Oriented 3D Geometric Algebra Multivectors

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Abstract

Three dimensional Euclidean geometric algebra can be implemented using 4x4 real matrices in six fundamentally unique representations. I first present these six representations, and note that all parts of a calculation need to use the same basis set for reliable results. This simple three space does not provide orientation independent of position. Such a orientation feature is necessary for particle electron models including magnetic moment.

Next, I look at the basis set in more detail, and show that these basis are part of a 4x4 implementation of a four dimensional Minkowski spacetime, with the Euclidean three space being the even rank subspace of the Minkowski space.

Finally, I look at this same set of matrices as a full, conventional three dimensional Euclidean algebra augmented by two sets of independent three vectors, with dedicated associated trivectors, sharing bivectors with the conventional Euclidean algebra.

3D Geometric Algebra Basis Product Table

	(+1)	(x)	(y)	(z)	(xy)	(zx)	(yz)	(xyz)
(+1)	(+1)	(x)	(y)	(z)	(xy)	(zx)	(yz)	(xyz)
(x)	(x)	(+1)	(xy)	(-zx)	(y)	(-z)	(xyz)	(yz)
(y)	(y)	(-xy)	(+1)	(yz)	(-x)	(xyz)	(z)	(zx)
(z)	(z)	(zx)	(-yz)	(+1)	(xyz)	(x)	(-y)	(xy)
(xy)	(xy)	(-y)	(x)	(xyz)	(-1)	(yz)	(-zx)	(-z)
(zx)	(zx)	(z)	(xyz)	(-x)	(-yz)	(-1)	(xy)	(-y)
(yz)	(yz)	(xyz)	(-z)	(y)	(zx)	(-xy)	(-1)	(-x)
(xyz)	(xyz)	(yz)	(zx)	(xy)	(-z)	(-y)	(-x)	(-1)

The Euclidean three dimensional geometric algebra basis truth table is shown on the previous page. Forefactors are the left column, while the top header are the postfactors. Vectors square to one, while bivectors and trivectors square to negative one. Vectors and bivectors anti-commute among themselves, while the trivector commutes with everything.

Six 4x4 Matrix Representations

Here are six sets of matrices implementing 3D Euclidean Geometric Algebra.

```
Set 1
      1            x            y            z
[ 1  0  0  0 ] [ 0 -1  0  0 ] [-1  0  0  0 ] [ 0  0  0  1 ]
[ 0  1  0  0 ] [-1  0  0  0 ] [ 0  1  0  0 ] [ 0  0  1  0 ]
[ 0  0  1  0 ] [ 0  0  0  1 ] [ 0  0 -1  0 ] [ 0  1  0  0 ]
[ 0  0  0  1 ] [ 0  0  1  0 ] [ 0  0  0  1 ] [ 1  0  0  0 ]

     xy           zx           yz           xyz
[ 0 -1  0  0 ] [ 0  0  1  0 ] [ 0  0  0 -1 ] [ 0  0 -1  0 ]
[ 1  0  0  0 ] [ 0  0  0  1 ] [ 0  0  1  0 ] [ 0  0  0  1 ]
[ 0  0  0  1 ] [-1  0  0  0 ] [ 0 -1  0  0 ] [ 1  0  0  0 ]
[ 0  0 -1  0 ] [ 0 -1  0  0 ] [ 1  0  0  0 ] [ 0 -1  0  0 ]


Set 2
      1            x            y            z
[ 1  0  0  0 ] [ 1  0  0  0 ] [ 0  1  0  0 ] [ 0  0 -1  0 ]
[ 0  1  0  0 ] [ 0 -1  0  0 ] [ 1  0  0  0 ] [ 0  0  0  1 ]
[ 0  0  1  0 ] [ 0  0 -1  0 ] [ 0  0  0  1 ] [-1  0  0  0 ]
[ 0  0  0  1 ] [ 0  0  0  1 ] [ 0  0  1  0 ] [ 0  1  0  0 ]

     xy           zx           yz           xyz
[ 0  1  0  0 ] [ 0  0  1  0 ] [ 0  0  0  1 ] [ 0  0  0  1 ]
[-1  0  0  0 ] [ 0  0  0  1 ] [ 0  0 -1  0 ] [ 0  0  1  0 ]
[ 0  0  0 -1 ] [-1  0  0  0 ] [ 0  1  0  0 ] [ 0 -1  0  0 ]
[ 0  0  1  0 ] [ 0 -1  0  0 ] [-1  0  0  0 ] [-1  0  0  0 ]
```

```
Set 3
       1               x               y               z
[ 1  0  0  0 ]  [ 0  0  1  0 ]  [ 0  0  0  1 ]  [-1  0  0  0 ]
[ 0  1  0  0 ]  [ 0  0  0  1 ]  [ 0  0 -1  0 ]  [ 0 -1  0  0 ]
[ 0  0  1  0 ]  [ 1  0  0  0 ]  [ 0 -1  0  0 ]  [ 0  0  1  0 ]
[ 0  0  0  1 ]  [ 0  1  0  0 ]  [ 1  0  0  0 ]  [ 0  0  0  1 ]

      xy              zx              yz              xyz
[ 0 -1  0  0 ]  [ 0  0 -1  0 ]  [ 0  0  0  1 ]  [ 0  1  0  0 ]
[ 1  0  0  0 ]  [ 0  0  0 -1 ]  [ 0  0 -1  0 ]  [-1  0  0  0 ]
[ 0  0  0  1 ]  [ 1  0  0  0 ]  [ 0  1  0  0 ]  [ 0  0  0  1 ]
[ 0  0 -1  0 ]  [ 0  1  0  0 ]  [-1  0  0  0 ]  [ 0  0 -1  0 ]
```

```
Set 4
       1               x               y               z
[ 1  0  0  0 ]  [ 0 -1  0  0 ]  [ 1  0  0  0 ]  [ 0  0  1  0 ]
[ 0  1  0  0 ]  [-1  0  0  0 ]  [ 0 -1  0  0 ]  [ 0  0  0  1 ]
[ 0  0  1  0 ]  [ 0  0  0  1 ]  [ 0  0 -1  0 ]  [ 1  0  0  0 ]
[ 0  0  0  1 ]  [ 0  0  1  0 ]  [ 0  0  0  1 ]  [ 0  1  0  0 ]

      xy              zx              yz              xyz
[ 0  1  0  0 ]  [ 0  0  0  1 ]  [ 0  0  1  0 ]  [ 0  0  0  1 ]
[-1  0  0  0 ]  [ 0  0  1  0 ]  [ 0  0  0 -1 ]  [ 0  0 -1  0 ]
[ 0  0  0  1 ]  [ 0 -1  0  0 ]  [-1  0  0  0 ]  [ 0  1  0  0 ]
[ 0  0 -1  0 ]  [-1  0  0  0 ]  [ 0  1  0  0 ]  [-1  0  0  0 ]
```

```
Set 5
       1               x               y               z
[ 1  0  0  0 ]  [-1  0  0  0 ]  [ 0  1  0  0 ]  [ 0  0  0  1 ]
[ 0  1  0  0 ]  [ 0  1  0  0 ]  [ 1  0  0  0 ]  [ 0  0 -1  0 ]
[ 0  0  1  0 ]  [ 0  0 -1  0 ]  [ 0  0  0  1 ]  [ 0 -1  0  0 ]
[ 0  0  0  1 ]  [ 0  0  0  1 ]  [ 0  0  1  0 ]  [ 1  0  0  0 ]

      xy              zx              yz              xyz
[ 0 -1  0  0 ]  [ 0  0  0  1 ]  [ 0  0 -1  0 ]  [ 0  0  1  0 ]
[ 1  0  0  0 ]  [ 0  0  1  0 ]  [ 0  0  0  1 ]  [ 0  0  0  1 ]
[ 0  0  0 -1 ]  [ 0 -1  0  0 ]  [ 1  0  0  0 ]  [-1  0  0  0 ]
[ 0  0  1  0 ]  [-1  0  0  0 ]  [ 0 -1  0  0 ]  [ 0 -1  0  0 ]
```

```
Set 6
       1              x              y              z
[ 1  0  0  0 ]  [ 0  0  0  1 ]  [ 0  0 -1  0 ]  [-1  0  0  0 ]
[ 0  1  0  0 ]  [ 0  0  1  0 ]  [ 0  0  0  1 ]  [ 0 -1  0  0 ]
[ 0  0  1  0 ]  [ 0  1  0  0 ]  [-1  0  0  0 ]  [ 0  0  1  0 ]
[ 0  0  0  1 ]  [ 1  0  0  0 ]  [ 0  1  0  0 ]  [ 0  0  0  1 ]

      xy             zx             yz             xyz
[ 0  1  0  0 ]  [ 0  0  0 -1 ]  [ 0  0 -1  0 ]  [ 0 -1  0  0 ]
[-1  0  0  0 ]  [ 0  0 -1  0 ]  [ 0  0  0  1 ]  [ 1  0  0  0 ]
[ 0  0  0  1 ]  [ 0  1  0  0 ]  [ 1  0  0  0 ]  [ 0  0  0  1 ]
[ 0  0 -1  0 ]  [ 1  0  0  0 ]  [ 0 -1  0  0 ]  [ 0  0 -1  0 ]
```

For implementing a 3D Euclidean geometric algebra, we can use any of these six representations, as long as we use the same set throughout the calculation.

Taken individually, each set of basis only uses eight of sixteen degrees of freedom associated with 4x4 matrices. The question arises: "Is there some unrecognized utility associated with the additional bits?"

A significant clue lies in the fact that the full set of unique matrices in the six sets above comes to sixteen matrices, each orthogonal to the others, which do cover all sixteen degrees of freedom associated with 4x4 real matrices.

4D Minkowski Geometric Algebra

The sixteen matrices form a unique representation for 4D Minkowski space-time with a signature (+,+,+,-).

Historically, I arrived at these matrices by numerically searching the 4x4 space looking for matrices which squared to unity or negative one. The numerical index from my software which identifies each matrix as a sixteen digit trinary number, and the nickname I use for the matrices, are shown above each matrix on the following page. Lower case nicknamed matrices square to one, while upper case matrices square to negative 1.

The nine unitary root matrices can be arranged in a 3x3 grid, where the three row elements anticommute among themselves, and the three column elements anticommute as well. In this fashion, the three rows and the three columns provide the six implementations for 3D Euclidean geometric algebra shown above.

The Sixteen Matrices

```
 35931560 (P1)
[  1  0  0  0]
[  0  1  0  0]
[  0  0  1  0]
[  0  0  0  1]

 16563328 (+a)  35813460 (+b)  22075240 (+c)  23108944 (+D)
[  0  1  0  0] [ -1  0  0  0] [  0  0  0  1] [  0  0  1  0]
[  1  0  0  0] [  0  1  0  0] [  0  0  1  0] [  0  0  0 -1]
[  0  0  0 -1] [  0  0 -1  0] [  0  1  0  0] [ -1  0  0  0]
[  0  0 -1  0] [  0  0  0  1] [  1  0  0  0] [  0  1  0  0]

 35812976 (+d)  26483560 (+e)  19933420 (+f)  22073728 (+E)
[  1  0  0  0] [  0  1  0  0] [  0  0  1  0] [  0  0  0 -1]
[  0 -1  0  0] [  1  0  0  0] [  0  0  0 -1] [  0  0 -1  0]
[  0  0 -1  0] [  0  0  0  1] [  1  0  0  0] [  0  1  0  0]
[  0  0  0  1] [  0  0  1  0] [  0 -1  0  0] [  1  0  0  0]

 23126440 (+g)  21012304 (+h)  35931072 (+i)  26129260 (+F)
[  0  0  1  0] [  0  0  0 -1] [ -1  0  0  0] [  0 -1  0  0]
[  0  0  0  1] [  0  0  1  0] [  0 -1  0  0] [  1  0  0  0]
[  1  0  0  0] [  0  1  0  0] [  0  0  1  0] [  0  0  0 -1]
[  0  1  0  0] [ -1  0  0  0] [  0  0  0  1] [  0  0  1  0]

 22035820 (+C)  23122048 (+B)  26129104 (+A)
[  0  0  0 -1] [  0  0 -1  0] [  0  1  0  0]
[  0  0  1  0] [  0  0  0 -1] [ -1  0  0  0]
[  0 -1  0  0] [  1  0  0  0] [  0  0  0 -1]
[  1  0  0  0] [  0  1  0  0] [  0  0  1  0]
```

The nine lower case matrices square to unity. The six upper case matrices square to negative one. We can arrange the nine unity roots as shown above in a 3x3 grid, such that the triple products across and down result in the imaginary roots shown right and below.

In nickname form, this multiplication format is shown below.

```
(+a)(+b)(+c)  |  (+D)
(+d)(+e)(+f)  |  (+E)
(+g)(+h)(+i)  |  (+F)
---------------------
(+C)(+B)(+A)
```

The full multiplication table among these elements is

Products Among the 16

	(P1)	(+a)	(+b)	(+c)	(+d)	(+e)	(+f)	(+g)	(+h)	(+i)	(+A)	(+B)	(+C)	(+D)	(+E)	(+F)
(P1)	(P1)	(+a)	(+b)	(+c)	(+d)	(+e)	(+f)	(+g)	(+h)	(+i)	(+A)	(+B)	(+C)	(+D)	(+E)	(+F)
(+a)	(+a)	(P1)	(+A)	(-B)	(+F)	(-i)	(+h)	(-E)	(+f)	(-e)	(+b)	(-c)	(+D)	(+C)	(-g)	(+d)
(+b)	(+b)	(-A)	(P1)	(+C)	(+i)	(+F)	(-g)	(-f)	(-E)	(+d)	(-a)	(+D)	(+c)	(+B)	(-h)	(+e)
(+c)	(+c)	(+B)	(-C)	(P1)	(-h)	(+g)	(+F)	(+e)	(-d)	(-E)	(+D)	(+a)	(-b)	(+A)	(-i)	(+f)
(+d)	(+d)	(-F)	(+i)	(-h)	(P1)	(+A)	(-B)	(+D)	(-c)	(+b)	(+e)	(-f)	(+E)	(+g)	(+C)	(-a)
(+e)	(+e)	(-i)	(-F)	(+g)	(-A)	(P1)	(+C)	(+c)	(+D)	(-a)	(-d)	(+E)	(+f)	(+h)	(+B)	(-b)
(+f)	(+f)	(+h)	(-g)	(-F)	(+B)	(-C)	(P1)	(-b)	(+a)	(+D)	(+E)	(+d)	(-e)	(+i)	(+A)	(-c)
(+g)	(+g)	(+E)	(-f)	(+e)	(-D)	(+c)	(-b)	(P1)	(+A)	(-B)	(+h)	(-i)	(+F)	(-d)	(+a)	(+C)
(+h)	(+h)	(+f)	(+E)	(-d)	(-c)	(-D)	(+a)	(-A)	(P1)	(+C)	(-g)	(+F)	(+i)	(-e)	(+b)	(+B)
(+i)	(+i)	(-e)	(+d)	(+E)	(+b)	(-a)	(-D)	(+B)	(-C)	(P1)	(+F)	(+g)	(-h)	(-f)	(+c)	(+A)
(+A)	(+A)	(-b)	(+a)	(+D)	(-e)	(+d)	(+E)	(-h)	(+g)	(+F)	(M1)	(+C)	(-B)	(-c)	(-f)	(-i)
(+B)	(+B)	(+c)	(+D)	(-a)	(+f)	(+E)	(-d)	(+i)	(+F)	(-g)	(-C)	(M1)	(+A)	(-b)	(-e)	(-h)
(+C)	(+C)	(+D)	(-c)	(+b)	(+E)	(-f)	(+e)	(+F)	(-i)	(+h)	(+B)	(-A)	(M1)	(-a)	(-d)	(-g)
(+D)	(+D)	(+C)	(+B)	(+A)	(-g)	(-h)	(-i)	(+d)	(+e)	(+f)	(-c)	(-b)	(-a)	(M1)	(-F)	(+E)
(+E)	(+E)	(+g)	(+h)	(+i)	(+C)	(+B)	(+A)	(-a)	(-b)	(-c)	(-f)	(-e)	(-d)	(+F)	(M1)	(-D)
(+F)	(+F)	(-d)	(-e)	(-f)	(+a)	(+b)	(+c)	(+C)	(+B)	(+A)	(-i)	(-h)	(-g)	(-E)	(+D)	(M1)

The subgroups of the eight 3D space matrices inside the sixteen matrices above are found to be even grade elements of a Minkowski 4D geometric algebra.

In symbolic form, a Minkowski 4D geometric algebra has a scalar, four vectors (x,y,z,t), six bivectors (xy, zx, yz, xt, yt, zt), four trivectors (xyz, xyt, xzt, yzt), and a trivector (xyzt). The novelty, is that vectors (x, y, z) square to one, while vector t squares to negative one. A consequence of the space time split, is that three bivectors xy, yz and zx square to negative one, as in 3D, while the time related bivectors xt, yt, and zt square to positive one, mimicing 3D linear vectors. Likewise, in the trivectors, xyz squares to negative one, while xyt, xzt and yzt square to positive one.

In mapping the sixteen matrices to the Minkowski geometric algebra, we have six choices for the x,y,z components using one the the 3x3 row or columns. For each x,y,z set, we have a choice of two anticommuting negative unity roots, giving the twelve implementations listed on the next page.

```
Twelve Assignments Generating Minkowski Algebra

              x y z t
              -------
              a b c E
              a b c F
              d e f D
              d e f F
              g h i D
              g h i E
              a d g B
              a d g A
              b e h C
              b e h A
              c f i B
              c f i C
```

In all these cases, the even grade subspace P1, xt, yt, zt, xy, yz, zx, xyzt maps into 3D Euclidean space. I mention in passing that the metric (+,+,+,-) maps into 4x4 real matrices, but the metric (-,-,-,+) does not.

This interpretation of the 4x4 matrices as a Minkowski space is similar to Hestenes and Doran's spacetime algebra, though using the (+,+,+,-) metric.

The idea of our three dimensional experience being a spinor subspace of Minkowski spacetime is nice. However, this is not the only way to interpret these matrices.

Internal Orientation and 3D

When working with these matrices, I cannot help but be impressed by the high degree of symmetry among unity root triad/negative root sets of four. The table below shows a multiplication process across and down, based upon Minkowski spacetime symbols. The upper left nine elements square to one, while the top right three elements, and bottom left three elements square to negative one. As we multiply across the rows, we see vectors, bivectors and trivectors. However, as we assign matrices to these representations, there is nothing tying a specific implementation to a type of element in the table.

```
  x   y   z -  xyz
 tx  ty  tz - txyz
xyt zxt yzt -    t
-----------------
 xy  zx  yz -   -1
```

I can, for example, assign matrices (a b c D) to (x y z xyz), or to (tx ty tz txyz), or to (xyt zxt yzt t), and things just work. Knowing that the matrices do not care about the role to which they are assigned, raises the question, can all the nine unitary roots be assigned to spatial duty. Consider the following table.

```
 x    y    z - xyz
 p    r    s - prs
 u    v    w - uvw
-----------------
xpr  yrv  zsw
```

As I look at this table, I can envision two implementations. One implementation really uses nine linear dimensions, with perhaps six compactified. The better implementation, I think, re-interprets these numbers as differences between three particle coordinates, such as A-B, B-C, C-A sets of differences. I can also see a system where x, y, z are the core of the particle, and (p,r,s) and (u,v,w) are deltas from this central position. I am guided in this expectation by one of Julian Schwinger's electron models, which had a electric charge bound to a literal magnetic dipole, being a +g magnetic charge and -g magnetic charge. The two magnetic charges define a spin axis and magnetic dipole moment. In this spirit, I can see a sixteen component multivector which incorporates a 3D Euclidean multivector, augmented by two delta sets of terms.

$$\begin{aligned} M &= q + xe_x + ye_y + ze_z + ae_{xy} + be_{zx} + ce_{yz} + ge_{xyz} \\ &\quad + pe_p + re_r + se_s + he_{prs} + ue_u + ve_v + we_w + je_{uvw} \end{aligned}$$

In this analogy, in the expression above, the first line of the multivector expresses the charge state, while the second line gives separation and moments for the two monopoles.

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