Pauli Equation and Geometric Algebra

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Abstract

The experimental work of Uhlenberg and Goldschmidt with a beam of silver atoms in a non-homogenous magnetic field strongly suggested inherent electron spin. Wolfgang Pauli, overcoming his initial bias against electron spin, developed a non-relativistic extension to the Schrodinger equation, based upon analogy with orbital angular momentum, using matrices as a mathematical extension. These notes step through the process of converting the Pauli equation from high level format to complex component level differential equations for the wavefunctions. I find that geometric algebra is present and equivalent to the sigma matrices used in formulating spin effects. However, the final equations for the wavefunctions have no dependencies upon geometric algebra, sigma matrices or any interpretation of the origin of spin.

3D Euclidean Geometric Algebra Basis

Three dimensional Euclidean geometrical algebra has a scalar (1), three vectors (e_x, e_y and e_z), three bivectors (e_xe_y, e_ze_x, and e_ye_z), and one trivector ($e_xe_ye_z$) defining the geometry. Multivector multiplication is associative, but not necessarily commutative.

In 3D Euclidean space, by definition, the three vector elements individually square to +1.

$$
\begin{aligned}
e_x * e_x &= e_xe_x = 1 \\
e_y * e_y &= e_ye_y = 1 \\
e_z * e_z &= e_ze_z = 1
\end{aligned}
$$

In contrast to the cross product, the product of different vector basis is an anti-commutating bivector.

$$\begin{aligned} e_x * e_y &= e_x e_y = -e_y e_x \\ e_y * e_z &= e_y e_z = -e_z e_y \\ e_z * e_x &= e_z e_x = -e_x e_z \end{aligned}$$

These bivectors square to -1, as illustrated by

$$\begin{aligned} (e_x e_y) * (e_x e_y) &= e_x e_y e_x e_y \\ &= -e_y e_x e_x e_y \\ &= -e_y 1 e_y = -e_y e_y \\ &= -1 \end{aligned}$$

The trivector $e_x e_y e_z$ also squares to negative one, and commutes will all multivector components. This trivector, mimicing the behavior of i, is commonly written as I, sometimes as i, sometimes as j in the literature. In our case, whenever I see an i in a parent equation prior geometric algebra, I will suspect this to translate into the trivector in the post geometric algebra format. When I want to emphasize the correlation to older equations, I will use the capital $I = e_x e_y e_z$.

With our bivectors, I have a preference to use $e_x e_y, e_y e_z, e_z e_x$ as the preferred order of products, which leads to component equations with obvious dot product and couple terms.

Table 1 presents the the order-sensitive geometric algebra product in multiplication table format, with prefactors on the left column and postfactors on top row.

3D Euclidean Wedge Product

The Grassman wedge product, widely used in differential forms, differs from the geometrical product by having squared vector basis elements be zero. (This is a Clifford algebra with metric 0.) Multiplication among vector elements is anticommutative. $e_x \wedge e_y = -e_y \wedge e_x$. However, the wedge of a bivector and vector is commutative. $(e_x e_y) \wedge e_z = e_z \wedge (e_x e_y) = e_x e_y e_z$. In

$*$	1	e_x	e_y	e_z	e_xe_y	e_ze_x	e_ye_z	$e_xe_ye_z$
1	1	e_x	e_y	e_z	e_xe_y	e_ze_x	e_ye_z	$e_xe_ye_z$
e_x	e_x	1	e_xe_y	-e_ze_x	e_y	$-e_z$	$e_xe_ye_z$	e_ye_z
e_y	e_y	$-e_xe_y$	1	e_ye_z	$-e_x$	$e_xe_ye_z$	e_z	e_ze_x
e_z	e_z	e_ze_x	$-e_ye_z$	1	$e_xe_ye_z$	e_x	$-e_y$	e_xe_y
e_xe_y	e_xe_y	$-e_y$	e_x	$e_xe_ye_z$	-1	e_ye_z	$-e_ze_x$	$-e_z$
e_ze_x	e_ze_x	e_z	$e_xe_ye_z$	$-e_x$	$-e_ye_z$	-1	e_xe_y	$-e_y$
e_ye_z	e_ye_z	$e_xe_ye_z$	$-e_z$	e_y	e_ze_x	$-e_xe_y$	-1	$-e_x$
$e_xe_ye_z$	$e_xe_ye_z$	e_ye_z	e_ze_x	e_xe_y	$-e_z$	$-e_y$	$-e_x$	-1

Table 1: Geometric Product in Three Dimensional Euclidean Space

$\wedge$	1	e_x	e_y	e_z	e_xe_y	e_ze_x	e_ye_z	$e_xe_ye_z$
1	1	e_x	e_y	e_z	e_xe_y	e_ze_x	e_ye_z	$e_xe_ye_z$
e_x	e_x	0	e_xe_y	-e_ze_x	0	0	$e_xe_ye_z$	0
e_y	e_y	$-e_xe_y$	0	e_ye_z	0	$e_xe_ye_z$	0	0
e_z	e_z	e_ze_x	$-e_ye_z$	0	$e_xe_ye_z$	0	0	0
e_xe_y	e_xe_y	0	0	$e_xe_ye_z$	0	0	0	0
e_ze_x	e_ze_x	0	$e_xe_ye_z$	0	0	0	0	0
e_ye_z	e_ye_z	$e_xe_ye_z$	0	0	0	0	0	0
$e_xe_ye_z$	$e_xe_ye_z$	0	0	0	0	0	0	0

Table 2: Wedge Product in Three Dimensional Euclidean Space

three dimensional space, any vector or bivector wedged with the trivector will be zero. Consequently, all wedge products of more than three vector elements are automatically zero.

Scalar terms commute with all elements in the wedge product. $a \wedge e_x = e_x \wedge a$.

Table 2 presents the the order-sensitive wedge product in multiplication table format, with prefactors on the left column and postfactors on top row.

Schrodinger Equation with Electromagnetism

The standard Schrodinger equation for an electron in an electromagnetic field, prior to Pauli's spin extension, can be written using a complex wavefunction as

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\vec{\nabla}^2\psi + \frac{iq\hbar}{m}(\vec{A}\cdot\vec{\nabla})\psi + \frac{iq\hbar}{2m}(\vec{\nabla}\cdot\vec{A})\psi + \frac{q^2}{2m}A^2\psi + q\phi\psi$$

Complex Wavefunction

The spin-free wavefunction $\psi = \psi_r + i\psi_i$ is a complex field. In three dimensional geometric algebra, this maps to a scalar plus trivector combination. As a multivector, this is missing vector and bivector terms. This is not surprising however, as this is an incomplete, simplest model. Adding spin, as in the Pauli and Dirac equations, will fill out this multivector.

Writing out the wavefunction in component form, illustrating the missing terms, we have

$$\psi = (\psi_r, 0, 0, 0, 0, 0, 0, \psi_i)$$

Returning to our Scrodinger equation, we now separate out the real and imaginary components for ψ.

Organizing by variables

$$\begin{aligned}
\frac{\partial}{\partial t}\psi_r &= +\frac{q}{m}(\vec{A}\cdot\vec{\nabla}\psi_r) + \frac{q}{2m}(\vec{\nabla}\cdot\vec{A})(\psi_r) - \frac{\hbar}{2m}\vec{\nabla}^2\psi_i + \frac{q^2}{2\hbar m}A^2(\psi_i) + \frac{q\phi}{\hbar}(\psi_i) \\
\frac{\partial}{\partial t}\psi_i &= \frac{\hbar}{2m}\vec{\nabla}^2\psi_r - \frac{q^2}{2\hbar m}A^2(\psi_r) - \frac{q\phi}{\hbar}(\psi_r) + \frac{q}{m}(\vec{A}\cdot\vec{\nabla}\psi_i) + \frac{q}{2m}(\vec{\nabla}\cdot\vec{A})(\psi_i)
\end{aligned}$$

The next step is the Pauli Schrodinger equation.

Orbital Angular Momentum

Angular momentum in quantum mechanics was well known before Pauli. The classical angular momentum of a particle in a central field, such as an electron around a massive proton in the hydrogen atom, is

$$
\begin{aligned}
\vec{L} &= \vec{r} \times \vec{p} \\
L_x &= y p_z - z p_y \\
L_y &= z p_x - x p_z \\
L_z &= x p_y - y p_x
\end{aligned}
$$

In classical mechanics, the cross product has been a source of ambiguity regarding parity transformations, due to polar versus axial vectors having three components, but different parity traits. In geometric algebra, this confusion is resolved by noting that axial or pseudovectors are, in reality, bivectors having even parity due to the presense of a product of two space terms (such as area) as part of their definition. In three dimensional space, there is a one to one mapping available of bivectors and their normal vector. Using this mapping to fold bivectors into vectors has been a source of confusion.

In geometric algebra, angular momentum will be recognized as a bivector. Instead of a cross product, which is folding bivectors back into vectors, we use the wedge product.

$$
\begin{aligned}
\mathbf{L} &= \vec{r} \wedge \vec{p} \\
L_{yz} &= y p_z - z p_y \\
L_{zx} &= z p_x - x p_z \\
L_{xy} &= x p_y - y p_x
\end{aligned}
$$

As a sometimes useful tool for translating cross products to wedge products, we have the mapping $\times \to I\wedge = e_x e_y e_z \wedge$. Demonstrating with our angular momentum expression, we have the translation

$$
\begin{aligned}
\vec{L} &= \vec{r} \times \vec{p} \\
&= \vec{r} I \wedge \vec{p} \\
&= I(\vec{r} \wedge \vec{p}) \\
&= I((x e_x + y e_y + z e_z) \wedge (p_x e_x + p_y e_y + p_z e_z)) \\
&= I\left[(x p_y - y p_x)\, e_x e_y + (y p_z - z p_y)\, e_y e_z + (z p_x - x p_z)\, e_z e_x\right] \\
&= e_x e_y e_z \left[(x p_y - y p_x)\, e_x e_y + (y p_z - z p_y)\, e_y e_z + (z p_x - x p_z)\, e_z e_x\right] \\
&= (y p_z - z p_y)\, e_x + (z p_x - x p_z)\, e_y + (x p_y - y p_x)\, e_z
\end{aligned}
$$

This demonstrates the recovery of a cross product formula using the wedge product. However, in practice, we really do not want this defective cross product, and are better advised to shift over to an apppropriate wedge product.

Returning to the discussion of classical orbital momentum, we now present the classical commutator relationships.

$$
\begin{aligned}
\vec{L} &= \vec{r} \times \vec{p} \\
p_x &= -i\hbar \frac{\partial}{\partial x} \\
p_y &= -i\hbar \frac{\partial}{\partial y} \\
p_z &= -i\hbar \frac{\partial}{\partial z} \\
L_x &= y p_z - z p_y = -i\hbar \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) \\
L_y &= z p_x - x p_z = -i\hbar \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) \\
L_z &= x p_y - y p_x = -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)
\end{aligned}
$$

In multivector format, the i shifts the pseudovector to the appropriate bivector slot, and we have instead the bivector terms

$$
\begin{aligned}
\mathbf{L} &= \vec{r} \wedge \vec{p} \\
L_{yz} &= y p_z - z p_y = -\hbar \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) \\
L_{zx} &= z p_x - x p_z = -\hbar \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) \\
L_{xy} &= x p_y - y p_x = -\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)
\end{aligned}
$$

A mathematical product can be expressed as a sum of symmetric and antisymmetric product components.

$$
AB = \left(\frac{AB + BA}{2} \right) + \left(\frac{AB - BA}{2} \right)
$$

The quantum mechanical commutator is $[A, B] = AB - BA$, and as such, is just twice the antisymmetric product. The factor of two is historically dropped, as we renormalize wavefunctions later, anyway, and we are often only concerned about zero versus nonzero commutator status.

Forming the commutator among the standard quantum components of $\vec{L}$, and absorbing the double negative products to reduce clutter, we have

$$\begin{aligned}
(L_x L_y - L_y L_x)\,\phi &= i\hbar \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) i\hbar \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) \phi \\
&\quad -i\hbar \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) i\hbar \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \phi \\
(L_x L_y - L_y L_x)\,\phi &= -\hbar^2 \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) \phi \\
&\quad +\hbar^2 \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \phi \\
(L_x L_y - L_y L_x)\,\phi &= -\hbar^2 \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \left(z\frac{\partial \phi}{\partial x} - x\frac{\partial \phi}{\partial z} \right) \\
&\quad +\hbar^2 \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) \left(y\frac{\partial \phi}{\partial z} - z\frac{\partial \phi}{\partial y} \right)
\end{aligned}$$

Notice that I am using a dummy function ϕ to reduce careless errors with operator algebra. I will remove this term later, once the desired result is achieved. Expanding out our parenthesis, we have

$$\begin{aligned}
(L_x L_y - L_y L_x)\,\phi &= -\hbar^2 \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \left(z\frac{\partial \phi}{\partial x} - x\frac{\partial \phi}{\partial z} \right) \\
&\quad +\hbar^2 \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) \left(y\frac{\partial \phi}{\partial z} - z\frac{\partial \phi}{\partial y} \right) \\
(L_x L_y - L_y L_x)\,\phi &= -\hbar^2 \left(+y\frac{\partial}{\partial z} z\frac{\partial \phi}{\partial x} - y\frac{\partial}{\partial z} x\frac{\partial \phi}{\partial z} - z\frac{\partial}{\partial y} z\frac{\partial \phi}{\partial x} + z\frac{\partial}{\partial y} x\frac{\partial \phi}{\partial z} \right) \\
&\quad +\hbar^2 \left(+z\frac{\partial}{\partial x} y\frac{\partial \phi}{\partial z} - z\frac{\partial}{\partial x} z\frac{\partial \phi}{\partial y} - x\frac{\partial}{\partial z} y\frac{\partial \phi}{\partial z} + x\frac{\partial}{\partial z} z\frac{\partial \phi}{\partial y} \right) \\
(L_x L_y - L_y L_x)\,\phi &= -\hbar^2 \left(+y\frac{\partial \phi}{\partial x} + yz\frac{\partial^2 \phi}{\partial x \partial z} - xy\frac{\partial^2 \phi}{\partial z^2} - z^2\frac{\partial^2}{\partial x \partial y} + zx\frac{\partial^2 \phi}{\partial y \partial z} \right) \\
&\quad +\hbar^2 \left(+yz\frac{\partial^2 \phi}{\partial x \partial z} - z^2\frac{\partial^2 \phi}{\partial x \partial y} - xy\frac{\partial^2 \phi}{\partial z^2} + x\frac{\partial \phi}{\partial y} + xz\frac{\partial^2 \phi}{\partial y \partial z} \right)
\end{aligned}$$

$$\begin{aligned}
(L_x L_y - L_y L_x)\,\phi &= +\hbar^2 \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \phi \\
(L_x L_y - L_y L_x) &= +\hbar^2 \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)
\end{aligned}$$

We explicitly see the nice cancellation of the commutating derivative terms in the anticommutative product. Given

$$L_z = x p_y - y p_x = -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$$

we have

$$\begin{aligned}
(L_x L_y - L_y L_x) &= +\hbar^2 \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \\
(L_x L_y - L_y L_x) &= i\hbar L_z
\end{aligned}$$

In a similar fashion, we collect the entire set

$$\begin{aligned}
(L_x L_y - L_y L_x) &= i\hbar L_z \\
(L_y L_z - L_z L_y) &= i\hbar L_x \\
(L_z L_x - L_x L_z) &= i\hbar L_y
\end{aligned}$$

Repeat Commutators using Geometric Algebra

Just for reference, we repeat the previous material, this time using bivector angular momentum and geometric algebra. We see that the linearization and cancellation work the very same way.

$$\begin{aligned}
\mathbf{L} &= \vec{r} \wedge \vec{p} \\
L_{yz} &= y p_z - z p_y = -\hbar \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) \\
L_{zx} &= z p_x - x p_z = -\hbar \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) \\
L_{xy} &= x p_y - y p_x = -\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)
\end{aligned}$$

Notice i has been absorbed into the bivector basis. Form our commutator, with dummy function ϕ.

$$
\begin{aligned}
(L_{yz}L_{zx} - L_{zx}L_{yz})\phi &= +\hbar^2 \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) \phi \\
&\quad -\hbar^2 \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \phi
\end{aligned}
$$

We notice the right hand side is the negative of our previous work. We fast forward to

$$
(L_{yz}L_{zx} - L_{zx}L_{yz})\phi = -\hbar^2 \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \phi
$$

Given

$$
L_{xy} = -\hbar \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right)
$$

We have the result

$$
\begin{aligned}
(L_{yz}L_{zx} - L_{zx}L_{yz})\phi &= -\hbar^2 \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \phi \\
(L_{yz}L_{zx} - L_{zx}L_{yz}) &= -\hbar^2 \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \\
&= \hbar L_{xy}
\end{aligned}
$$

Our complete set in geometric algebra is

$$
\begin{aligned}
(L_{yz}L_{zx} - L_{zx}L_{yz}) &= \hbar L_{xy} \\
(L_{zx}L_{xy} - L_{xy}L_{zx}) &= \hbar L_{yz} \\
(L_{xy}L_{yz} - L_{yz}L_{xy}) &= \hbar L_{zx}
\end{aligned}
$$

As I read these bivector terms, I mentally translate to antivector notation [10].

```
[not x  not y] => hbar not z
```

Pauli Intrinsic Spin Analogy

So, how did Wolfgang Pauli hammer intrinsic spin into the Schrodinger equation? In the absense of magnetic fields, the extended Schrodinger equation must coincide with the original Schrodinger equation. However, the presense of a magnetic field, interacting with the magnetic moment due to electron spin, will diminish or increase the electron energy depending upon the relative alignment of the external field and the internal spin. Pauli realized that the system had two extreme cases, field align or anti-aligned, and that all intermediate cases can be modeled as a scaled superposition of these two extreme cases. His solution was to double the wavefunction to have two complex components, which in the absense of a magnetic field, would coincide.

In matrix terms, the field became a two element column. For the time dependent Schrodinger equation, the field evolution is governed by the Hamiltonian operating on the current state. This, in turn, meant that the Hamiltonian has become a two by two matrix.

The next issue was how to deal with the direction of the spin axis, as in general, the spin axis would be independent of the position. Pauli was adamant about keeping the spin axis distinct from the coordinate space. Perhaps he was influenced by considerations of four dimensional space, where we can have two independent, orthogonal planes of rotation. Perhaps he viewed the electron as a point particle, and felt that a zero dimensional point particle, lacking extent, could not rotate, generate a magnetic field or possess angular momentum. Unlike Louis de Broglie, Pauli was not open to the idea of an electron in continuous motion (action), self-interacting, following a curved path even in the absense of external magnetic fields.

Pauli's starting point was to assert a mathematical model for spin S, analogous to orbital angular momentum L. Intrinsic spin S had a set of commutator relationships mimicing L.

$$\begin{aligned}
[S_x, S_y] &= S_x S_y - S_y S_x = i\hbar S_z \\
[S_y, S_z] &= S_y S_z - S_z S_y = i\hbar S_x \\
[S_z, S_x] &= S_z S_x - S_x S_z = i\hbar S_y
\end{aligned}$$

The spin state is described by a two component spinor.

$$\begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$*$	$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\sigma_x^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\sigma_x\sigma_y = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\sigma_x\sigma_z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\sigma_y\sigma_x = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$	$\sigma_y^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\sigma_y\sigma_z = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$
$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\sigma_z\sigma_x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\sigma_z\sigma_y = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$	$\sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Table 3: Pauli Matrix Multiplication Table

where spin up is $\chi_+ = (1 \;\; 0)^T$ and spin down is $\chi_- = (0 \;\; 1)^T$.

The spin operators are

$$S_x = \frac{\hbar}{2}\sigma_x \qquad S_y = \frac{\hbar}{2}\sigma_y \qquad S_z = \frac{\hbar}{2}\sigma_z$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli Matrices and Geometric Algebra

At this point in textbooks, we typically see a demonstration or assignment showing the pairwise products of the Pauli matrices. In Table 3, forefactors are on the left, postfactors on the top. We notice that each vector squares to unity, while the dissimilar product terms are antisymmetric. This happens to match our geometric algebra vector multiplication table.

We decide to expand our matrix set.

$$\begin{array}{llll} \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \sigma_{xy} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \sigma_{zx} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \sigma_{yz} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} & \sigma_{xyz} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = \sigma_i \end{array}$$

The multiplication table amoungst the σ elements exactly matches the three dimensional, Euclidean geometric product table at the start of this paper.

Table 4 presents the Pauli matrix multiplication table, explicitly showing each product, such as $\sigma_x\sigma_y$

*	1	σ_x	σ_y	σ_z	$\sigma_x\sigma_y$	$\sigma_z\sigma_x$	$\sigma_y\sigma_z$	$\sigma_x\sigma_y\sigma_z$
1	1	σ_x	σ_y	σ_z	$\sigma_x\sigma_y$	$\sigma_z\sigma_x$	$\sigma_y\sigma_z$	$\sigma_x\sigma_y\sigma_z$
σ_x	σ_x	1	$\sigma_x\sigma_y$	-$\sigma_z\sigma_x$	σ_y	$-\sigma_z$	$\sigma_x\sigma_y\sigma_z$	$\sigma_y\sigma_z$
σ_y	σ_y	$-\sigma_x\sigma_y$	1	$\sigma_y\sigma_z$	$-\sigma_x$	$\sigma_x\sigma_y\sigma_z$	σ_z	$\sigma_z\sigma_x$
σ_z	σ_z	$\sigma_z\sigma_x$	$-\sigma_y\sigma_z$	1	$\sigma_x\sigma_y\sigma_z$	σ_x	$-\sigma_y$	$\sigma_x\sigma_y$
$\sigma_x\sigma_y$	$\sigma_x\sigma_y$	$-\sigma_y$	σ_x	$\sigma_x\sigma_y\sigma_z$	-1	$\sigma_y\sigma_z$	$-\sigma_z\sigma_x$	$-\sigma_z$
$\sigma_z\sigma_x$	$\sigma_z\sigma_x$	σ_z	$\sigma_x\sigma_y\sigma_z$	$-\sigma_x$	$-\sigma_y\sigma_z$	-1	$\sigma_x\sigma_y$	$-\sigma_y$
$\sigma_y\sigma_z$	$\sigma_y\sigma_z$	$\sigma_x\sigma_y\sigma_z$	$-\sigma_z$	σ_y	$\sigma_z\sigma_x$	$-\sigma_x\sigma_y$	-1	$-\sigma_x$
$\sigma_x\sigma_y\sigma_z$	$\sigma_x\sigma_y\sigma_z$	$\sigma_y\sigma_z$	$\sigma_z\sigma_x$	$\sigma_x\sigma_y$	$-\sigma_z$	$-\sigma_y$	$-\sigma_x$	-1

Table 4: Pauli's Spin Matrices are Geometric Algebra

*	1	σ_x	σ_y	σ_z	σ_{xy}	σ_{zx}	σ_{yz}	σ_{xyz}
1	1	σ_x	σ_y	σ_z	σ_{xy}	σ_{zx}	σ_{yz}	σ_{xyz}
σ_x	σ_x	1	σ_{xy}	-σ_{zx}	σ_y	$-\sigma_z$	σ_{xyz}	σ_{yz}
σ_y	σ_y	$-\sigma_{xy}$	1	σ_{yz}	$-\sigma_x$	σ_{xyz}	σ_z	σ_{zx}
σ_z	σ_z	σ_{zx}	$-\sigma_{yz}$	1	σ_{xyz}	σ_x	$-\sigma_y$	σ_{xy}
σ_{xy}	σ_{xy}	$-\sigma_y$	σ_x	σ_{xyz}	-1	σ_{yz}	$-\sigma_{zx}$	$-\sigma_z$
σ_{zx}	σ_{zx}	σ_z	σ_{xyz}	$-\sigma_x$	$-\sigma_{yz}$	-1	σ_{xy}	$-\sigma_y$
σ_{yz}	σ_{yz}	σ_{xyz}	$-\sigma_z$	σ_y	σ_{zx}	$-\sigma_{xy}$	-1	$-\sigma_x$
σ_{xyz}	σ_{yz}	σ_{yz}	σ_{zx}	σ_{xy}	$-\sigma_z$	$-\sigma_y$	$-\sigma_x$	-1

Table 5: Pauli's Spin Matrices are Geometric Algebra in Compact Notation

In Table 5, we use the more compact notation for the spin matrices product names, such as $\sigma_{xy} = \sigma_x\sigma_y$.

In standard quantum mechanics, the next step is to write the expression

$$\sigma = \mathbf{i}\sigma_x + \mathbf{j}\sigma_y + \mathbf{k}\sigma_z$$

From my point of view, this expression is not a vector,but rather a dyad. A fine reference for dyads and dyadics is Morse and Feshbach [11], §1.6. In practice, this dyad is dotted into a vector to obtain another vector. Using $\vec{a}$

as a generic vector, we have

$$
\begin{aligned}
(\sigma \cdot \vec{a}) &= (\mathbf{i}\sigma_x + \mathbf{j}\sigma_y + \mathbf{k}\sigma_z) \cdot (\mathbf{i}a_x + \mathbf{j}a_y + \mathbf{k}a_z) \\
&= a_x\sigma_x + a_y\sigma_y + a_z\sigma_z
\end{aligned}
$$

The next standard exercise, for example in Arfken [1] is to show

$$
(\sigma \cdot \vec{a})(\sigma \cdot \vec{b}) = \vec{a} \cdot \vec{b} + i\sigma \cdot (\vec{a} \times \vec{b})
$$

This is easy from the geometric algebra point of view. We recognize $i = \sigma_x\sigma_y\sigma_z$ as the trivector or pseudoscalar. We recognize the sigma dyad dot products as vectors. We have

$$
\begin{aligned}
(\sigma \cdot \vec{a})(\sigma \cdot \vec{b}) &= \vec{a} \cdot \vec{b} + i\sigma \cdot (\vec{a} \times \vec{b}) \\
(\sigma \cdot \vec{a}) &= (a_x\sigma_x + a_y\sigma_y + a_z\sigma_z) \\
(\sigma \cdot \vec{b}) &= (b_x\sigma_x + b_y\sigma_y + b_z\sigma_z) \\
\sigma \cdot (\vec{a} \times \vec{b}) &= (a_yb_z - a_zb_y)\sigma_x + (a_zb_x - a_xb_z)\sigma_y + (a_xb_y - a_yb_x)\sigma_z \\
i &= \sigma_x\sigma_y\sigma_z \\
i\sigma \cdot (\vec{a} \times \vec{b}) &= (a_yb_z - a_zb_y)\sigma_y\sigma_z + (a_zb_x - a_xb_z)\sigma_z\sigma_x + (a_xb_y - a_yb_x)\sigma_x\sigma_y \\
&= \vec{a} \wedge \vec{b}
\end{aligned}
$$

We now carry out our multiplication.

$$
\begin{aligned}
(\sigma \cdot \vec{a})(\sigma \cdot \vec{b}) &= (a_x\sigma_x + a_y\sigma_y + a_z\sigma_z)(b_x\sigma_x + b_y\sigma_y + b_z\sigma_z) \\
&= a_x\sigma_xb_x\sigma_x + a_y\sigma_yb_y\sigma_y + a_z\sigma_zb_z\sigma_z \\
&\quad +(a_yb_z - a_zb_y)\sigma_y\sigma_z + (a_zb_x - a_xb_z)\sigma_z\sigma_x + (a_xb_y - a_yb_x)\sigma_x\sigma_y \\
&= a_xb_x + a_yb_y + a_zb_z \\
&\quad +(a_yb_z - a_zb_y)\sigma_y\sigma_z + (a_zb_x - a_xb_z)\sigma_z\sigma_x + (a_xb_y - a_yb_x)\sigma_x\sigma_y \\
&= \vec{a} \cdot \vec{b} + \vec{a} \wedge \vec{b}
\end{aligned}
$$

This tautalogical result is a standard geometric algebra definition for the geometric product of two vectors.

Pauli Equation in Matrix Notation

We now work through the Pauli Schrodinger equation in standard format. For reference, the Schrodinger equation for a particle in an electromagnetic field is commonly written as

$$E\psi = \left[\frac{(\vec{p}-q\vec{A})^2}{2m} + q\phi\right]\psi$$

The Pauli equation is

$$E\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} = \left[\frac{(\sigma\cdot(\vec{p}-q\vec{A}))^2}{2m} + q\phi\right]\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}$$

We are going to work a bit with the operator $(\sigma\cdot(\vec{p}-q\vec{A}))^2$.

$$(\sigma\cdot(\vec{p}-q\vec{A}))^2 = (\sigma\cdot(\vec{p}-q\vec{A}))\cdot(\sigma\cdot(\vec{p}-q\vec{A}))$$

Using

$$(\sigma\cdot\vec{a})(\sigma\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\sigma\cdot(\vec{a}\times\vec{b})$$

we have

$$\begin{aligned}(\sigma\cdot(\vec{p}-q\vec{A}))\cdot(\sigma\cdot(\vec{p}-q\vec{A})) &= (\vec{p}-q\vec{A})\cdot(\vec{p}-q\vec{A}) + i\sigma\cdot((\vec{p}-q\vec{A})\times(\vec{p}-q\vec{A}))\\ &= (\vec{p}-q\vec{A})^2 + i\sigma\cdot((\vec{p}-q\vec{A})\times(\vec{p}-q\vec{A}))\end{aligned}$$

We see we have the Schrodinger term plus a cross product. Let's work with the cross product, using a dummy function ϕ to reduce confusion with the operator. We start by substituting $p = -i\hbar\vec{\nabla}$

$$\begin{aligned}(\vec{p}-q\vec{A})\times(\vec{p}-q\vec{A})\phi &= (-i\hbar\vec{\nabla}-q\vec{A})\times(-i\hbar\vec{\nabla}-q\vec{A})\phi\\ &= (-i\hbar\vec{\nabla}-q\vec{A})\times(-i\hbar\vec{\nabla}\phi-q\vec{A}\phi)\\ &= (i\hbar\vec{\nabla}+q\vec{A})\times(i\hbar\vec{\nabla}\phi+q\vec{A}\phi)\\ &= -\hbar^2\vec{\nabla}\times\vec{\nabla}\phi + i\hbar\vec{\nabla}\times q(\vec{A}\phi) + q\vec{A}\times(i\hbar\vec{\nabla}\phi) + q\vec{A}\times(q\vec{A}\phi)\end{aligned}$$

Now, we lose the curl of a gradient, and we lose the cross of $\vec{A}$ with itself. Continuing to simplify,

$$\begin{aligned}(\vec{p}-q\vec{A})\times(\vec{p}-q\vec{A})\phi &= +iq\hbar\vec{\nabla}\times(\vec{A}\phi) + iq\hbar\vec{A}\times(\vec{\nabla}\phi)\\ &= +iq\hbar(\phi\vec{\nabla}\times\vec{A} - \vec{A}\times\nabla\phi) + iq\hbar\vec{A}\times\vec{\nabla}\phi\\ &= iq\hbar\phi\vec{\nabla}\times\vec{A}\\ &= iq\hbar\vec{B}\phi\end{aligned}$$

Having served its purpose, I now dispense with the dummy function ϕ, and return to the earlier equation.

$$\begin{aligned}(\sigma\cdot(\vec{p}-q\vec{A}))\cdot(\sigma\cdot(\vec{p}-q\vec{A})) &= (\vec{p}-q\vec{A})^2 + i\sigma\cdot((\vec{p}-q\vec{A})\times(\vec{p}-q\vec{A})) \\ &= (\vec{p}-q\vec{A})^2 + i\sigma\cdot(iq\hbar\vec{B}) \\ &= (\vec{p}-q\vec{A})^2 - q\hbar\sigma\cdot\vec{B}\end{aligned}$$

We now substitute into the Pauli equation, and have

$$\begin{aligned}E\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} &= \left[\frac{(\sigma\cdot(\vec{p}-q\vec{A}))^2}{2m}+q\phi\right]\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} \\ E\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} &= \left[\frac{(\vec{p}-q\vec{A})^2 - q\hbar\sigma\cdot\vec{B}}{2m}+q\phi\right]\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} \\ E\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} &= \left[\frac{(\vec{p}-q\vec{A})^2}{2m}+q\phi\right]\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} - \left[\frac{q\hbar\sigma\cdot\vec{B}}{2m}\right]\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}\end{aligned}$$

This is a standard form for the Pauli equation. The terms in black are the standard Schrodinger equation, applied to the two component spinor. In the absense of a magnetic field, the two terms ψ_+ and ψ_- coincide. The red terms are the magnetic field terms related to spin.

Spin Related Terms

I already know the component level equations for the Schrodinger equation. I now want to look at the spin terms.

$$\left[\frac{q\hbar\sigma\cdot\vec{B}}{2m}\right]\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} = \frac{q\hbar}{2m}(B_x\sigma_x + B_y\sigma_y + B_z\sigma_z)\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}$$

Repeating for convenience

$$\sigma_1 = \begin{pmatrix}1&0\\0&1\end{pmatrix} \quad \sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix} \quad \sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix} \quad \sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$
$$\sigma_{xy} = \begin{pmatrix}i&0\\0&-i\end{pmatrix} \quad \sigma_{zx} = \begin{pmatrix}0&1\\-1&0\end{pmatrix} \quad \sigma_{yz} = \begin{pmatrix}0&i\\i&0\end{pmatrix} \quad \sigma_{xyz} = \begin{pmatrix}i&0\\0&i\end{pmatrix} = \sigma_i$$

We thus have

$$B_x\sigma_x + B_y\sigma_y + B_z\sigma_z = \begin{pmatrix} (B_z) & (B_x - iB_y) \\ (B_x + iB_y) & (-B_z) \end{pmatrix}$$

We can now write

$$\begin{aligned}
\left[\frac{q\hbar\sigma\cdot\vec{B}}{2m}\right]\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} &= \frac{q\hbar}{2m}(B_x\sigma_x + B_y\sigma_y + B_z\sigma_z)\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} \\
&= \frac{q\hbar}{2m}\begin{pmatrix} (B_z) & (B_x - iB_y) \\ (B_x + iB_y) & (-B_z) \end{pmatrix}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} \\
&= \frac{q\hbar}{2m}\begin{pmatrix} (B_z)\psi_+ + (B_x - iB_y)\psi_- \\ (B_x + iB_y)\psi_+ + (-B_z)\psi_- \end{pmatrix}
\end{aligned}$$

We back substitute into the Pauli equation and get

$$\begin{aligned}
E\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} &= \left[\frac{(\vec{p}-q\vec{A})^2}{2m} + q\phi\right]\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} - \left[\frac{q\hbar\sigma\cdot\vec{B}}{2m}\right]\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} \\
E\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} &= \left[\frac{(\vec{p}-q\vec{A})^2}{2m} + q\phi\right]\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} - \frac{q\hbar}{2m}\begin{pmatrix} B_z\psi_+ + (B_x - iB_y)\psi_- \\ (B_x + iB_y)\psi_+ - B_z\psi_- \end{pmatrix}
\end{aligned}$$

At this point, we can write down the complex differential equations for ψ_+ and ψ_-. Before I do so, I point out that all traces of the sigma matrices have disappeared from the system of equations.

I had not anticipated this at all.

In effect, the sigma matrices and the geometric algebra was but a scaffolding to incorporate spin. Having served its purpose, it is quietly removed. From this point of view, I think I can better understand Wolfgang Pauli. Slightly restating his position, he made no assumptions about the nature of spin. It could be local geometric circulation. It could be some isospace separate from our experience. From the point of view of the wavefunctions, the origin does not matter.

Pauli Wavefunction Component Equations

The translation process to component equations matches the process done with the Schrodinger equation. We first substitute for energy and momentum operators.

$$\begin{aligned}
E &= i\hbar\frac{\partial}{\partial t} \\
E\psi &= \frac{\partial}{\partial t}(i\hbar\psi) \\
\vec{p} &= -i\hbar\vec{\nabla} \\
\vec{p}\psi &= -i\hbar\vec{\nabla}\psi
\end{aligned}$$

The wavefunction has real and imaginary components. We separate these out as two equations.

In our equations above, the black terms correspond to the standard Schrodinger equation, and translate directly to the black terms below. The magnetic terms, in red above and below, get separated by wavefunction and real versus imaginary status.

All that said and done, here are the four component equations for the Pauli equation.

For the positive wavefunction, we have

$$\begin{aligned}
\frac{\partial}{\partial t}\psi_{r+} &= +\frac{q}{m}(\vec{A}\cdot\vec{\nabla}\psi_{r+}) + \frac{q}{2m}(\vec{\nabla}\cdot\vec{A})(\psi_{r+}) \\
&\quad -\frac{\hbar}{2m}\vec{\nabla}^2\psi_{i+} + \frac{q^2}{2\hbar m}A^2(\psi_{i+}) + \frac{q\phi}{\hbar}(\psi_{i+}) \\
&\quad -\frac{q\hbar}{2m}\left(B_x\psi_{r-} + B_y\psi_{i-} + B_z\psi_{r+}\right) \\
\frac{\partial}{\partial t}\psi_{i+} &= \frac{\hbar}{2m}\vec{\nabla}^2\psi_{r+} - \frac{q^2}{2\hbar m}A^2(\psi_{r+}) - \frac{q\phi}{\hbar}(\psi_{r+}) \\
&\quad +\frac{q}{m}(\vec{A}\cdot\vec{\nabla}\psi_{i+}) + \frac{q}{2m}(\vec{\nabla}\cdot\vec{A})(\psi_{i+}) \\
&\quad -\frac{q\hbar}{2m}\left(B_x\psi_{i-} - B_y\psi_{r-} + B_z\psi_{i+}\right)
\end{aligned}$$

For the negative wavefunction, we have

$$\begin{aligned}
\frac{\partial}{\partial t}\psi_{r-} &= +\frac{q}{m}(\vec{A}\cdot\vec{\nabla}\psi_{r-}) + \frac{q}{2m}(\vec{\nabla}\cdot\vec{A})(\psi_{r-}) \\
&\quad -\frac{\hbar}{2m}\vec{\nabla}^2\psi_{i-} + \frac{q^2}{2\hbar m}A^2(\psi_{i-}) + \frac{q\phi}{\hbar}(\psi_{i-}) \\
&\quad -\frac{q\hbar}{2m}\left(B_x\psi_{r+} - B_y\psi_{i+}B_z\psi_{r-}\right) \\
\frac{\partial}{\partial t}\psi_{i-} &= \frac{\hbar}{2m}\vec{\nabla}^2\psi_{r-} - \frac{q^2}{2\hbar m}A^2(\psi_{r-}) - \frac{q\phi}{\hbar}(\psi_{r-}) \\
&\quad +\frac{q}{m}(\vec{A}\cdot\vec{\nabla}\psi_{i-}) + \frac{q}{2m}(\vec{\nabla}\cdot\vec{A})(\psi_{i-}) \\
&\quad -\frac{q\hbar}{2m}\left(B_x\psi_{i+} + B_y\psi_{r+} - B_z\psi_{i-}\right)
\end{aligned}$$

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