yz}	e_{xyz}
e_q	e_q	e_x	e_y	e_z	e_{xy}	e_{xz}	e_{yz}	e_{xyz}
e_x	e_x	e_q	e_{xy}	e_{xz}	e_y	e_z	e_{xyz}	e_{yz}
e_y	e_y	-e_{xy}	e_q	e_{yz}	-e_x	-e_{xyz}	e_z	-e_{xz}
e_z	e_z	-e_{xz}	-e_{yz}	e_q	e_{xyz}	-e_x	-e_y	e_{xy}
e_{xy}	e_{xy}	-e_y	e_x	e_{xyz}	-e_q	-e_{yz}	e_{xz}	-e_z
e_{xz}	e_{xz}	-e_z	-e_{xyz}	e_x	e_{yz}	-e_q	-e_{xy}	e_y
e_{yz}	e_{yz}	e_{xyz}	-e_z	e_y	-e_{xz}	e_{xy}	-e_q	-e_x
e_{xyz}	e_{xyz}	e_{yz}	-e_{xz}	e_{xy}	-e_z	e_y	-e_x	-e_q

Table 17: Re-organized Condensed $\mathbb{C} \otimes \mathbb{H}$ Direct Product

biquaternions and three dimensional Clifford algebras, with six mappings. In retrospect, this is not surprising, as the ordered pairs of the direct product merely require consistency in a definition set, but the actual implementation is your choice. I look forward to the same exercise with non-associative octonions, to see if the same behavior is present.

Commentary on Notation

In my work above, I have a preference for, and have focused on Clifford Algebras over real numbers, specifically $C\ell_3 = Cl(3) = Cl(3,0) = Cl(3,0,0)$. By contrast, Cohl Furey has a preference for Clifford Algebras over complex numbers, such as $\mathbb{C}l(2)$, which also maps to $\mathbb{C} \otimes \mathbb{H}$. In speech, both are pronounced the same, as in 'seal 2' or 'seal 3'. Be aware of the context in speech, and notice the font in written publications.

Also be aware that there are multiple mapping of basis elements from $\mathbb{C} \otimes \mathbb{H}$ to $C\ell_3$. From my previous development of $\mathbb{C} \otimes \mathbb{H} = C\ell_3$, I prefer to use

$$\begin{aligned}
e_q &= 1 \\
e_x &= -Ij \\
e_y &= +Ik \\
e_z &= -Ii \\
e_{xy} &= i \\
e_{yz} &= j \\
e_{xz} &= k \\
e_{xyz} &= I
\end{aligned}$$

By contrast, Furey (and I recall Dixon), use the very reasonable

$$\begin{aligned}
e_q &= 1 \\
e_x &= Ii \\
e_y &= Ij \\
e_z &= Ik \\
e_{xy} &= -ij = -k \\
e_{zx} &= -ki = -j \\
e_{yz} &= -jk = -i \\
e_{xyz} &= IiIjIk = (-I)(ijk) = +I
\end{aligned}$$

Finally, when I want to emphasis the similarities between quaternions, octonions, and Clifford algebras, I will usually not use k or K, but instead leave $k = ij$ as ij.

Evaluation of $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O} = C\ell_3 \otimes \mathbb{O}$

Having built my confidence with the evaluation of $\mathbb{C} \otimes \mathbb{H}$, I jumped directly to the evaluation of $C\ell_3 \otimes \mathbb{O}$, correctly reasoning that $\mathbb{C} \otimes \mathbb{O}$ and $\mathbb{H} \otimes \mathbb{O}$ will be simple, easily implemented, subsets of the above.

Before getting started, I want to document the octonion labelling scheme I like to use. Previously, I have enumerated 2^{19} sign schemes for octonion variations which satisfy sum of squares product formulas. It is my hope to connect the nineteen free variable bits to the nineteen parameters of the standard model at a later time.

Cayley versus Baez Octonion Labelling Schemes

John Baez, in his classic paper [3], uses a base identification scheme based on the Fano plane, with the feature that indices double, modulo 7, upon 120 rotation of the triangular diagram. I, on the other hand, prefer Cayley's table, with a bit-wise mapping between basis elements and bits of a binary number. This allows the product of the basis to be split into a bitwise XOR operation to identify the basis of the product, and a separate operation to determine the sign of the product. This approach also clearly shows the inheritance of complex numbers in quaternions, and quaternions inside octonions. Bit assignments are 'Eji' for three bit binary.

```
 * | q  i  j  k | E  I  J  K                Sign Table         q = 000
-----------------------------                                  i = 001
 q | q  i  j  k | E  I  J  K             + + + +  + + + +      j = 010
 i | i -q  k -j | I -E -K  J             + - + -  + - - +      k = 011
 j | j -k -q  i | J  K -E -I             + - - +  + + - -      E = 100
 k | k  j -i -q | K -J  I -E             + + - -  + - + -      I = 101
-----------------------------                                  J = 110
 E | E -I -J -K |-q  i  j  k             + - - -  - + + +      K = 111
 I | I  E -K  J |-i -q -k  j             + + - +  - - - +
 J | J  K  E -I |-j  k -q -i             + + + -  - + - -
 K | K -J  I  E |-k -j  i -q             + - + +  - - + -
```

$C\ell_3$ Labelling Scheme

In a similar fashion, the Clifford algebra for three dimensional Euclidean space can be mapped to a bitmap and associated sign matrix.

```
  * |  q    x    y   xy |   z   xz   yz  xyz       Sign Table      q = 000
---------------------------------------------                      x = 001
  q |  q    x    y   xy |   z   xz   yz  xyz    + + + +  + + + +   y = 010
  x |  x    q   xy    y |  xz    z  xyz   yz    + + + +  + + + +  xy = 011
  y |  y  -xy    q   -x |  yz -xyz    z  -xz    + - + -  + - + -   z = 100
 xy | xy   -y    x   -q | xyz  -yz   xz   -z    + - + -  + - + -  xz = 101
---------------------------------------------                     yz = 110
  z |  z  -xz  -yz  xyz |   q   -x   -y   xy    + - - +  + - - +  xyz = 111
 xz | xz   -z -xyz   yz |   x   -q  -xy    y    + - - +  + - - +
 yz | yz  xyz   -z  -xz |   y   xy   -q   -x    + + - -  + + - -
xyz |xyz   yz  -xz   -z |  xy    y   -x   -q    + + - -  + + - -
```

Since both the $C\ell_3$ and octonion basis use three bit XOR logic for base calculation, the direct product of the two uses a concatenated six bit field with XOR logic as well. Similarly, the sign of the direct product of the bases is the product of the appropriate signs of the $C\ell_3$ and octonion factors.

The general CHO has 64 components, which I generally print as an 8x8 grid, ordered as below.

```
q      i      j      ij     E      iE     jE     ijE
x      xi     xj     xij    xE     xiE    xjE    xijE
y      yi     yj     yij    yE     yiE    yjE    yijE
xy     xyi    xyj    xyij   xyE    xyiE   xyjE   xyijE
z      zi     zj     zij    zE     ziE    zjE    zijE
xz     xzi    xzj    xzij   xzE    xziE   xzjE   xzijE
yz     yzi    yzj    yzij   yzE    yziE   yzjE   yzijE
xyz    xyzi   xyzj   xyzij  xyzE   xyziE  xyzjE  xyzijE
```

Formatted CHO Sign Table

```
++++++++ ++++++++ ++++++++ ++++++++ ++++++++ ++++++++ ++++++++ ++++++++
+-+-+--+ +-+-+--+ +-+-+--+ +-+-+--+ +-+-+--+ +-+-+--+ +-+-+--+ +-+-+--+
+--+++-- +--+++-- +--+++-- +--+++-- +--+++-- +--+++-- +--+++-- +--+++--
++--+-+- ++--+-+- ++--+-+- ++--+-+- ++--+-+- ++--+-+- ++--+-+- ++--+-+-
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```

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```

With this little bit of formatting, the direct product nature of the 64x64 sign matrix is apparent.

Properties

Only 1 and e_{xyz} commute with all elements. The equation set for general CHO multiplication is at `http://www.kurtnalty.com/CHO_Routines.c`

Evaluation of $\mathbb{C} \otimes \mathbb{O}$

The general CO has 16 components, which I generally print as an 2x8 grid.

```
q      i      j      ij     E      iE     jE     ijE
xyz    xyzi   xyzj   xyzij  xyzE   xyziE  xyzjE  xyzijE
```

The sign matrix for $\mathbb{C} \otimes \mathbb{O}$ is

```
+ + + + + + + +  + + + + + + + +
+ - + - + - - +  + - + - + - - +
+ - - + + + - -  + - - + + + - -
+ + - - + - + -  + + - - + - + -
+ - - - - + + +  + - - - - + + +
+ + - + - - - +  + + - + - - - +
+ + + - - + - -  + + + - - + - -
+ - + + - - + -  + - + + - - + -
```

```
+ + + + + + + +  - - - - - - - -
+ - + - + - - +  - + - + - + + -
+ - - + + + - -  - + + - - - + +
+ + - - + - + -  - - + + - + - +
+ - - - - + + +  - + + + + - - -
+ + - + - - - +  - - + - + + + -
+ + + - - + - -  - - - + + - + +
+ - + + - - + -  - + - - + + - +
```

Properties

Only 1 and e_{xyz} commute with all elements. The equation set for general CHO multiplication is at `http://www.kurtnalty.com/CHO_Routines.c`

Evaluation of $\mathbb{H} \otimes \mathbb{O}$

The general HO has 32 components, which I generally print as an 4x8 grid.

```
q      i      j      ij     E      iE     jE     ijE
xy     xyi    xyj    xyij   xyE    xyiE   xyjE   xyijE
xz     xzi    xzj    xzij   xzE    xziE   xzjE   xzijE
yz     yzi    yzj    yzij   yzE    yziE   yzjE   yzijE
```

The sign array for HO is

```
+ + + + + + + +  + + + + + + + +  + + + + + + + +  + + + + + + + +
+ - + - + - - +  + - + - + - - +  + - + - + - - +  + - + - + - - +
+ - - + + + - -  + - - + + + - -  + - - + + + - -  + - - + + + - -
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```

```
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+ + - - + - + -  + + - - + - + -  - - + + - + - +  - - + + - + - +
+ - - - - + + +  + - - - - + + +  - + + + + - - -  - + + + + - - -
+ + - + - - - +  + + - + - - - +  - - + - + + + -  - - + - + + + -
+ + + - - + - -  + + + - - + - -  - - - + + - + +  - - - + + - + +
+ - + + - - + -  + - + + - - + -  - + - - + + - +  - + - - + + - +
```

Properties

Only 1 and e_{xyz} commute with all elements. Given the basis above, the equation set for general HO multiplication is easy enough to write, and found at `http://www.kurtnalty.com/CHO_Routines.c`

$\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ Conjugations

Algebra is filled with many types of conjugation, most given silly names which obscure their geometric significance. In three dimensional geometric algebra, we have involutions, which invert the sign of the x, y and z components, preserve the xy, yz and zx components (due to the square of -1), and invert the xyz component. We also have the reverse, which reverses the order of basis multiplication, with the effect of preserving x, y, and z while complementing xy, yz, xz and xyz. Combining the two gives the Clifford conjugation (see the obscuring name) which inverts x, y, z, xy, yz and xyz components. Taking the product of a CH or GA3E multivector with the Clifford conjugate yields a multivector with six zero elements in vector and bivector slots, with the scalar and trivector components playing the role of a complex number. Similarly, for octonions, the octonion complement changes the sign of all elements except the scalar. The product of an octonion with its complement yields a pure scalar, with the remaining seven components zero.

The general CHO has 64 components, which I generally print as an 8x8 grid, ordered as below.

```
q      i      j      ij     E      iE     jE     ijE
x      xi     xj     xij    xE     xiE    xjE    xijE
y      yi     yj     yij    yE     yiE    yjE    yijE
xy     xyi    xyj    xyij   xyE    xyiE   xyjE   xyijE
z      zi     zj     zij    zE     ziE    zjE    zijE
xz     xzi    xzj    xzij   xzE    xziE   xzjE   xzijE
yz     yzi    yzj    yzij   yzE    yziE   yzjE   yzijE
xyz    xyzi   xyzj   xyzij  xyzE   xyziE  xyzjE  xyzijE
```

Many times, I only care about the zero/non-zero status of a product. I have written a software routine "CHO_Zero_Grid(CHO a)" which examines the CHO and prints either an x for non-zero, or 0 for zero components. I can specify a particular conjugation pattern for a CHO as another 8x8 grid, this time indicating + or - factors for the conjugation.

Using this notation, the combined Clifford/Octonion conjugation for CHO is

```
+ - - - - - - -                x 0 0 0 0 0 0 0
- + + + + + + +      with      0 x x x x x x x
- + + + + + + +   conjugate    0 x x x x x x x
- + + + + + + +    product     0 x x x x x x x
- + + + + + + +                0 x x x x x x x
- + + + + + + +                0 x x x x x x x
- + + + + + + +                0 x x x x x x x
+ - - - - - - -                x 0 0 0 0 0 0 0
```

The Product of Pure $\mathbb{C}\otimes\mathbb{H}$ times Pure $\mathbb{O}$

For the special case of a CHO which is the product of a pure CH times a pure O, we get even better, with 62 zeroes and only scalar and xyz component non-zero terms.

```
CHO4 =  CHO2*CHO3 =
(c00*b00 , b01*c00 , b02*c00 , c00*b03 , b04*c00 , b05*c00 , c00*b06 , c00*b07 ,
 c10*b00 , c10*b01 , c10*b02 , c10*b03 , b04*c10 , c10*b05 , c10*b06 , c10*b07 ,
 c20*b00 , b01*c20 , b02*c20 , c20*b03 , b04*c20 , b05*c20 , c20*b06 , c20*b07 ,
 c30*b00 , c30*b01 , c30*b02 , c30*b03 , c30*b04 , c30*b05 , c30*b06 , c30*b07 ,
 c40*b00 , b01*c40 , b02*c40 , c40*b03 , b04*c40 , b05*c40 , c40*b06 , c40*b07 ,
 b00*c50 , b01*c50 , b02*c50 , b03*c50 , b04*c50 , b05*c50 , b06*c50 , c50*b07 ,
 c60*b00 , b01*c60 , c60*b02 , c60*b03 , b04*c60 , b05*c60 , c60*b06 , c60*b07 ,
 b00*c70 , b01*c70 , b02*c70 , b03*c70 , b04*c70 , b05*c70 , b06*c70 , c70*b07)
```

```
CHO4*CHO_Signed_Conjugation(CHO4); =
x 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0
x 0 0 0 0 0 0 0
```

More details at `http://www.kurtnalty.com/CHO_Explore_Conjugates.cp`

Translating Quantum Mechanic's Notation

I am doing a small side track here as a translation guide from quantum mechanics to geometric algebra for various common operators and bases. I am tracking the presentation of C. Furey's thesis, chapters three and four.

$C\ell_3 = \mathbb{C} \otimes \mathbb{H}$

My preferred mapping between $C\ell_3$ and $\mathbb{C} \otimes \mathbb{H}$ is

$$\begin{aligned}
e_q &= 1 \\
e_x &= -Ij \\
e_y &= +Ik = +Iij \\
e_z &= -Ii \\
e_{xy} &= i \\
e_{yz} &= j \\
e_{xz} &= k = ij \\
e_{xyz} &= I
\end{aligned}$$

I emphasize the bivector nature of the quaternion i, j, and k.

GA3E Reverse

The reverse operator reverses the order of multiplication of the basis elements, effectively calculating the reciprocal of that basis element. For GA3E, the reverse changes the sign of the bivector and trivector elements. The reverse corresponds to both the transpose of the matrix implementation, as well as the Hermitian $()^\dagger$ conjugation.

$$\begin{aligned}
\texttt{Reverse}(ae_q + be_x + ce_y + de_z + ee_{xy} + fe_{yz} + ge_{xz} + he_{xyz}) &= \\
(ae_q + be_x + ce_y + de_z - ee_{xy} - fe_{yz} - ge_{xz} - he_{xyz}) & \\
(ae_q + be_x + ce_y + de_z + ee_{xy} + fe_{yz} + ge_{xz} + he_{xyz})^\dagger &= \\
(ae_q + be_x + ce_y + de_z - ee_{xy} - fe_{yz} - ge_{xz} - he_{xyz}) &
\end{aligned}$$

Parity Conjugate

The parity conjugation replace each space coordinate by its negative, with the same process applied to higher grade elements, resulting in a sign change of space and trivector terms.

$$\begin{aligned}
\texttt{Parity}(ae_q + be_x + ce_y + de_z + ee_{xy} + fe_{yz} + ge_{xz} + he_{xyz}) &= \\
(ae_q - be_x - ce_y - de_z + ee_{xy} + fe_{yz} + ge_{xz} - he_{xyz}) &
\end{aligned}$$

Clifford Conjugate and Multivector Magnitude

The Clifford conjugation is a composite of the `Reverse` and `Parity` operations. This changes the sign of the vector and bivector components. `Clifford() = Reverse(Parity()) = Parity(Reverse())`.

$$\begin{aligned}
\texttt{Clifford}(ae_q + be_x + ce_y + de_z + ee_{xy} + fe_{yz} + ge_{xz} + he_{xyz}) &= \\
(ae_q - be_x - ce_y - de_z - ee_{xy} - fe_{yz} - ge_{xz} + he_{xyz}) &
\end{aligned}$$

The product of a multivector times its Clifford conjugate yields a complex number which equals the determinant of the 2x2 complex matrix implementation of GA3E. The sum of squares of this complex determinant yields a real number which is the determinant of the 4x4 real matrix implementation of GA3E. This sum is the fourth power of the magnitude for the GA3E multivector.

Complex Numbers

The trivector basis e_{xyz} commutes with all GA3E elements, and squares to -1, and thus faithfully maps $I = \sqrt{-1} = e_{xyz}$. The generic complex number $a + Ib \rightarrow a + be_{xyz}$.

Complex conjugation corresponds to the reverse operator. $(a + be_{xyz})^* = (a + be_{zyx}) = (a - be_{xyz})$.

Quaternions

Quaternions map to the even grade subspace of GA3E. The confusion of bivector and vector elements has been a recurring problem in 3D, as we have three vectors, and three bivectors. For rotations in particular, we can confuse the wheel with the axle in three space, and still make things work (in an awkward sort of way).

The generic quaternion $a+bi+cj+dk$ maps to GA3E as $a+be_{xy}+ce_{yz}+de_{xz}$. The quaternion conjugation is seen to be the GA3E reverse operation, just like complex conjugation.

$$\begin{aligned}
(a + bi + cj + dk)\tilde{} &= (a - bi - cj - dk) \\
\texttt{Reverse}(a + be_{xy} + ce_{yz} + de_{xz}) &= (a - be_{xy} - ce_{yz} - de_{xz})
\end{aligned}$$

Projection Operators

Projection operators satisfy $P^2 = P$. Furey uses a specific example projection operator $P = (1+I\epsilon_z)/2$ in her thesis. The corresponding projection operator in geometric algebra is $P = (1 + e_z)/2$.

The most general GA3E projection operator that I know is the idempotent

$$\begin{aligned}
P &= \frac{1}{2} \pm \frac{1}{2}\left(\vec{u}\cosh(t) + U\sinh(t)\right) \\
&= a + (be_x + ce_y + de_z) + (ee_{xy} + fe_{yz} + ge_{xz})
\end{aligned}$$

where

$$\begin{aligned}
a &= 1/2 \\
0 &= U\vec{u} + \vec{u}U \\
\vec{u} &= \frac{be_x + ce_y + de_z}{\sqrt{b^2 + c^2 + d^2}} \\
\cosh(t) &= 2\sqrt{b^2 + c^2 + d^2} \\
\sinh(t) &= 2\sqrt{e^2 + f^2 + g^2} \\
W &= \vec{u}\vec{w} - \vec{w}\vec{u} \qquad \text{arbitrary vector } \vec{w} \\
U &= \frac{W}{|W|}
\end{aligned}$$

Spin Basis

Cohl Furey defines

$$\begin{aligned}\epsilon_{\uparrow\uparrow} &= \frac{1}{2}\left(1+i\epsilon_z\right)\\ \epsilon_{\downarrow\uparrow} &= \frac{1}{2}\left(\epsilon_y+i\epsilon_x\right)\\ \epsilon_{\uparrow\downarrow} &= \frac{1}{2}\left(-\epsilon_y+i\epsilon_x\right)\\ \epsilon_{\downarrow\downarrow} &= \frac{1}{2}\left(1-i\epsilon_z\right)\end{aligned}$$

(I am not happy with the nilpotent middle two lines below. Recheck! Expect four idempotents, or four nilpotents. Do not expect mix.) Translating to geometric algebra, this becomes

$$\begin{aligned}e_{\uparrow\uparrow} &= \frac{1}{2}\left(1+e_z\right)\\ e_{\downarrow\uparrow} &= \frac{1}{2}\left(e_{xz}+e_x\right)\\ e_{\uparrow\downarrow} &= \frac{1}{2}\left(-e_{xz}+e_x\right)\\ e_{\downarrow\downarrow} &= \frac{1}{2}\left(1-e_z\right)\end{aligned}$$

Furey gives the old basis in terms of the new one,

$$\begin{aligned}1 &= e_{\uparrow\uparrow}+e_{\downarrow\downarrow}\\ e_x &= \left(e_{\downarrow\uparrow}+e_{\uparrow\downarrow}\right)\\ e_y &= -e_{xyz}\left(e_{\downarrow\uparrow}-e_{\uparrow\downarrow}\right)\\ e_z &= \left(e_{\uparrow\uparrow}-e_{\downarrow\downarrow}\right).\end{aligned}$$

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