Wedge Product Differentials and Derivatives

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Abstract

The wedge product and partial derivatives are well established in physics as the differential form. Here I look at the wedge product with differentials and derivatives, and various powers of the wedge product.

Wedge Product

In the wedge product, the directional vector basis anti-commute, while an overall scalar term commutes (reference David Hestenes [7] (p. 6))

The wedge product in component form (Alan Bromborsky [1] and Eric Langyel [9]) is

```
GA3DE Wedge(const GA3DE &a, const GA3DE &b) // wedge product
{
    GA3DE c;

    c.q   = + a.q*b.q;
    c.x   = + a.q*b.x   + a.x*b.q;
    c.y   = + a.q*b.y   + a.y*b.q;
    c.z   = + a.q*b.z   + a.z*b.q;
    c.xy  = + a.q*b.xy  + a.x*b.y  - a.y*b.x  + a.xy*b.q;
    c.zx  = + a.q*b.zx  - a.x*b.z  + a.z*b.x  + a.zx*b.q;
    c.yz  = + a.q*b.yz  + a.y*b.z  - a.z*b.y  + a.yz*b.q;
    c.xyz = + a.q*b.xyz + a.x*b.yz + a.y*b.zx + a.xy*b.z
            + a.z*b.xy  + a.zx*b.y + a.yz*b.x + a.xyz*b.q;
    return c;
}
```

Wedge Products Preserve Spatial Dimensional Grade Units

The wedge product has the property of preserving reasonable units of measure in its product. For a pair of multivectors where the scalar is a pure number, the vector is measured in meters, the bivector is measured in square meters, and the trivector is measure in cubic meters, their product has the same units for the corresponding components, meaning that the scalar product is still dimensionless, the vector portion is in meters, bivector portion in square meters, and trivector in cubic meters. Consequently, multivectors using the wedge product and graded units of measure are closed under both addition and multiplication.

```
A = (a.q, L*a.x, L*a.y, L*a.z, L^2*a.xy, L^2*a.zx, L^2*a.yz, L^3*a.xyz)
B = (b.q, L*b.x, L*b.y, L*b.z, L^2*b.xy, L^2*b.zx, L^2*b.yz, L^3*b.xyz)
Wedge Product with Units (L) C = A^B
C.q   =       + a.q*b.q
C.x   =    L*( + a.x*b.q + a.q*b.x )
C.y   =    L*( + a.y*b.q + a.q*b.y )
C.z   =    L*( + a.z*b.q + a.q*b.z )
C.xy  = L^2*( + a.q*b.xy + a.x*b.y - a.y*b.x + a.xy*b.q )
C.zx  = L^2*( + a.q*b.zx - a.x*b.z + a.z*b.x + a.zx*b.q )
C.yz  = L^2*( + a.q*b.yz + a.y*b.z - a.z*b.y + a.yz*b.q )
C.xyz = L^3*( + a.q*b.xyz + a.x*b.yz + a.y*b.zx + a.z*b.xy
              + a.xy*b.z + a.zx*b.y + a.yz*b.x + a.xyz*b.q )
```

Repeated Operators

In the work that follows, A will be an operator, and B will be a multivector field. We now look at the generic form of repeated wedged operators. Due to the length of the equations, and my desire to do a suggestive layout of terms, I've implemented the following sets of equations in sidewise table format.

As we examine tables 1-4, we see that only the scalar part $a.q$ of the operator A get raised to any power. We also see that significant simplications occur for the important cases of $a.q = 0$ and $a.q = 1$

Pure Wedge Product

The case of $a.q = 0$ is a pure wedge product. Due to the pure antisymmetry of this product, and the limitation three dimensional space, all wedge powers

```
A = (a.q, a.x, a.y, a.z, a.xy, a.zx, a.yz, a.xyz)
B = (b.q, b.x, b.y, b.z, b.xy, b.zx, b.yz, b.xyz)
Wedge Product C = (A^B)
C.q   =  + a.q*b.q
C.x   =  + a.q*b.x   + a.x*b.q
C.y   =  + a.q*b.y   + a.y*b.q
C.z   =  + a.q*b.z   + a.z*b.q
C.xy  =  + a.q*b.xy  + a.xy*b.q  + a.x*b.y  - a.y*b.x
C.zx  =  + a.q*b.zx  + a.zx*b.q  + a.z*b.x  - a.x*b.z
C.yz  =  + a.q*b.yz  + a.yz*b.q  + a.y*b.z  - a.z*b.y
C.xyz =  + a.q*b.xyz + a.xyz*b.q + a.x*b.yz + a.y*b.zx + a.z*b.xy
                                 + a.yz*b.x + a.zx*b.y + a.xy*b.z
```

Table 1: Simple Operator Wedge Product

```
A = (a.q, a.x, a.y, a.z, a.xy, a.zx, a.yz, a.xyz)
B = (b.q, b.x, b.y, b.z, b.xy, b.zx, b.yz, b.xyz)
Wedge Product D = A^(A^B)
D.q   =  + a.q^2*b.q
D.x   =  + a.q^2*b.x   + 2*a.q*a.x*b.q
D.y   =  + a.q^2*b.y   + 2*a.q*a.y*b.q
D.z   =  + a.q^2*b.z   + 2*a.q*a.z*b.q
D.xy  =  + a.q^2*b.xy  + 2*a.q*a.xy*b.q  + 2*a.q*a.x*b.y  - 2*a.q*a.y*b.x
D.zx  =  + a.q^2*b.zx  + 2*a.q*a.zx*b.q  + 2*a.q*a.z*b.x  - 2*a.q*a.x*b.z
D.yz  =  + a.q^2*b.yz  + 2*a.q*a.yz*b.q  + 2*a.q*a.y*b.z  - 2*a.q*a.z*b.y
D.xyz =  + a.q^2*b.xyz + 2*a.q*a.xyz*b.q + 2*a.q*a.x*b.yz + 2*a.q*a.y*b.zx + 2*a.q*a.z*b.xy
                                         + 2*a.q*a.yz*b.x + 2*a.q*a.zx*b.y + 2*a.q*a.xy*b.z
         + 2*a.x*a.yz*b.q + 2*a.y*a.zx*b.q + 2*a.z*a.xy*b.q
```

Table 2: Squared Operator Wedge Product

```
A = (a.q, a.x, a.y, a.z, a.xy, a.zx, a.yz, a.xyz)
B = (b.q, b.x, b.y, b.z, b.xy, b.zx, b.yz, b.xyz)
Wedge Product E = A^(A^(A^B))
E.q   =  + a.q^3*b.q
E.x   =  + a.q^3*b.x   + 3*a.q^2*a.x*b.q
E.y   =  + a.q^3*b.y   + 3*a.q^2*a.y*b.q
E.z   =  + a.q^3*b.z   + 3*a.q^2*a.z*b.q
E.xy  =  + a.q^3*b.xy  + 3*a.q^2*a.xy*b.q  + 3*a.q^2*a.x*b.y  - 3*a.q^2*a.y*b.x
E.zx  =  + a.q^3*b.zx  + 3*a.q^2*a.zx*b.q  + 3*a.q^2*a.z*b.x  - 3*a.q^2*a.x*b.z
E.yz  =  + a.q^3*b.yz  + 3*a.q^2*a.yz*b.q  + 3*a.q^2*a.y*b.z  - 3*a.q^2*a.z*b.y
E.xyz =  + a.q^3*b.xyz + 3*a.q^2*a.xyz*b.q + 3*a.q^2*a.x*b.yz + 3*a.q^2*a.y*b.zx + 3*a.q^2*a.z*b.xy
                                           + 3*a.q^2*a.yz*b.x + 3*a.q^2*a.zx*b.y + 3*a.q^2*a.xy*b.z
         + 6*a.q*a.x*a.yz*b.q + 6*a.q*a.y*a.zx*b.q + 6*a.q*a.z*a.xy*b.q
```

Table 3: Cubed Operator Wedge Product

```
A = (a.q, a.x, a.y, a.z, a.xy, a.zx, a.yz, a.xyz)
B = (b.q, b.x, b.y, b.z, b.xy, b.zx, b.yz, b.xyz)
Wedge Product F = A^(A^(A^(A^B)))
F.q   =  + a.q^4*b.q
F.x   =  + a.q^4*b.x   + 4*a.q^3*a.x*b.q
F.y   =  + a.q^4*b.y   + 4*a.q^3*a.y*b.q
F.z   =  + a.q^4*b.z   + 4*a.q^3*a.z*b.q
F.xy  =  + a.q^4*b.xy  + 4*a.q^3*a.xy*b.q  + 4*a.q^3*a.x*b.y  - 4*a.q^3*a.y*b.x
F.zx  =  + a.q^4*b.zx  + 4*a.q^3*a.zx*b.q  + 4*a.q^3*a.z*b.x  - 4*a.q^3*a.x*b.z
F.yz  =  + a.q^4*b.yz  + 4*a.q^3*a.yz*b.q  + 4*a.q^3*a.y*b.z  - 4*a.q^3*a.z*b.y
F.xyz =  + a.q^4*b.xyz + 4*a.q^3*a.xyz*b.q + 4*a.q^3*a.x*b.yz + 4*a.q^3*a.y*b.zx + 4*a.q^3*a.z*b.xy
                                           + 4*a.q^3*a.yz*b.x + 4*a.q^3*a.zx*b.y + 4*a.q^3*a.xy*b.z
         + 12*a.q^2*a.y*a.zx*b.q + 12*a.q^2*a.z*a.xy*b.q + 12*a.q^2*a.x*a.yz*b.q
```

Table 4: Quartic Operator Wedge Product

over two go to zero. Physical systems which obey this type of law display remarkable closure and simplicity.

```
A = (0, a.x, a.y, a.z, a.xy, a.zx, a.yz, a.xyz)
B = (b.q, b.x, b.y, b.z, b.xy, b.zx, b.yz, b.xyz)
Wedge Product C = (A^B)
C.q   =  + 0
C.x   =  + a.x*b.q
C.y   =  + a.y*b.q
C.z   =  + a.z*b.q
C.xy  =  + a.xy*b.q  + a.x*b.y  - a.y*b.x
C.zx  =  + a.zx*b.q  + a.z*b.x  - a.x*b.z
C.yz  =  + a.yz*b.q  + a.y*b.z  - a.z*b.y
C.xyz =  + a.xyz*b.q + a.x*b.yz + a.y*b.zx + a.z*b.xy
                     + a.yz*b.x + a.zx*b.y + a.xy*b.z

A = (0, a.x, a.y, a.z, a.xy, a.zx, a.yz, a.xyz)
B = (b.q, b.x, b.y, b.z, b.xy, b.zx, b.yz, b.xyz)
Wedge Product D = A^(A^B)
D.q   =  + 0
D.x   =  + 0
D.y   =  + 0
D.z   =  + 0
D.xy  =  + 0
D.zx  =  + 0
D.yz  =  + 0
D.xyz =  + 2*a.x*a.yz*b.q + 2*a.y*a.zx*b.q + 2*a.z*a.xy*b.q

Wedge Product E = A^(A^(A^B)) = 0
```

Unit Scalar Wedge Product

The next specialization uses $a.q = 1$. We no longer have a simple closed system, but what we have is stubbornly linear. These formulas are split into two tables for typesetting convenience.

Differentials

My generic multivector operator in positive linear powers is

$$A = (a, dx, dy, dz, dx\ dy, dz\ dx, dy\ dz, dx\ dy\ dz)$$

My target multifield is

$$B = (b_q, b_x, b_y, b_z, b_{xy}, b_{zx}, b_{yz}, b_{xyz})$$

```
A = (1, a.x, a.y, a.z, a.xy, a.zx, a.yz, a.xyz)
B = (b.q, b.x, b.y, b.z, b.xy, b.zx, b.yz, b.xyz)
Wedge Product C = (A^B)
C.q   =  + b.q
C.x   =  + b.x   + a.x*b.q
C.y   =  + b.y   + a.y*b.q
C.z   =  + b.z   + a.z*b.q
C.xy  =  + b.xy  + a.xy*b.q  + a.x*b.y  - a.y*b.x
C.zx  =  + b.zx  + a.zx*b.q  + a.z*b.x  - a.x*b.z
C.yz  =  + b.yz  + a.yz*b.q  + a.y*b.z  - a.z*b.y
C.xyz =  + b.xyz + a.xyz*b.q + a.x*b.yz + a.y*b.zx + a.z*b.xy
                             + a.yz*b.x + a.zx*b.y + a.xy*b.z

A = (1, a.x, a.y, a.z, a.xy, a.zx, a.yz, a.xyz)
B = (b.q, b.x, b.y, b.z, b.xy, b.zx, b.yz, b.xyz)
Wedge Product D = A^(A^B)
D.q   =  + b.q
D.x   =  + b.x   + 2*a.x*b.q
D.y   =  + b.y   + 2*a.y*b.q
D.z   =  + b.z   + 2*a.z*b.q
D.xy  =  + b.xy  + 2*a.xy*b.q  + 2*a.x*b.y  - 2*a.y*b.x
D.zx  =  + b.zx  + 2*a.zx*b.q  + 2*a.z*b.x  - 2*a.x*b.z
D.yz  =  + b.yz  + 2*a.yz*b.q  + 2*a.y*b.z  - 2*a.z*b.y
D.xyz =  + b.xyz + 2*a.xyz*b.q + 2*a.x*b.yz + 2*a.y*b.zx + 2*a.z*b.xy
                               + 2*a.yz*b.x + 2*a.zx*b.y + 2*a.xy*b.z
         + 2*a.x*a.yz*b.q + 2*a.y*a.zx*b.q + 2*a.z*a.xy*b.q
```

Table 5: Unit Scalar Wedge Products for Powers One and Two

```
A = (1, a.x, a.y, a.z, a.xy, a.zx, a.yz, a.xyz)
B = (b.q, b.x, b.y, b.z, b.xy, b.zx, b.yz, b.xyz)
Wedge Product E = A^(A^(A^B))
E.q   =  + b.q
E.x   =  + b.x   + 3*a.x*b.q
E.y   =  + b.y   + 3*a.y*b.q
E.z   =  + b.z   + 3*a.z*b.q
E.xy  =  + b.xy  + 3*a.xy*b.q  + 3*a.x*b.y  - 3*a.y*b.x
E.zx  =  + b.zx  + 3*a.zx*b.q  + 3*a.z*b.x  - 3*a.x*b.z
E.yz  =  + b.yz  + 3*a.yz*b.q  + 3*a.y*b.z  - 3*a.z*b.y
E.xyz =  + b.xyz + 3*a.xyz*b.q + 3*a.x*b.yz + 3*a.y*b.zx + 3*a.z*b.xy
                               + 3*a.yz*b.x + 3*a.zx*b.y + 3*a.xy*b.z
         + 6*a.x*a.yz*b.q + 6*a.y*a.zx*b.q + 6*a.z*a.xy*b.q

A = (1, a.x, a.y, a.z, a.xy, a.zx, a.yz, a.xyz)
B = (b.q, b.x, b.y, b.z, b.xy, b.zx, b.yz, b.xyz)
Wedge Product F = A^(A^(A^(A^B)))
F.q   =  + b.q
F.x   =  + b.x   + 4*a.x*b.q
F.y   =  + b.y   + 4*a.y*b.q
F.z   =  + b.z   + 4*a.z*b.q
F.xy  =  + b.xy  + 4*a.xy*b.q  + 4*a.x*b.y  - 4*a.y*b.x
F.zx  =  + b.zx  + 4*a.zx*b.q  + 4*a.z*b.x  - 4*a.x*b.z
F.yz  =  + b.yz  + 4*a.yz*b.q  + 4*a.y*b.z  - 4*a.z*b.y
F.xyz =  + b.xyz + 4*a.xyz*b.q + 4*a.x*b.yz + 4*a.y*b.zx + 4*a.z*b.xy
                               + 4*a.yz*b.x + 4*a.zx*b.y + 4*a.xy*b.z
         + 12*a.y*a.zx*b.q + 12*a.z*a.xy*b.q + 12*a.x*a.yz*b.q
```

Table 6: Unit Scalar Wedge Products for Powers Three and Four

The first order generic wedge is

$$\begin{aligned}
A &= (a_q, dx, dy, dz, (dxdy), (dzdx), (dydz), (dxdydz)) \\
B &= (b_q, b_x, b_y, b_z, b_{xy}, b_{zx}, b_{yz}, b_{xyz}) \\
C &= (A \wedge B) \\
C_q &= a_q b_q \\
C_x &= a_q b_x + b_q(dx) \\
C_y &= a_q b_y + b_q(dy) \\
C_z &= a_q b_z + b_q(dz) \\
C_{xy} &= a_q b_{xy} + b_q(dxdy) + b_y(dx) - b_x(dy) \\
C_{zx} &= a_q b_{zx} + b_q(dzdx) + b_x(dz) - b_z(dx) \\
C_{yz} &= a_q b_{yz} + b_q(dydz) + b_z(dy) - b_y(dz) \\
C_{xyz} &= a_q b_{xyz} + b_q(dxdydz) + b_{yz}(dx) + b_{zx}(dy) + b_{xy}(dz) \\
&\quad + b_x(dydz) + b_y(dzdx) + b_z(dxdy)
\end{aligned}$$

Accumulating sums over these terms will have different domains, according to the actual differential at hand. However, no integrand is more complex than a volume element.

Jumping to the fourth order generic wedge, where the coefficient pattern is well established, we have

$$
\begin{aligned}
A &= (a_q, (dx), (dy), (dz), (dxdy), (dzdx), (dydz), (dxdydz)) \\
B &= (b_q, b_x, b_y, b_z, b_{xy}, b_{zx}, b_{yz}, b_{xyz}) \\
F &= A \wedge (A \wedge (A \wedge (A \wedge B))) \\
F_q &= +a_q^4 b_q \\
F_x &= +a_q^4 b_x + 4a_q^3 b_q (dx) \\
F_y &= +a_q^4 b_y + 4a_q^3 b_q (dy) \\
F_z &= +a_q^4 b_z + 4a_q^3 b_q (dz) \\
F_{xy} &= +a_q^4 b_{xy} + 4a_q^3 b_q (dxdy) + 4a_q^3 b_y (dx) - 4a_q^3 b_x (dy) \\
F_{zx} &= +a_q^4 b_{zx} + 4a_q^3 b_q (dzdx) + 4a_q^3 b_x (dz) - 4a_q^3 b_z (dx) \\
F_{yz} &= +a_q^4 b_{yz} + 4a_q^3 b_q (dydz) + 4a_q^3 b_z (dy) - 4a_q^3 b_y (dz) \\
F_{xyz} &= +a_q^4 b_{xyz} + 4a_q^3 b_q (dxdydz) + 4a_q^3 b_{yz} (dx) + 4a_q^3 b_{zx} (dy) + 4a_q^3 b_{xy} (dz) \\
&\quad +4a_q^3 b_x (dydz) + 4a_q^3 b_y (dzdx) + 4a_q^3 b_z (dxdy) \\
&\quad +12a_q^2 b_q (dy)(dzdx) + 12a_q^2 b_q (dz)(dxdy) + 12a_q^2 b_q (dx)(dydz)
\end{aligned}
$$

The last row of F_{xyz} has been left uncombined to leave the coefficients formula apparent.

Pure Wedge Product

If we specialize to the pure wedge product, our two expressions of interest are

$$
\begin{aligned}
A &= (0, dx, dy, dz, (dxdy), (dzdx), (dydz), (dxdydz)) \\
B &= (b_q, b_x, b_y, b_z, b_{xy}, b_{zx}, b_{yz}, b_{xyz}) \\
C &= (A \wedge B) \\
C_q &= 0 \\
C_x &= +b_q(dx) \\
C_y &= +b_q(dy) \\
C_z &= +b_q(dz) \\
C_{xy} &= +b_q(dxdy) + b_y(dx) - b_x(dy) \\
C_{zx} &= +b_q(dzdx) + b_x(dz) - b_z(dx) \\
C_{yz} &= +b_q(dydz) + b_z(dy) - b_y(dz) \\
C_{xyz} &= +b_q(dxdydz) + b_{yz}(dx) + b_{zx}(dy) + b_{xy}(dz) \\
&\quad +b_x(dydz) + b_y(dzdx) + b_z(dxdy)
\end{aligned}
$$

and

$$
\begin{aligned}
A &= (0, dx, dy, dz, (dxdy), (dzdx), (dydz), (dxdydz)) \\
B &= (b_q, b_x, b_y, b_z, b_{xy}, b_{zx}, b_{yz}, b_{xyz}) \\
D &= (A \wedge (A \wedge B)) \\
D_q &= 0 \\
D_x &= 0 \\
D_y &= 0 \\
D_z &= 0 \\
D_{xy} &= 0 \\
D_{zx} &= 0 \\
D_{yz} &= 0 \\
D_{xyz} &= +2b_q(dx)(dydz) + 2b_q(dy)(dzdx) + 2b_q(dz)(dxdy) = 6b_q(dxdydz)
\end{aligned}
$$

Unit Scalar Wedge Product

If we specialize to the unit scalar wedge product, our fourth order template becomes

$$
\begin{aligned}
A &= (1, (dx), (dy), (dz), (dxdy), (dzdx), (dydz), (dxdydz)) \\
B &= (b_q, b_x, b_y, b_z, b_{xy}, b_{zx}, b_{yz}, b_{xyz}) \\
F &= A \wedge (A \wedge (A \wedge (A \wedge B))) \\
F_q &= +b_q \\
F_x &= +b_x + 4b_q(dx) \\
F_y &= +b_y + 4b_q(dy) \\
F_z &= +b_z + 4b_q(dz) \\
F_{xy} &= +b_{xy} + 4b_q(dxdy) + 4b_y(dx) - 4b_x(dy) \\
F_{zx} &= +b_{zx} + 4b_q(dzdx) + 4b_x(dz) - 4b_z(dx) \\
F_{yz} &= +b_{yz} + 4b_q(dydz) + 4b_z(dy) - 4b_y(dz) \\
F_{xyz} &= +b_{xyz} + 4b_q(dxdydz) + 4b_{yz}(dx) + 4b_{zx}(dy) + 4b_{xy}(dz) \\
&\quad +4b_x(dydz) + 4b_y(dzdx) + 4b_z(dxdy) \\
&\quad +12b_q(dy)(dzdx) + 12b_q(dz)(dxdy) + 12b_q(dx)(dydz)
\end{aligned}
$$

Derivatives

Inverse units, such as m^{-1}, m^{-2}, and m^{-3} are preserved in the wedge product as well. The natural item of interest here are derivative and gradients. The inverse units for the B term are a natural for fit for potential between charges m^{-1}, stress m^{-2} and pressures m^{-3}.

The first order generic derivative is

$$
\begin{aligned}
A &= \left(a_q, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial^2}{\partial x \partial y}, \frac{\partial^2}{\partial z \partial x}, \frac{\partial^2}{\partial y \partial z}, \frac{\partial^3}{\partial x \partial y \partial z}\right) \\
B &= (b_q, b_x, b_y, b_z, b_{xy}, b_{zx}, b_{yz}, b_{xyz}) \\
C &= (A \wedge B) \\
C_q &= +a_q * b_q; \\
C_x &= +a_q * b_x + \frac{\partial b_q}{\partial x}; \\
C_y &= +a_q * b_y + \frac{\partial b_q}{\partial y}; \\
C_z &= +a_q * b_z + \frac{\partial b_q}{\partial z}; \\
C_{xy} &= +a_q * b_{xy} + \frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y} + \frac{\partial^2 b_q}{\partial x \partial y}; \\
C_{zx} &= +a_q * b_{zx} + \frac{\partial b_x}{\partial z} - \frac{\partial b_z}{\partial x} + \frac{\partial^2 b_q}{\partial z \partial x}; \\
C_{yz} &= +a_q * b_{yz} + \frac{\partial b_z}{\partial y} - \frac{\partial b_y}{\partial z} + \frac{\partial^2 b_q}{\partial y \partial z}; \\
C_{xyz} &= +a_q * b_{xyz} + \frac{\partial b_{yz}}{\partial x} + \frac{\partial b_{zx}}{\partial y} + \frac{\partial b_{xy}}{\partial z} + \frac{\partial^2 b_z}{\partial x \partial y} + \frac{\partial^2 b_y}{\partial z \partial x} + \frac{\partial^2 b_x}{\partial y \partial z} + \frac{\partial^3 b_q}{\partial x \partial y \partial z};
\end{aligned}
$$

Jumping to the fourth order generic derivative, we have

$$
\begin{aligned}
A &= \left(a_q, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial^2}{\partial x \partial y}, \frac{\partial^2}{\partial z \partial x}, \frac{\partial^2}{\partial y \partial z}, \frac{\partial^3}{\partial x \partial y \partial z}\right) \\
B &= (b_q, b_x, b_y, b_z, b_{xy}, b_{zx}, b_{yz}, b_{xyz}) \\
F &= (A \wedge (A \wedge (A \wedge (A \wedge B)))) \\
F_q &= +a_q^4 b_q \\
F_x &= +a_q^4 b_x + 4a_q^3 \frac{\partial}{\partial x} b_q \\
F_y &= +a_q^4 b_y + 4a_q^3 \frac{\partial}{\partial y} b_q \\
F_z &= +a_q^4 b_z + 4a_q^3 \frac{\partial}{\partial z} b_q \\
F_{xy} &= +a_q^4 b_{xy} + 4a_q^3 \frac{\partial^2}{\partial x \partial y} b_q + 4a_q^3 \frac{\partial}{\partial x} b_y - 4a_q^3 \frac{\partial}{\partial y} b_x \\
F_{zx} &= +a_q^4 b_{zx} + 4a_q^3 \frac{\partial^2}{\partial z \partial x} b_q + 4a_q^3 \frac{\partial}{\partial z} b_x - 4a_q^3 \frac{\partial}{\partial x} b_z \\
F_{yz} &= +a_q^4 b_{yz} + 4a_q^3 \frac{\partial^2}{\partial y \partial z} b_q + 4a_q^3 \frac{\partial}{\partial y} b_z - 4a_q^3 \frac{\partial}{\partial z} b_y \\
F_{xyz} &= +a_q^4 b_{xyz} + 4a_q^3 \frac{\partial^3}{\partial x \partial y \partial z} b_q \\
&\quad +4a_q^3 \frac{\partial}{\partial x} b_{yz} + 4a_q^3 \frac{\partial}{\partial y} b_{zx} + 4a_q^3 \frac{\partial}{\partial z} b_{xy} \\
&\quad +4a_q^3 \frac{\partial^2}{\partial y \partial z} b_x + 4a_q^3 \frac{\partial^2}{\partial z \partial x} b_y + 4a_q^3 \frac{\partial^2}{\partial x \partial y} b_z \\
&\quad +12a_q^2 \frac{\partial}{\partial y} \frac{\partial^2}{\partial z \partial x} b_q + 12a_q^2 \frac{\partial}{\partial z} \frac{\partial^2}{\partial x \partial y} b_q + 12a_q^2 \frac{\partial}{\partial x} \frac{\partial^2}{\partial y \partial z} b_q
\end{aligned}
$$

Unit Scalar Wedge Product

For the specialize case of the unit scalar $a_q = 1$, we have

$$
\begin{aligned}
A &= \left(1, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial^2}{\partial x \partial y}, \frac{\partial^2}{\partial z \partial x}, \frac{\partial^2}{\partial y \partial z}, \frac{\partial^3}{\partial x \partial y \partial z}\right) \\
B &= (b_q, b_x, b_y, b_z, b_{xy}, b_{zx}, b_{yz}, b_{xyz}) \\
F &= (A \wedge (A \wedge (A \wedge (A \wedge B)))) \\
F_q &= +b_q \\
F_x &= +b_x + 4\frac{\partial}{\partial x} b_q \\
F_y &= +b_y + 4\frac{\partial}{\partial y} b_q \\
F_z &= +b_z + 4\frac{\partial}{\partial z} b_q \\
F_{xy} &= +b_{xy} + 4\frac{\partial^2}{\partial x \partial y} b_q + 4\frac{\partial}{\partial x} b_y - 4\frac{\partial}{\partial y} b_x \\
F_{zx} &= +b_{zx} + 4\frac{\partial^2}{\partial z \partial x} b_q + 4\frac{\partial}{\partial z} b_x - 4\frac{\partial}{\partial x} b_z \\
F_{yz} &= +b_{yz} + 4\frac{\partial^2}{\partial y \partial z} b_q + 4\frac{\partial}{\partial y} b_z - 4\frac{\partial}{\partial z} b_y \\
F_{xyz} &= +b_{xyz} + 4\frac{\partial^3}{\partial x \partial y \partial z} b_q \\
&\quad +4\frac{\partial}{\partial x} b_{yz} + 4\frac{\partial}{\partial y} b_{zx} + 4\frac{\partial}{\partial z} b_{xy} \\
&\quad +4\frac{\partial^2}{\partial y \partial z} b_x + 4\frac{\partial^2}{\partial z \partial x} b_y + 4\frac{\partial^2}{\partial x \partial y} b_z \\
&\quad +12\frac{\partial}{\partial y}\frac{\partial^2}{\partial z \partial x} b_q + 12\frac{\partial}{\partial z}\frac{\partial^2}{\partial x \partial y} b_q + 12\frac{\partial}{\partial x}\frac{\partial^2}{\partial y \partial z} b_q
\end{aligned}
$$

Pure Wedge Product

If we specialize to the pure wedge product, our two expressions of interest are

$$\begin{aligned}
A &= \left(0, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial^2}{\partial x \partial y}, \frac{\partial^2}{\partial z \partial x}, \frac{\partial^2}{\partial y \partial z}, \frac{\partial^3}{\partial x \partial y \partial z}\right) \\
B &= (b_q, b_x, b_y, b_z, b_{xy}, b_{zx}, b_{yz}, b_{xyz}) \\
C &= (A \wedge B) \\
C_q &= 0 \\
C_x &= +\frac{\partial b_q}{\partial x} \\
C_y &= +\frac{\partial b_q}{\partial y} \\
C_z &= +\frac{\partial b_q}{\partial z} \\
C_{xy} &= +\frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y} + \frac{\partial^2 b_q}{\partial x \partial y} \\
C_{zx} &= +\frac{\partial b_x}{\partial z} - \frac{\partial b_z}{\partial x} + \frac{\partial^2 b_q}{\partial z \partial x} \\
C_{yz} &= +\frac{\partial b_z}{\partial y} - \frac{\partial b_y}{\partial z} + \frac{\partial^2 b_q}{\partial y \partial z} \\
C_{xyz} &= +\frac{\partial b_{yz}}{\partial x} + \frac{\partial b_{zx}}{\partial y} + \frac{\partial b_{xy}}{\partial z} + \frac{\partial^2 b_z}{\partial x \partial y} + \frac{\partial^2 b_y}{\partial z \partial x} + \frac{\partial^2 b_x}{\partial y \partial z} + \frac{\partial^3 b_q}{\partial x \partial y \partial z}
\end{aligned}$$

and

$$
\begin{aligned}
A &= \left(0, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial^2}{\partial x \partial y}, \frac{\partial^2}{\partial z \partial x}, \frac{\partial^2}{\partial y \partial z}, \frac{\partial^3}{\partial x \partial y \partial z}\right) \\
B &= (b_q, b_x, b_y, b_z, b_{xy}, b_{zx}, b_{yz}, b_{xyz}) \\
D &= (A \wedge (A \wedge B)) \\
D_q &= 0 \\
D_x &= 0 \\
D_y &= 0 \\
D_z &= 0 \\
D_{xy} &= 0 \\
D_{zx} &= 0 \\
D_{yz} &= 0 \\
D_{xyz} &= +2\frac{\partial^3}{\partial x \partial y \partial z} b_q + 2\frac{\partial^3}{\partial x \partial y \partial z} b_q + 2\frac{\partial^3}{\partial x \partial y \partial z} b_q = 6\frac{\partial^3}{\partial x \partial y \partial z} b_q
\end{aligned}
$$

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