Potent Exercises

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Abstract

This is a formula summary sheet for nilpotents, idempotents and related variations in three dimensional, Euclidean, geometric algebra.

Formulas

Basic functional definitions: z is a nilpotent, P is an idempotent, $(1 - P)$ is also an idempotent, * is multivector multiplication. Idempotents and nilpotents come in pairs.

$$\begin{aligned} z * z &= 0 \\ P * P &= P \\ P^2 - P &= 0 \\ P(P-1) &= P(1-P) = 0 \\ (1-P) * (1-P) &= (1-P) \\ P &= Az \\ 1 - P &= zB \end{aligned}$$

Nilpotent scale independence: The product of a nilpotent scaled by a complex number is also a nilpotent.

$$[(a + ib) * (z)]^2 = ((a + ib)^2 z^2) = 0$$

In general, however, a the product of a multivector and a nilpotent won't be a nilpotent, but can be a factor of zero.

Idempotents have a natural scale, due to the scalar component of 1/2. While nilpotents are independent of scale, I prefer the following form due to the relationship between nilpotents and idempotents. In the following formulas, the two unit vectors $\vec{u}$ and $\vec{v}$ are orthogonal, meaning $\vec{u}\vec{v} = -\vec{v}\vec{u}$. $P_{\pm}$ is shorthand for both P_{+} and P_{-}.

Periodic formats

$$\begin{aligned} \vec{u}\vec{v} &= -\vec{v}\vec{u} \quad (\vec{u} \perp \vec{v}) \\ z_{\pm} &= \frac{1}{2}(\vec{u}\sin\theta + \vec{v}\cos\theta \pm \vec{u}\vec{v}) \\ P_{\pm} &= \frac{1}{2}(1 \pm \vec{u}\sin\theta \pm \vec{v}\cos\theta) \end{aligned}$$

Hyperbolic formats

$$\begin{aligned} P_{+} &= \frac{1}{2}(1 + (\vec{u}\cosh(\alpha) + \vec{u}\vec{v}\sinh(\alpha))) = P \\ P_{-} &= \frac{1}{2}(1 - (\vec{u}\cosh(\alpha) + \vec{u}\vec{v}\sinh(\alpha))) = (1 - P) \\ P_{+}P_{-} &= 0 \\ P_{\pm} &= \frac{1}{2}(1 \pm (\vec{u}\cosh(\alpha) + \vec{u}\vec{v}\sinh(\alpha))) \\ P_{\pm} &= \vec{v}z_{\pm} \\ z_{\pm} &= \vec{v}P_{\pm} \\ &= \frac{1}{2}(\vec{v} \pm \vec{v}\vec{u}\cosh(\alpha) \mp \vec{u}\sinh(\alpha)) \end{aligned}$$

I have three angles and one parameter associated with nilpotents and idempotents. The unit vector $\vec{u}$ has two angles (θ and ϕ) associated with latitude and longitude. The vector $\vec{v}$, which is perpedicular to $\vec{u}$, in three space, has an angle γ that rotates around $\vec{u}$. In a sense, the multivector $\vec{u}\vec{v}$ is like a flag at the end of a flagpole, free to orient in the wind, but always perpendicular to the pole (given enough wind). By contrast, the argument of the hyperbolic functions α is a real parameter sweeping from $-\infty$ to $+\infty$.

$$
\begin{aligned}
\vec{u} &= \cos\theta\sin\phi e_x + \sin\theta\sin\phi e_y + \cos\phi e_z \\
\vec{v} &= +(\cos\gamma\cos\theta\cos\phi - \sin\gamma\sin\theta)e_x \\
&\quad +(\cos\gamma\sin\theta\cos\phi + \sin\gamma\cos\theta)e_y \\
&\quad +(-\cos\gamma\sin\phi)e_z \\
\vec{u}\vec{v} &= +(\sin\gamma\sin\phi)e_x e_y \\
&\quad +(\cos\gamma\cos\theta - \sin\gamma\sin\theta\cos\phi)e_z e_x \\
&\quad +(-\cos\gamma\sin\theta - \sin\gamma\cos\theta\cos\phi)e_y e_z
\end{aligned}
$$

Writing $P_\pm$ in component form in this basis, we have

$$
\begin{aligned}
P_\pm &= \frac{1}{2}\left(1 \pm \left(\vec{u}\cosh(\alpha) + \vec{u}\vec{v}\sinh(\alpha)\right)\right) \\
&= +\frac{1}{2} \\
&\quad \pm\frac{1}{2}\left[\cosh(\alpha)\cos\theta\sin\phi\right] e_x \\
&\quad \pm\frac{1}{2}\left[\cosh(\alpha)\sin\theta\sin\phi\right] e_y \\
&\quad \pm\frac{1}{2}\left[\cosh(\alpha)\cos\phi\right] e_z \\
&\quad \pm\frac{1}{2}\left[\sinh(\alpha)\left(\sin\gamma\sin\phi\right)\right] e_x e_y \\
&\quad \pm\frac{1}{2}\left[\sinh(\alpha)\left(\cos\gamma\cos\theta - \sin\gamma\sin\theta\cos\phi\right)\right] e_z e_x \\
&\quad \pm\frac{1}{2}\left[\sinh(\alpha)\left(-\cos\gamma\sin\theta - \sin\gamma\cos\theta\cos\phi\right)\right] e_y e_z
\end{aligned}
$$

Writing $z_\pm$ in component form in this basis, we have

$$\begin{aligned}
z_\pm &= \frac{1}{2}\left(\vec{v} \pm \vec{v}\vec{u}\cosh(\alpha) \mp \vec{u}\sinh(\alpha)\right) \\
&= \frac{1}{2}\left(\vec{v} \mp \vec{u}\sinh(\alpha) \pm \vec{v}\vec{u}\cosh(\alpha)\right) \\
&= +\frac{1}{2}\left[(\cos\gamma\cos\theta\cos\phi - \sin\gamma\sin\theta) \mp \sinh(\alpha)\cos\theta\sin\phi\right] e_x \\
&\quad +\frac{1}{2}\left[(\cos\gamma\sin\theta\cos\phi + \sin\gamma\cos\theta) \mp \sinh(\alpha)\sin\theta\sin\phi\right] e_y \\
&\quad +\frac{1}{2}\left[(-\cos\gamma\sin\phi) \mp \sinh(\alpha)\cos\phi\right] e_z \\
&\quad \pm\frac{1}{2}\left[\cosh(\alpha)(\sin\gamma\sin\phi)\right] e_x e_y \\
&\quad \pm\frac{1}{2}\left[\cosh(\alpha)(\cos\gamma\cos\theta - \sin\gamma\sin\theta\cos\phi)\right] e_z e_x \\
&\quad \pm\frac{1}{2}\left[\cosh(\alpha)(-\cos\gamma\sin\theta - \sin\gamma\cos\theta\cos\phi)\right] e_y e_z
\end{aligned}$$

A real number r combined with z has the following powers.

$$\begin{aligned}
(r+z)^2 &= r^2 + 2rz + z^2 \\
&= r^2 + 2rz \\
(r+z)^3 &= (r+z) * (r^2 + 2rz) \\
&= r^3 + 3r^2 z \\
(r+z)^4 &= r^4 + 4r^3 z
\end{aligned}$$

How about taking the exponential of this combination?

$$e^{(r+z)} = e^r + ze^r = (1+z)e^r$$

Now we look at powers of real and idempotent combinations.

$$\begin{aligned}
(r+P)^2 &= r^2 + 2rP + P^2 \\
&= r^2 + (2r+1)P \\
(r+P)^3 &= r^3 + 3r^2 P + 3rP^2 + P^3 \\
&= r^3 + (3r^2 + 3r + 1)P \\
(r+P)^4 &= r^4 + 4r^3 P + 6r^2 P^2 + 4rP^3 + P^4 \\
&= r^4 + (4r^3 + 6r^2 + 4r + 1)P
\end{aligned}$$

Let's take the exponential of P.

$$\begin{aligned} e^{P} &= 1 + P(e-1) \\ e^{(r+P)} &= e^{r} + Pe^{r}(e-1) \end{aligned}$$

Now look at a general combination of two orthogonal idempotents.

$$\begin{aligned} (aP_{+} + bP_{-})^{2} &= a^{2}P_{+}^{2} + 2abP_{+}P_{-} + b^{2}P_{-}^{2} \\ &= a^{2}P_{+} + b^{2}P_{-} \\ (aP_{+} + bP_{-})^{3} &= a^{3}P_{+}^{3} + 3a^{2}P_{+}^{2}bP_{-} + 3aP_{+}b^{2}P_{-}^{2} + b^{3}P_{-}^{3} \\ &= a^{3}P_{+} + b^{3}P_{-} \\ (aP_{+} + bP_{-})^{n} &= a^{n}P_{+} + b^{n}P_{-} \end{aligned}$$

Let's take the exponential of this combination.

$$e^{(aP_{+}+bP_{-})} = e^{a}P_{+} + e^{b}P_{-}$$

These formulas depend upon the commutativity of real numbers and multivectors. Given four degrees of freedom in specifying potents, we may be able to 'tailor make' potents for specific multivectors.

Variations on Nilpotents and Idempotents

Scalar and trivectors (pseudoscalars) commute with everything. Consequently, $z * (a + be_{x}e_{y}e_{z})$ is also a nilpotent.

In a similar fashion, the sandwich product $z * M * z$ also yields nilpotents.

Anti-idempotents (A) square to their negative.

$$\begin{aligned} A &= -P \\ (-P)(-P) &= P \\ A * A &= -A \\ A_{\pm} &= -\frac{1}{2} \pm \frac{1}{2}\left(\vec{u}\cosh(t) + U\sinh(t)\right) \end{aligned}$$

The unit trivector $e_{x}e_{y}e_{z}$ commutes with all multivectors, and, squaring to -1, is indistinguishible from i. We now define imagpotents

$$\begin{aligned} i &= e_{x}e_{y}e_{z} \\ iP &= (e_{x}e_{y}e_{z})\left[\frac{1}{2} \pm \frac{1}{2}\left(\vec{u}\cosh(t) + U\sinh(t)\right)\right] \\ (iP)(iP) &= -P^{2} = -P = A \end{aligned}$$

3D Euclidean Geometric Algebra Basis

Three dimensional Euclidean geometrical algebra has a scalar (1), three vectors (e_x, e_y and e_z), three bivectors (e_xe_y, e_ze_x, and e_ye_z), and one trivector ($e_xe_ye_z$) defining the geometry. In multiplication table format, the order-sensitive multiplication among these elements, with prefactors on the left column and postfactors on top row, is

	1	e_x	e_y	e_z	e_xe_y	e_ze_x	e_ye_z	$e_xe_ye_z$
1	1	e_x	e_y	e_z	e_xe_y	e_ze_x	e_ye_z	$e_xe_ye_z$
e_x	e_x	1	e_xe_y	-e_ze_x	e_y	$-e_z$	$e_xe_ye_z$	e_ye_z
e_y	e_y	$-e_xe_y$	1	e_ye_z	$-e_x$	$e_xe_ye_z$	e_z	e_ze_x
e_z	e_z	e_ze_x	$-e_ye_z$	1	$e_xe_ye_z$	e_x	$-e_y$	e_xe_y
e_xe_y	e_xe_y	$-e_y$	e_x	$e_xe_ye_z$	-1	e_ye_z	$-e_ze_x$	$-e_z$
e_ze_x	e_ze_x	e_z	$e_xe_ye_z$	$-e_x$	$-e_ye_z$	-1	e_xe_y	$-e_y$
e_ye_z	e_ye_z	$e_xe_ye_z$	$-e_z$	e_y	e_ze_x	$-e_xe_y$	-1	$-e_x$
$e_xe_ye_z$	$e_xe_ye_z$	e_ye_z	e_ze_x	e_xe_y	$-e_z$	$-e_y$	$-e_x$	-1

In this algebra, scalar multiplication is commutative and associative, basis vectors square to scalar one, and the product of two basis (orthogonal) vectors resulting in a bivector is anti-commutative, associative, squares to negative one, while the trivector basis commutes with everything, yet squares to negative one.

Component Level Equations for Multivector Product Using xy yz zx Convention

```
c.q   = a.q*b.q   + a.x*b.x   + a.y*b.y   + a.z*b.z
      - a.xy*b.xy  - a.zx*b.zx  - a.yz*b.yz  - a.xyz*b.xyz;
c.x   = a.q*b.x   + a.x*b.q   - a.y*b.xy  + a.z*b.zx
      + a.xy*b.y   - a.zx*b.z   - a.yz*b.xyz - a.xyz*b.yz ;
c.y   = a.q*b.y   + a.x*b.xy  + a.y*b.q   - a.z*b.yz
      - a.xy*b.x   - a.zx*b.xyz + a.yz*b.z   - a.xyz*b.zx ;
c.z   = a.q*b.z   - a.x*b.zx  + a.y*b.yz  + a.z*b.q
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```
      - a.xy*b.xyz + a.zx*b.x   - a.yz*b.y   - a.xyz*b.xy ;
c.xy  = a.q*b.xy  + a.x*b.y   - a.y*b.x   + a.z*b.xyz
      + a.xy*b.q   + a.zx*b.yz  - a.yz*b.zx  + a.xyz*b.z  ;
c.zx  = a.q*b.zx  - a.x*b.z   + a.y*b.xyz + a.z*b.x
      - a.xy*b.yz  + a.zx*b.q   + a.yz*b.xy  + a.xyz*b.y  ;
c.yz  = a.q*b.yz  + a.x*b.xyz + a.y*b.z   - a.z*b.y
      + a.xy*b.zx  - a.zx*b.xy  + a.yz*b.q   + a.xyz*b.x  ;
c.xyz = a.q*b.xyz + a.x*b.yz  + a.y*b.zx  + a.z*b.xy
      + a.xy*b.z   + a.zx*b.y   + a.yz*b.x   + a.xyz*b.q  ;
```