From Ellipse to Ogg

Kurt Nalty

August 16, 2014

Abstract

The ellipse is commonly presented in parametric form using sines and cosines. It is easy, but wrong, to assume that this is related to polar format. This informal note presents the parametric ellipse, then shows the conversion to CRC format polar form, then to double angle polar form. From the double angle polar form of the ellipse, I introduce the closely related Ogg curve, which has connections to the harmonic-geometric mean, and elliptic integrals of the second kind.

Stretched Circle Representation for the Ellipse

The most common parametric representation for an ellipse with major diameter $2a$ along the x axis,and minor diameter $2b$ along the y axis is

$$\begin{aligned} x &= a\cos\theta \\ y &= b\sin\theta \end{aligned}$$

This formula is easy to remember, easy to implement and works well. However, θ is not the polar angle, but truly a parameter used to generate the curve.

Polar Representation for the Ellipse

To get the polar representation for the ellipse, we take the above formulas, and find the expression for the polar angle from the coordinates x and y.

$$\begin{aligned}
\tan\phi &= \frac{y}{x} \\
&= \frac{b\sin\theta}{a\cos\theta} \\
&= \frac{b}{a}\tan\theta \\
\phi &= \tan^{-1}\left(\frac{b}{a}\tan\theta\right) \quad \text{and} \\
\theta &= \tan^{-1}\left(\frac{a}{b}\tan\phi\right)
\end{aligned}$$

Figure 1 shows the polar angle against the parametric angle for a variety of ratios for a and b. We see that the end points 0 and $\pi/2$ align, but the intermediate portions of the curves deviate significantly from a line.

To express our equations in polar form, we will need to take sines and cosines of inverse tangents. Here is a short derivation of the formulas we will need.

$$\begin{aligned}
\alpha &= \tan\left(\tan^{-1}(\alpha)\right) \\
&= \frac{\sin\left(\tan^{-1}(\alpha)\right)}{\cos\left(\tan^{-1}(\alpha)\right)} \\
\alpha^2 &= \frac{\sin^2\left(\tan^{-1}(\alpha)\right)}{\cos^2\left(\tan^{-1}(\alpha)\right)} \\
&= \frac{1-\cos^2\left(\tan^{-1}(\alpha)\right)}{\cos^2\left(\tan^{-1}(\alpha)\right)} \\
\alpha^2+1 &= \frac{1}{\cos^2\left(\tan^{-1}(\alpha)\right)} \\
\cos\left(\tan^{-1}(\alpha)\right) &= \sqrt{\frac{1}{1+\alpha^2}} \\
\sin\left(\tan^{-1}(\alpha)\right) &= \sqrt{\frac{\alpha^2}{1+\alpha^2}}
\end{aligned}$$

Ellipse Angles.png

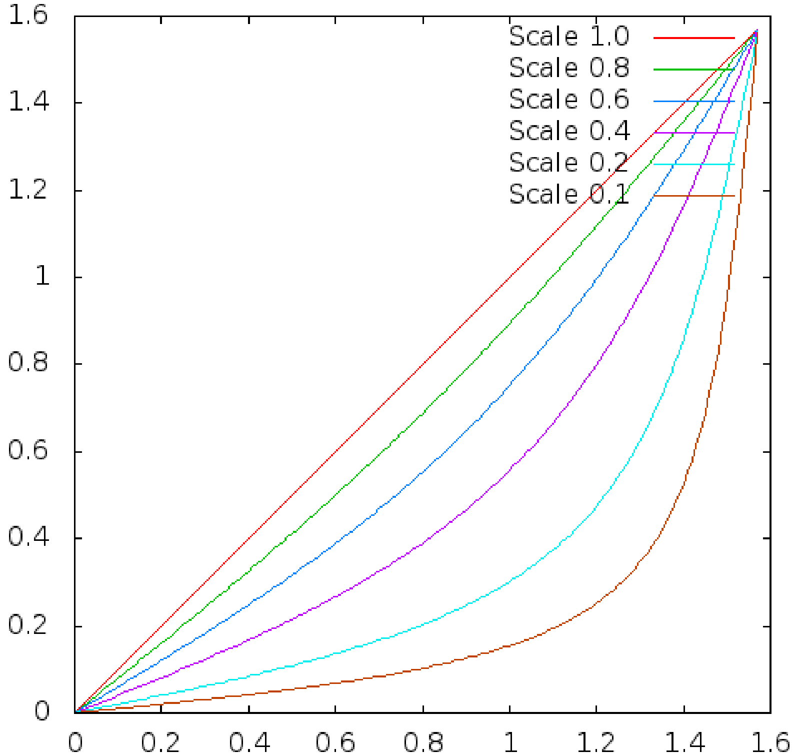

Figure 1: Comparision of Polar versus Parametric Angle

Starting with our parametric form, we have

$$\begin{aligned}
x &= a\cos\theta \\
&= a\cos\left(\tan^{-1}\left(\frac{a}{b}\tan\phi\right)\right) \\
&= a\sqrt{\frac{1}{1+\left(\frac{a}{b}\tan\phi\right)^2}} \\
&= a\sqrt{\frac{b^2}{b^2+a^2\tan^2\phi}} \\
&= a\sqrt{\frac{b^2\cos^2\phi}{b^2\cos^2\phi+a^2\sin^2\phi}} \\
&= \frac{ab\cos\phi}{\sqrt{a^2\sin^2\phi+b^2\cos^2\phi}}
\end{aligned}$$

From the polar format definitions of $x = r\cos\phi$ and $y = r\sin\phi$, we recognize

$$\begin{aligned}
x &= \frac{ab\cos\phi}{\sqrt{a^2\sin^2\phi+b^2\cos^2\phi}} \\
y &= \frac{ab\sin\phi}{\sqrt{a^2\sin^2\phi+b^2\cos^2\phi}} \\
r(\phi) &= \frac{ab}{\sqrt{a^2\sin^2\phi+b^2\cos^2\phi}}
\end{aligned}$$

This is the square root of the format on page 203 of the 27th edition of the CRC Standard Math Tables, so I commonly call this the CRC formula.

Double Angle Format for Polar Form

Seeing the square of the sines and cosines above, we see we can do a double angle subsitution in the above equation.

$$\begin{aligned}
\sin^2\phi &= \frac{1}{2} - \frac{\cos(2\phi)}{2} \\
\cos^2\phi &= \frac{1}{2} + \frac{\cos(2\phi)}{2} \\
r(\phi) &= \frac{ab}{\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}} \\
&= \frac{ab\sqrt{2}}{\sqrt{a^2\left(1-\cos\left(2\phi\right)\right) + b^2\left(1+\cos\left(2\phi\right)\right)}} \\
r(\phi) &= \frac{ab\sqrt{2}}{\sqrt{(a^2+b^2) - (a^2-b^2)\cos(2\phi)}} \quad \text{Standard Ellipse}
\end{aligned}$$

We know from the shape of the ellipse, as well as this formula above, that the radius completes two excursions out and in during one 360 degree cycle.

Polar Representation for the Ogg Curve

Doing integrals with the elliptic curves, we can often use the symmetry of the ellipse to change variables eliminating the double frequency term from above. The resulting curve is no longer an ellipse, but due to the egg shape, is called an Ogg curve, facetiously attributed to James Ogg (1792-1744).

$$r(\phi) = \frac{ab\sqrt{2}}{\sqrt{(a^2+b^2) - (a^2-b^2)\cos(\phi)}} \quad \text{Ogg Curve}$$

Figure 2 shows a 3:1 ellipse, associated Ogg curve, as well as unity radius circle.

With the Ogg curve, the right most x coordinate is a, the left most x coordinate is $-b$. In the derivation that follows, we will see the maximum and minimum y coordinates are $\pm(2ab)/(a+b)$, which is the harmonic mean of a and b.

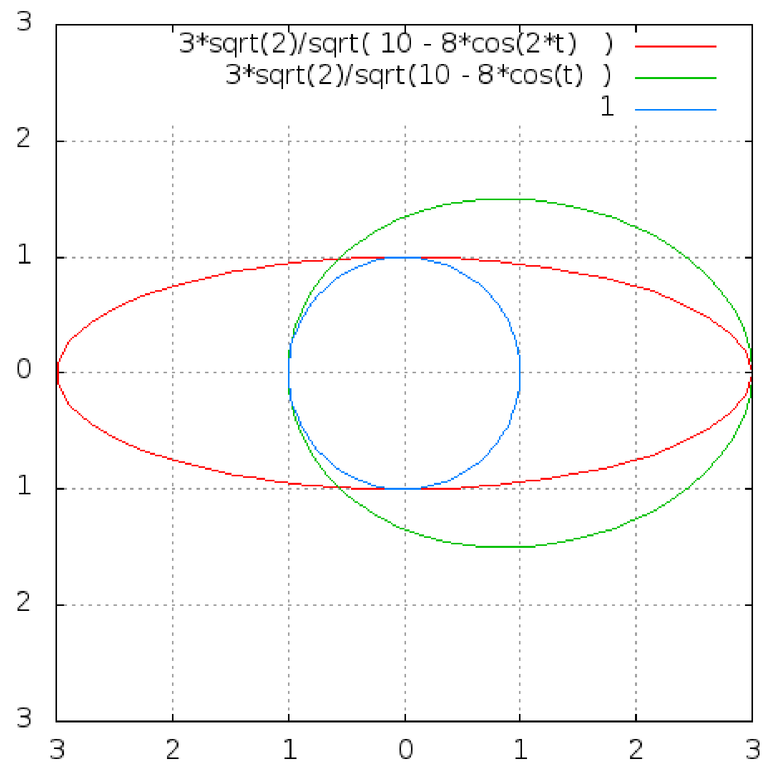

Figure 2: Ellipse and Ogg Curve with $a = 3$ and $b = 1$

Ogg curve summary:

$$
\begin{aligned}
x_{\text{max}} &= a \\
x_{\text{min}} &= -b \\
x_{\text{span}} &= 2\left(\frac{a+b}{2}\right) \qquad \text{twice arithmetic mean} \\
y_{\text{max}} &= \left(\frac{2ab}{a+b}\right) \\
y_{\text{min}} &= -\left(\frac{2ab}{a+b}\right) \\
y_{\text{span}} &= 2\left(\frac{2ab}{a+b}\right) \qquad \text{twice harmonic mean} \\
\phi_{\text{ymax}} &= \cos^{-1}\left(\frac{a-b}{a+b}\right)
\end{aligned}
$$

Derivation of Y_{max} for Ogg Curve

We begin with the definition of y for the polar form.

$$
\begin{aligned}
y &= r\sin\phi \\
&= \frac{ab\sqrt{2}\sin\phi}{\sqrt{(a^2+b^2)-(a^2-b^2)\cos(\phi)}}
\end{aligned}
$$

The maximum of y is also at the maximum of y^2, and our math is much cleaner.

$$
y^2 = \frac{2a^2b^2\sin^2\phi}{(a^2+b^2)-(a^2-b^2)\cos(\phi)}
$$

Take the derivative with respect to ϕ, and set to zero.

$$
\begin{aligned}
y^2 &= \frac{2a^2b^2\sin^2\phi}{(a^2+b^2)-(a^2-b^2)\cos(\phi)} \\
\frac{\partial(y^2)}{\partial\phi} &= \frac{4a^2b^2\sin\phi\cos\phi}{(a^2+b^2)-(a^2-b^2)\cos(\phi)} \\
&\quad - \frac{2a^2b^2\sin^2\phi}{\left[(a^2+b^2)-(a^2-b^2)\cos(\phi)\right]^2}\big(-(a^2-b^2)\left(-\sin\left(\phi\right)\right)\big) = 0
\end{aligned}
$$

We have

$$\frac{4a^2b^2\sin\phi\cos\phi}{(a^2+b^2)-(a^2-b^2)\cos(\phi)} = \frac{2a^2b^2(a^2-b^2)\sin^3\phi}{\left[(a^2+b^2)-(a^2-b^2)\cos(\phi)\right]^2}$$

Cancelling common factors, we get

$$\begin{aligned}
2\cos\phi &= \frac{(a^2-b^2)\sin^2\phi}{\left[(a^2+b^2)-(a^2-b^2)\cos(\phi)\right]} \\
2\cos\phi &= \frac{(a^2-b^2)(1-\cos^2\phi)}{\left[(a^2+b^2)-(a^2-b^2)\cos(\phi)\right]} \\
2\cos\phi\big[(a^2+b^2)-(a^2-b^2)\cos(\phi)\big] &= (a^2-b^2)(1-\cos^2\phi) \\
2\cos\phi(a^2+b^2)-2(a^2-b^2)\cos^2(\phi) &= (a^2-b^2)-(a^2-b^2)\cos^2\phi \\
2(a^2+b^2)\cos\phi-(a^2-b^2)\cos^2(\phi) &= (a^2-b^2)
\end{aligned}$$

Collecting powers of $\cos\phi$, we get

$$\cos^2(\phi) - 2\left(\frac{a^2+b^2}{a^2-b^2}\right)\cos(\phi) + 1 = 0$$

which has solutions

$$\cos\phi = \frac{a \pm b}{a \mp b}$$

Assuming $a > b$, we choose

$$\begin{aligned}
\cos\phi_{\text{ymax}} &= \frac{a-b}{a+b} \\
\sin\phi_{\text{ymax}} &= \frac{2\sqrt{ab}}{a+b}
\end{aligned}$$

We find our radius at y_{max} to be

$$\begin{aligned}
r(\phi_{\text{ymax}}) &= \frac{\sqrt{2}ab}{\sqrt{(a^2+b^2)-(a^2-b^2)\left(\frac{a-b}{a+b}\right)}} \\
&= \frac{\sqrt{2}ab}{\sqrt{(a^2+b^2)-(a+b)(a-b)\left(\frac{a-b}{a+b}\right)}} \\
&= \frac{\sqrt{2}ab}{\sqrt{(a^2+b^2)-(a-b)^2}} \\
&= \frac{\sqrt{2}ab}{\sqrt{2ab}} \\
&= \sqrt{ab}
\end{aligned}$$

We get the result

$$\begin{aligned}
y_{\text{max}} &= r\sin\phi \\
&= \sqrt{ab}\left(\frac{2\sqrt{ab}}{a+b}\right) \\
y_{\text{max}} &= \frac{2ab}{a+b}
\end{aligned}$$

This value for y is the harmonic mean of a and b.

We have the result, that the x span for the Ogg curve is twice the arithmetic mean of a and b, and that the y span is twice the harmonic mean of a and b, and the radius to y_{max} is the geometric mean of a and b.

The Ogg Curve is Not a Cartesian Oval

The polar format for the Ogg curve is

$$\text{Ogg}(a,b,\theta) = \frac{ab\sqrt{2}}{\sqrt{(a^2+b^2)-(a^2-b^2)\cos(\theta)}} \quad \text{Ogg Curve}$$

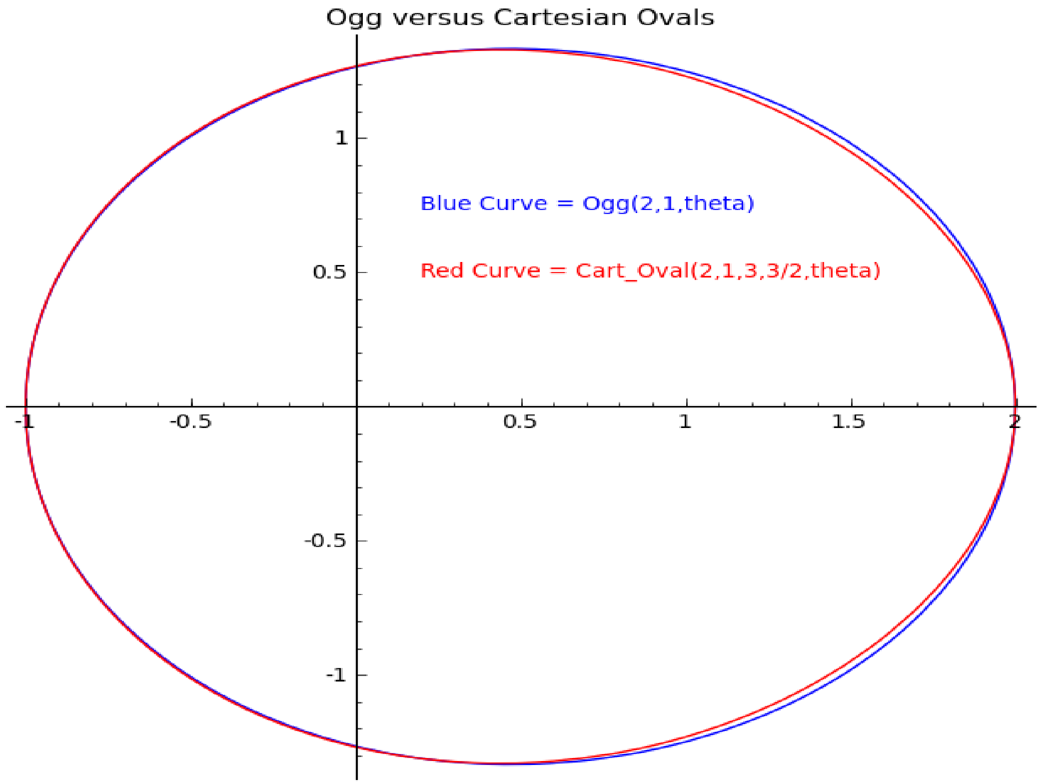

Figure 3: Close Match, but not Perfect

The polar format for the interior Cartesian Oval is

$$\text{Cart\_Oval}(a, b, c, d, \theta) = d\left[\frac{(a/c) - (b/c)^2\cos(\theta) - (b/c)\sqrt{1 + (a/c)^2 - 2(a/c)\cos(\theta) - (b/c)^2\sin^2(\theta)}}{(a/c)^2 - (b/c)^2}\right]$$

At casual glance, the Ogg curves appears to be a Cartesian Oval. Figure 3 illustrates the close fit between Ogg(2,1,theta) and Cart_Oval(2,1,3,1.5,theta). This is the closest fit I was able to get manually. Small changes in any parameter increase the mismatch.

The equations for the two curves do not match, but it is easy to be fooled by appearances when there is x or y translation of a polar curve. (As an example, checkout the polar equation for a circle centered at the origin, versus a circle tangent to the origin.)

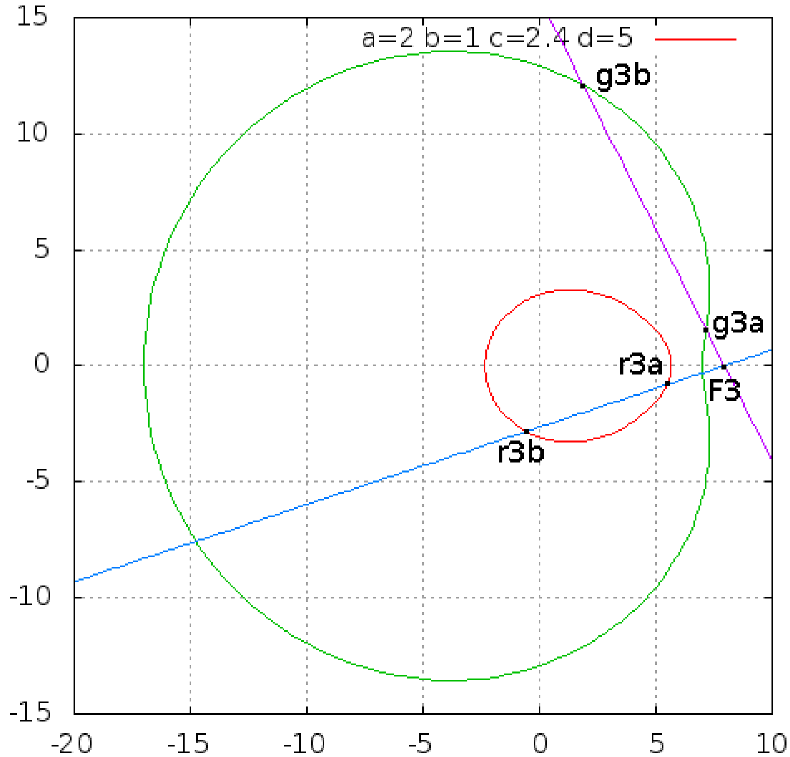

Figure 4: For Constant Product Relationship for F3, use both radii on the same curve.

To determine that the Ogg curve is not a special case of the Cartesian oval, I used the property of the third focus of the of Cartesian oval. A line from the third focus piercing or tangent to the Cartesian oval has a fixed constant for the product of the line segments from the focus to the intersections of the curve. Figure 4 illustrates this product with F3, r3a and r3b.

For the counter example, I drew an Ogg curve, and extended a tangent from the curve to the x axis, increasing x until the length of the tangent squared was (F3-a)*(F3+b), which is the constant derived from the pierce along the x axis. I then chose another point on the curve, calculated the distance to this point from F3, and calculated the distance expected for the second penetration based upon the F3 focus. Drawing this line, proved no fit. This is illustrated by the slate lines in Figure 5.

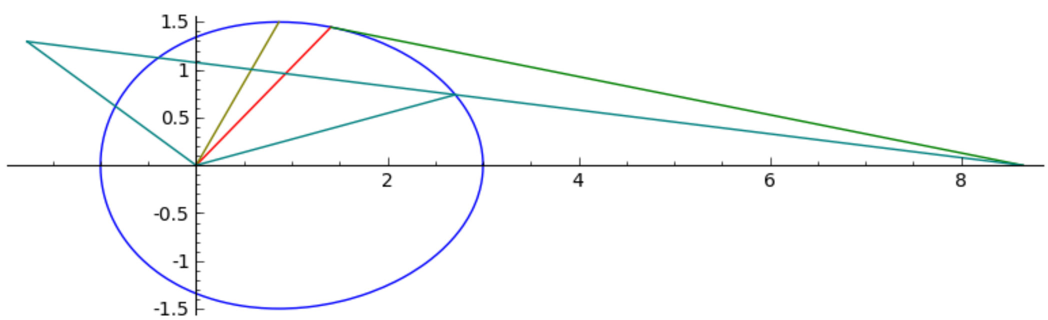

Figure 5: Calculated Second Pierce Does Not Align.