Nilpotents and Idempotents in 2D and 3D Euclidean Geometric Algebra

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Abstract

I present general homework level formulas for nilpotents (non-zero expressions which square to zero) and idempotents (expressions which square to themselves) in 2D and 3D Euclidean Geometric Algebras. I present both periodic as well as hyperbolic forms for both nilpotents and idempotents. In 2D, the generic nilpotent z and idempotent $p_{\pm}$ for unit vector $\vec{u}$ and unit bivector U is

$$\begin{aligned} z &= be_x + ce_y \pm \sqrt{(b^2+c^2)}e_xe_y \\ p_{\pm} &= \frac{1}{2} \pm \frac{1}{2}\left(\vec{u}\cosh(t) + \sinh(t)\, e_xe_y\right) \end{aligned}$$

Similarly in 3D, the generic nilpotent z and idempotents $p_{\pm}$ are

$$\begin{aligned} z &= k(\vec{u}+U), \qquad \text{with} \quad \vec{u}U + U\vec{u} = 0 \\ p_{\pm} &= \frac{1}{2} \pm \frac{1}{2}\left(\vec{u}\cosh(t) + U\sinh(t)\right) \end{aligned}$$

Specific periodic formats are

$$\begin{aligned} \vec{u}\vec{v} &= -\vec{v}\vec{u} \qquad (\vec{u} \perp \vec{v}) \\ z_{\pm} &= \frac{1}{2}\left(\vec{u}\sin\theta + \vec{v}\cos\theta \pm \vec{u}\vec{v}\right) \\ p_{\pm} &= \frac{1}{2}\left(1 \pm \vec{u}\sin\theta \pm \vec{v}\cos\theta\right) \end{aligned}$$

2D Euclidean Geometric Algebra Basis

Start with a two dimensional, Euclidean geometric algebra. We have two vector directions, e_x and e_y which correspond to our standard x and y directions. Our geometric multivector elements are scalars, vectors e_x and e_y, and a single bivector $e_x \wedge e_y = e_{xy} = e_x e_y$.

In this algebra, scalar multiplication is commutative and associative, vectors square to scalar one, and the product of two vectors resulting in a bivector is anti-commutative, associative and squares to negative one. The order sensitive multiplication table for this algebra is shown in the following table at the top of the next page.

	1	e_x	e_y	$e_x e_y$
1	1	e_x	e_y	$e_x e_y$
e_x	e_x	1	$e_x e_y$	e_y
e_y	e_y	$-e_x e_y$	1	$-e_x$
$e_x e_y$	$e_x e_y$	$-e_y$	e_x	-1

Given this notation, the generic 2D multivector can be written as

$$g = a + be_x + ce_y + de_x e_y$$

The square of this generic 2D multivector is

$$\begin{aligned} g^2 = gg &= a^2 + abe_x + ace_y + ade_x e_y \\ &\quad be_x a + be_x be_x + be_x ce_y + be_x de_x e_y \\ &\quad ce_y a + ce_y be_x + ce_y ce_y + ce_y de_x e_y \\ &\quad de_x e_y a + de_x e_y be_x + de_x e_y ce_y + de_x e_y de_x e_y \end{aligned}$$

2D Euclidean Square

Using our table above, and collecting common factors by grade, we have

$$\begin{aligned} g^2 = gg &= (a^2 + b^2 + c^2 - d^2) \\ &\quad + e_x(2ab) \\ &\quad + e_y(2ac) \\ &\quad + e_x e_y(2ad) \end{aligned}$$

2D Nilpotents

For the nilpotent z, which square to zero, we see the easy solution $a = 0$, $d^2 = b^2 + c^2$.

$$z = be_x + ce_y \pm \sqrt{(b^2 + c^2)}e_x e_y$$

2D Idempotent

Our square is

$$\begin{aligned} g^2 = gg &= (a^2 + b^2 + c^2 - d^2) \\ &\quad +e_x(2ab) \\ &\quad +e_y(2ac) \\ &\quad +e_x e_y(2ad) \end{aligned}$$

Idempotents square to themselves. As these are projection operators, I will call the generic idempotent p. We see a very easy solution for idempotents p, where $a = 1/2$, and $b^2 + c^2 - d^2 = 1/4$, or equivalently, $d = \pm\sqrt{b^2 + c^2 - 0.25}$. We see we have two independent numbers in this solution.

Our generic 2D Euclidean geometry idempotent is

$$\begin{aligned} p &= 0.5 + be_x + ce_y \pm \sqrt{(b^2 + c^2 - 0.25)}e_x e_y \\ &= \frac{1}{2}\left(1 + 2be_x + 2ce_y \pm \sqrt{(4b^2 + 4c^2 - 1)}e_x e_y\right) \end{aligned}$$

There are several specialized forms for this idempotent. Garret Sobczyk chose $c = 0, b = \pm 1/2$ to obtain two mutually annihilating idempotents, $u_+ = (1 + e_x)/2$ and $e_- = (1 - e_x)/2$. Another interesting solution is to set all values to 1/2, as in $p = (1 + e_x + e_y + e_x e_y)/2$. The main point here is that we have a rich set of projection operators, not simply restricted to number plus vector. A particularly nice form, in my opinion, is the geometric representation for arbitrary vector $\vec{v}$.

$$p_+ = \left(\frac{1}{2} + \vec{v}\right) \pm e_x e_y \sqrt{v^2 - \left(\frac{1}{2}\right)^2}$$

This idempotent has the related annihilator idempotent

$$p_- = \left(\frac{1}{2} - \vec{v}\right) \mp e_x e_y \sqrt{v^2 - \left(\frac{1}{2}\right)^2}$$

As foreshadowing, I want to point out that vectors and the bivector are mutually perpendicular in 2D. $(B\vec{v} + \vec{v}B)/2 = 0$.

Using hyperbolic functions to implement these formulas (per suggestion of Nicolas Le Bihan), we have

$$p_{\pm} = \frac{1}{2} \pm \frac{1}{2}\left(\vec{u}\cosh(t) + \sinh(t)\, e_x e_y\right)$$

where $\vec{u} = \vec{v}/|v|$, and the $\pm$ associated with the bivector term has been absorbed into the odd function $\sinh(t)$.

3D Euclidean Geometric Algebra Basis

Three dimensional Euclidean geometrical algebra has a scalar (1), three vectors (e_x, e_y and e_z), three bivectors (e_xe_y, e_ze_x, and e_ye_z), and one trivector ($e_xe_ye_z$) defining the geometry. In multiplication table format, the order-sensitive multiplication among these elements, with prefactors on the left column and postfactors on top row, is

	1	e_x	e_y	e_z	e_xe_y	e_ze_x	e_ye_z	$e_xe_ye_z$
1	1	e_x	e_y	e_z	e_xe_y	e_ze_x	e_ye_z	$e_xe_ye_z$
e_x	e_x	1	e_xe_y	-e_ze_x	e_y	$-e_z$	$e_xe_ye_z$	e_ye_z
e_y	e_y	$-e_xe_y$	1	e_ye_z	$-e_x$	$e_xe_ye_z$	e_z	e_ze_x
e_z	e_z	e_ze_x	$-e_ye_z$	1	$e_xe_ye_z$	e_x	$-e_y$	e_xe_y
e_xe_y	e_xe_y	$-e_y$	e_x	$e_xe_ye_z$	-1	e_ye_z	$-e_ze_x$	$-e_z$
e_ze_x	e_ze_x	e_z	$e_xe_ye_z$	$-e_x$	$-e_ye_z$	-1	e_xe_y	$-e_y$
e_ye_z	e_ye_z	$e_xe_ye_z$	$-e_z$	e_y	e_ze_x	$-e_xe_y$	-1	$-e_x$
$e_xe_ye_z$	$e_xe_ye_z$	e_ye_z	e_ze_x	e_xe_y	$-e_z$	$-e_y$	$-e_x$	-1

In this algebra, scalar multiplication is commutative and associative, basis vectors square to scalar one, and the product of two vectors resulting in a bivector is anti-commutative, associative, squares to negative one, and trivector basis commute with everything, yet square to negative one.

3D Euclidean Square

In 3D, I will use a capital G for the generic multivector to reduce confusion with a lower case g component. In component form, we can write the generic three dimensional multivector as

$$G = a + be_x + ce_y + de_z + ee_xe_y + fe_ze_x + ge_ye_z + he_xe_ye_z$$

Upon squaring this expression followed by simplifications (and using 'c' structure notation for the components), we get

$$\begin{aligned}
s &= GG \\
s.q &= a*a + (b*b + c*c + d*d) - (e*e + f*f + g*g) - h*h \\
s.x &= 2*(a*b - g*h) \\
s.y &= 2*(a*c - f*h) \\
s.z &= 2*(a*d - e*h) \\
s.xy &= 2*(a*e + d*h) \\
s.zx &= 2*(a*f + c*h) \\
s.yz &= 2*(a*g + b*h) \\
s.xyz &= 2*(a*h + b*g + c*f + d*e)
\end{aligned}$$

3D Euclidean Nilpotents

Looking at the above expression, with an eye toward patterns, we see that the multivector can be written as the sum of two four vectors, $q = (a + be_x + ce_y + de_z) + e_xe_ye_z(ee_z + fe_y + ge_x + h)$. For a zero scalar component, the magnitudes of these two fourvectors must be equal. Likewise, for a zero in the trivector component, these two four vectors must be perpendicular. Looking at the middle terms, we see an easy path where $a = 0$ and $h = 0$.

Putting all this together, we have a simple criteria. We set the scalar and trivector portions to zero. We make the vector and bivector components mutually perpendicular, with same magnitude.

Here is a simple algorithm which generates 3D nilpotents. Start with an arbitrary bivector $B = (0, 0, 0, 0, e, f, g, 0)$, and an arbitrary vector $\vec{v} = (0, b, c, d, 0, 0, 0, 0)$. Create a normal vector with regard to this bivector by $\vec{w} = (B\vec{v} - \vec{v}B)/2$. Cross scale these two terms, so they each have the same magnitude. Our nilpotent multivector is then $n = \vec{w}|B| + B|w|$. In component form, we have

$$\begin{aligned}
\vec{v} &= (0, b, c, d, 0, 0, 0, 0) \\
B &= (0, 0, 0, 0, e, f, g, 0) \\
\vec{w} &= (0, (ec - fd), (gd - eb), (fb - gc), 0, 0, 0, 0) \\
n.q &= 0 \\
n.x &= (ec - fd)\sqrt{e^2 + f^2 + g^2} \\
n.y &= (gd - eb)\sqrt{e^2 + f^2 + g^2} \\
n.z &= (fb - gc)\sqrt{e^2 + f^2 + g^2} \\
n.xy &= e\sqrt{(ec - fd)^2 + (gd - eb)^2 + (fb - gc)^2} \\
n.zx &= f\sqrt{(ec - fd)^2 + (gd - eb)^2 + (fb - gc)^2} \\
n.yz &= g\sqrt{(ec - fd)^2 + (gd - eb)^2 + (fb - gc)^2} \\
n.xyz &= 0
\end{aligned}$$

3D Euclidean Idempotents

We now find the idempotents in a fashion very similar to the 2D exercise. Our basic definition is $pp = p$. Working in component form, we have

$$\begin{aligned}
p &= pp \\
p.q &= a = a*a + (b*b + c*c + d*d) - (e*e + f*f + g*g) - h*h \\
p.x &= b = 2*(a*b - g*h) \\
p.y &= c = 2*(a*c - f*h) \\
p.z &= d = 2*(a*d - e*h) \\
p.xy &= e = 2*(a*e + d*h) \\
p.zx &= f = 2*(a*f + c*h) \\
p.yz &= g = 2*(a*g + b*h) \\
p.xyz &= h = 2*(a*h + b*g + c*f + d*e)
\end{aligned}$$

For the easy approach, we see by inspection that $h = 0$ gives us the same basic form as the 2D case. Keeping all values real, we set $a = 1/2$, $h = 0$, $(b, c, d) \perp (g, f, e)$, $(b^2 + c^2 + d^2) - (e^2 + f^2 + g^2) = 1/4$. Using multivector formulation, we can template our idempotent as

$$\begin{aligned} p &= \left(\frac{1}{2} + \vec{v}\right) + B \quad \text{with} \\ B^2 &= v^2 - \frac{1}{4} \quad \text{and} \\ B \perp \vec{v} &\rightarrow B\vec{v} + \vec{v}B = 0 \end{aligned}$$

Hyperbolic Function Format

Nicolas Le Bihan suggested the use of hyperbolic functions to Sangwine and Alfsmann. This is an excellent idea. Using the hyperbolic format, our generic projection idempotent with unit vector $\vec{u}$ and orthogonal unit bivector U becomes

$$\begin{aligned} p &= \frac{1}{2} \pm \frac{1}{2}\left(\vec{u}\cosh(t) + U\sinh(t)\right) \\ U \perp \vec{u} &\rightarrow U\vec{u} + \vec{u}U = 0 \end{aligned}$$

where

$$\begin{aligned} \vec{u} &= \frac{be_x + ce_y + de_z}{\sqrt{b^2 + c^2 + d^2}} \\ \cosh(t) &= 2\sqrt{b^2 + c^2 + d^2} \\ \sinh(t) &= 2\sqrt{e^2 + f^2 + g^2} \\ W &= \vec{u}\vec{w} - \vec{w}\vec{u} \qquad \text{arbitrary vector } \vec{w} \\ U &= \frac{W}{|W|} \end{aligned}$$

Like the 2D case, we have annihilating idempotents.

$$\begin{aligned} p_+ &= \frac{1}{2} + \frac{1}{2}\left(\vec{u}\cosh(t) + U\sinh(t)\right) \\ p_- &= \frac{1}{2} - \frac{1}{2}\left(\vec{u}\cosh(t) + U\sinh(t)\right) \end{aligned}$$

The main difference between the 2D and 3D cases here lie with the direction of the bivector in 3 space, versus the lack of choice of direction in the 2D case. In the 2D case, we had two free variables. In the 3D case, we have six being three for the unit vector $\vec{u}$, effectively two more for the normal bivector U, and one for the parameter t.

How about an example?

Sure thing. A simple example is $p_+ = (1/2) + e_x + e_{zx}\sqrt{3/4}$. Verification follows.

$$
\begin{aligned}
p_+ &= \frac{1}{2} + e_x + e_{zx}\sqrt{\frac{3}{4}} \\
(p_+)^2 &= \frac{1}{4} + \frac{1}{2}e_x + e_{zx}\frac{1}{2}\sqrt{\frac{3}{4}} \\
&\quad + e_x\frac{1}{2} + e_xe_x + e_xe_{zx}\sqrt{\frac{3}{4}} \\
&\quad + e_{zx}\frac{1}{2}\sqrt{\frac{3}{4}} + e_{zx}\sqrt{\frac{3}{4}}e_x + e_{zx}\sqrt{\frac{3}{4}}e_{zx}\sqrt{\frac{3}{4}} \\
(p_+)^2 &= \frac{1}{4} + \frac{1}{2}e_x + e_{zx}\frac{1}{2}\sqrt{\frac{3}{4}} \\
&\quad + e_x\frac{1}{2} + 1 - e_z\sqrt{\frac{3}{4}} \\
&\quad + e_{zx}\frac{1}{2}\sqrt{\frac{3}{4}} + e_z\sqrt{\frac{3}{4}} - \frac{3}{4} \\
(p_+)^2 &= \frac{1}{4} + 1 - \frac{3}{4} \\
&\quad + \frac{1}{2}e_x + e_x\frac{1}{2} \\
&\quad + e_{zx}\frac{1}{2}\sqrt{\frac{3}{4}} + e_{zx}\frac{1}{2}\sqrt{\frac{3}{4}} \\
&\quad - e_z\sqrt{\frac{3}{4}} + e_z\sqrt{\frac{3}{4}} \\
(p_+)^2 &= \frac{1}{2} + e_x + e_{zx}\sqrt{\frac{3}{4}}
\end{aligned}
$$

Comparison to Biquaternions

In 2010, Stephen Sangwine and Daniel Alfsmann wrote a delightful paper, *Determination of the biquaternion divisors of zero, including the idempotents and nilpotents*, which finds the idempotents and nilpotents for biquaternions.

Biquaternions are Euclidean 3D geometric algebra in a slightly repackaged format. Consequently, it is no surprise that they find similar results.

Define our generic multivector as

$$G = a + be_x + ce_y + de_z + ee_xe_y + fe_ze_x + ge_ye_z + he_xe_ye_z$$

Quaternions are the even order subset of this multivector. Addition, subtraction, multiplication and division are well defined among this subgroup. A generic quaternion is

$$Q = a + ee_xe_y + fe_ze_x + ge_ye_z$$

The trivector $e_xe_ye_z$ commutes with all elements of Euclidean 3D geometric algebra, and squares to negative one. We see that $e_xe_ye_z$ acts just like i in complex numbers. The leftover terms of our generic multivector, after identifying quaternion terms, can be modelled as another quaternion multiplied by this factor $e_xe_ye_z$. In common quaternion notation, the letter i corresponds to our bivector e_ye_z. To reduce confusion with the quaternion basis, we use $I = e_xe_ye_z$. Our net result, is that our standard generic multivector can be modelled as either a complexified quaternion, or as a quaternion with complex components. Both forms are fundamentally the same, and called biquaternions.

Write our multivector as a quaternion plus a different quaternion times the pseudoscalar $I = e_xe_ye_z$.

$$\begin{aligned} G &= a + be_x + ce_y + de_z + ee_xe_y + fe_ze_x + ge_ye_z + he_xe_ye_z \\ &= (a + ee_xe_y + fe_ze_x + ge_ye_z) + (be_x + ce_y + de_z + he_xe_ye_z) \\ &= (a + ee_xe_y + fe_ze_x + ge_ye_z) + (h - de_xe_y - ce_ze_x - be_ye_z)e_xe_ye_z \end{aligned}$$

Notice the negative signs on the b, c and d terms. This is a difference in sign convention between the biquaternion and multivector descriptions.

Write our multivector as a quaternion with complex coefficients.

$$\begin{aligned} G &= a + be_x + ce_y + de_z + ee_xe_y + fe_ze_x + ge_ye_z + he_xe_ye_z \\ &= (a + ee_xe_y + fe_ze_x + ge_ye_z) + (h - de_xe_y - ce_ze_x - be_ye_z)e_xe_ye_z \\ &= (a + hI) + (e - dI)e_xe_y + (f - cI)e_ze_x + (g - bI)e_ye_z \\ q_r &= (a + ee_xe_y + fe_ze_x + ge_ye_z) \\ q_i &= (h - de_xe_y - ce_ze_x - be_ye_z) \end{aligned}$$

Semi-norm

Sangwine and Alfsmann define the conventional norm of a biquaternion as

$$N = (a^2 + e^2 + f^2 + g^2) + (h^2 + b^2 + c^2 + d^2)$$

Sangwine and Alfsmann define a different norm, the semi-norm of a biquaternion as

$$\begin{aligned}
S &= (q_r\bar{q}_r - q_i\bar{q}_i) + I(q_r\bar{q}_i + q_r\bar{q}_r) \\
S &= a^2 + e^2 + f^2 + g^2 - h^2 - b^2 - c^2 - d^2 + 2I(ah - bg - cf - de) \\
S &= (a^2 - h^2) + (e^2 + f^2 + g^2) - (b^2 + c^2 + d^2) + 2I(ah - bg - cf - de)
\end{aligned}$$

It is interesting to compare and contrast the semi-norm with the real and trivector terms of the square of the generic multivector.

$$\begin{aligned}
G &= a + be_x + ce_y + de_z + ee_xe_y + fe_ze_x + ge_ye_z + he_xe_ye_z \\
s &= GG \\
s.q &= a^2 + (b^2 + c^2 + d^2) - (e^2 + f^2 + g^2) - h^2 \\
s.q &= (a^2 - h^2) - (e^2 + f^2 + g^2) + (b^2 + c^2 + d^2) \\
s.xyz &= 2(ah + bg + cf + de)
\end{aligned}$$

We see that these are close, but not exactly the same. The biquaternion imaginary component and the square trivector component are both dot products, taking into account the sign convention change mentioned above. However, the real components have a change in detail of the sign associated with the difference in magnitude of the space and bivector magnitudes. However, both expressions go to zero when the real and trivector components are equal in magnitude, and the vector and bivector components are equal in magnitude.

I emphasize that the semi-norm is not the same as the square of the general multivector. The purpose of my comparison is to show that the condition of the semi-norm being zero can coincide with the condition of the real and trivector terms of the multivector square being zero.

Biquaternion Nilpotent

For the biquaternion nilpotent, Sangwine and Alfsmann find that scalar and trivector components must be zero, and that the vector and bivector components must be orthogonal. This is in agreement with my results.

Biquaternion Idempotent

Sangwine and Alfsmann provide a generic formula for biquaternion idempotents, that has three specialized cases. Their formula for the generic idempotent P uses a biquaternion ξ which is a root of -1

$$\begin{aligned} P &= \frac{1}{2} \pm \frac{1}{2} I\xi \qquad \text{where} \\ \xi^2 &= -1 \end{aligned}$$

This equation specializes into three different forms. In these forms, $\vec{u}$ and $\vec{\nu}$ are mutually orthogonal, unit pure quaternions, which map into unit bivectors in geometric algebra.

$$\begin{aligned} P_1 &= \frac{1}{2} \pm \frac{1}{2} (\cosh(t)\vec{u}I - \sinh(t)\vec{\nu}) \\ P_2 &= \frac{1}{2} \pm \frac{1}{2} I\vec{u} \\ P_3 &= \frac{1}{2} \pm \frac{1}{2} I^2 = 0,\ 1 \end{aligned}$$

These are also in agreement with the previous results.

References

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