Multivectors and Eigenvectors

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August 7, 2018

Abstract

This note investigates the eigenvalues, polynomials and potents associated with multivectors in two and three dimensions, as well as in Minkowski spacetime.

Potents in Geometric Algebra

Nilpotents are mathematical objects which, while non-zero themself, square to zero. The cross product of vector algebra, and the wedge product of geometric algebra are examples of nilpotents. Similarly, idempotents are items which square to themselves. The numbers zero and one are examples of idempotents. For both nilpotents and idempotents, we have a similarity relationship $M^2 = \lambda M$, between the original item and the square. For nilpotents, $\lambda = 0$ and for idempotents $\lambda = 1$. It is easy to use this generalized relationship to define generic potents, where λ is not restricted to just zero or one.

Geometric algebra can always be mapped to matrix algebra. Consequently, matrix properties such as determinants can be readily defined for geometric algebra. In particular, the determinant is a measure of the metric dependent magnitude of the multivector. This magnitude can be zero, and this magnitude is correctly composited across multiplication, just like the determinant of matrix multiplication, to which it is related.

In matrix arithmetic, every matrix has associated eigenvalues λ, defined by

$$\det (M - \lambda) = 0$$

This is the defining equation of an eigenvalue. We can directly use this definition for multivectors. The corresponding reduced multivector $M - \lambda$, I will call an eigenfactor.

Two Dimensional Geometric Eigenvalues

The generic two dimensional Euclidean multivector is

$$M = q + ae_x + be_y + ce_{xy}$$

with determinant (in four by four matrix implementation)

$$\det(M) = (q^2 - a^2 - b^2 + c^2)^2$$

The related eigenvalue equation applies an offset λ against the scalar term q. The eigenvalue equation is

$$\det(M - \lambda) = \left((q - \lambda)^2 - a^2 - b^2 + c^2\right)^2 = 0$$

This equation is easy to solve, with double roots.

$$\begin{aligned}
(q - \lambda)^2 - a^2 - b^2 + c^2 &= 0 \\
(q - \lambda)^2 &= a^2 + b^2 - c^2 \\
q - \lambda &= \pm\sqrt{a^2 + b^2 - c^2} \\
\lambda &= q \mp \sqrt{a^2 + b^2 - c^2}
\end{aligned}$$

We will have real eigenvalues if $a^2 + b^2 \geq c^2$. Imaginary eigenvalues will require a third dimension for a geometrical interpretation. The associated eigenfactors are

$$M - \lambda = \left(\pm\sqrt{a^2 + b^2 - c^2}\right) + ae_x + be_y + ce_{xy}$$

Numerical Examples for 2D Euclidean Space

For our first example, choose $M = 5 + 6e_x + 7e_y + 9e_{xy}$ where the values have been pre-selected for easy computation. We now do some simple calculations with M.

$$
\begin{aligned}
M &= 5 + 6e_x + 7e_y + 9e_{xy} \\
M^2 &= 29 + 60e_x + 70e_y + 90e_{xy} \\
\det(M) &= 441
\end{aligned}
$$

In this case, it is interesting to see the vector and bivector portions of the square parallel with the original with a $2q = 10$ factor. This, of course makes sense, as $M^2 = (q^2 + b^2 + a^2 - c^2) + 2q(ae_x + be_y + ce_{xy})$.

Our eigenvalues are

$$
\begin{aligned}
\lambda_1 &= 5 + \sqrt{36 + 49 - 81} = 7 \\
\lambda_2 &= 5 - \sqrt{36 + 49 - 81} = 3
\end{aligned}
$$

Our eigenfactors are

$$
\begin{aligned}
E_1 &= -2 + 6e_x + 7e_y + 9e_{xy} \\
E_2 &= +2 + 6e_x + 7e_y + 9e_{xy}
\end{aligned}
$$

We check the properties of these eigenfactors. The determinants of E_1 and E_2 are zero, as expected. As E_1 and E_2 are multivectors, they will also have their own eigenvalues and eigenfactors. From the definition of the eigenvalue, one eigenvalue will be zero, while the other will be twice the scalar component. In a similar fashion, the eigenfactor of E_1 is E_2 and vice-versus.

$$
\begin{aligned}
E_1 &= -2 + 6e_x + 7e_y + 9e_{xy} \\
\lambda(E_1) &= -4 \text{ and } 0 \\
\text{Eigen}(E_1) &= E_1 - \lambda(E_1) = E_1 - 4 = E_2 \\
E_1^2 &= +8 - 24e_x - 28e_y - 36e_{xy} = \lambda(E_2) * E_2 \\
E_1 * M &= -6 + 18e_x + 21e_y + 27e_{xy} = \lambda_2 E_1 \\
M * E_1 &= -6 + 18e_x + 21e_y + 27e_{xy} = \lambda_2 E_1 \\
E_1 * E_2 &= 0
\end{aligned}
$$

Likewise, for E_2,

$$\begin{aligned}
E_2 &= +2 + 6e_x + 7e_y + 9e_{xy} \\
\lambda(E_2) &= 4 \text{ and } 0 \\
\text{Eigen}(E_2) &= E_2 - \lambda(E_2) = E_2 - (-4) = E_1 \\
E_2^2 &= +8 + 24e_x + 28e_y + 36e_{xy} = \lambda(E_1) * E_1 \\
E_2 * M &= +14 + 42e_x + 49e_y + 63e_{xy} = \lambda_1 E_2 \\
M * E_2 &= +14 + 42e_x + 49e_y + 63e_{xy} = \lambda_1 E_2 \\
E_2 * E_1 &= 0
\end{aligned}$$

We do another easy example.

$$\begin{aligned}
M &= 4 + 3e_x + 11e_y + 7e_{xy} \\
M^2 &= 97 + 24e_x + 88e_y + 56e_{xy} \\
\det(M) &= 4225
\end{aligned}$$

Again, we see the vector and bivector portions of the square parallel with the original with an $2q = 8$ factor.

Our eigenvalues are

$$\begin{aligned}
\lambda_1 &= 4 + \sqrt{9 + 121 - 49} = 13 \\
\lambda_2 &= 4 - \sqrt{9 + 121 - 49} = -5
\end{aligned}$$

Our eigenfactors are

$$\begin{aligned}
E_1 &= -9 + 3e_x + 11e_y + 7e_{xy} \\
E_2 &= +9 + 3e_x + 11e_y + 7e_{xy}
\end{aligned}$$

We check the properties of these eigenfactors. The determinants of E_1 and E_2 are zero, as expected.

$$\begin{aligned}
E_1 &= -9 + 3e_x + 11e_y + 7e_{xy} \\
\lambda(E_1) &= -18 \text{ and } 0 \\
\text{Eigen}(E_1) &= E_1 - \lambda(E_1) = E_1 + 18 = E_2 \\
E_1^2 &= +162 - 54e_x - 198e_y - 126e_{xy} = \lambda(E_1) * E_1 \\
E_1 * M &= +45 - 15e_x - 55e_y - 35e_{xy} = \lambda_2 E_1 \\
M * E_1 &= +45 - 15e_x - 55e_y - 35e_{xy} = \lambda_2 E_1 \\
E_1 * E_2 &= 0
\end{aligned}$$

Likewise, for E_2,

$$\begin{aligned}
E_2 &= +9 + 3e_x + 11e_y + 7e_{xy} \\
\lambda(E_2) &= 18 \text{ and } 0 \\
\text{Eigen}(E_2) &= E_2 - \lambda(E_2) = E_2 - 18 = E_1 \\
E_2^2 &= +162 + 54e_x + 198e_y + 126e_{xy} = \lambda(E_2) * E_2 \\
E_2 * M &= +117 + 39e_x + 143e_y + 91e_{xy} = \lambda_1 E_2 \\
M * E_2 &= +117 + 39e_x + 143e_y + 91e_{xy} = \lambda_1 E_2 \\
E_2 * E_1 &= 0
\end{aligned}$$

Now we do a complex answer example. We will repeat this example again, once we have a third dimension.

$$\begin{aligned}
M &= 5 + 3e_x + 4e_y + 13e_{xy} \\
M^2 &= -119 + 30e_x + 40e_y + 130e_{xy} \\
\det(M) &= 28561
\end{aligned}$$

Our eigenvalues are

$$\begin{aligned}
\lambda_1 &= 5 + 12i \\
\lambda_2 &= 5 - 12i
\end{aligned}$$

Our eigenfactors are

$$\begin{aligned}
E_1 &= -12i + 3e_x + 4e_y + 13e_{xy} \\
E_2 &= +12i + 3e_x + 4e_y + 13e_{xy}
\end{aligned}$$

Three Dimensional Geometric Eigenvalues

The generic three dimensional Euclidean multivector is

$$M = q + ae_x + be_y + ce_z + de_{xy} + ee_{xz} + fe_{yz} + ge_{xyz}$$

Note that about half the authors in this field choose an e_{zx} sign convention, which is important when comparing results.

We have a more complicated determinant (in four by four matrix implementation)

$$\det(M) = (+q^2 - a^2 - b^2 - c^2 + d^2 + e^2 + f^2 - g^2)^2 + (2qg - 2af + 2be - 2cd)^2$$

The related eigenvalue equation applies an offset λ against the scalar term q. The eigenvalue equation is

$$\begin{aligned} \det(M - \lambda) &= \left(+(q-\lambda)^2 - a^2 - b^2 - c^2 + d^2 + e^2 + f^2 - g^2\right)^2 \\ &\quad + \left(2(q-\lambda)g - 2af + 2be - 2cd\right)^2 = 0 \end{aligned}$$

In general, this would be a quartic equation with four usually complex roots. Noting, however, that this expression is the sum of two squares, and each term must independently be zero for real values of λ, we find two simultaneous equations

$$\begin{aligned} \left(+(q-\lambda)^2 - a^2 - b^2 - c^2 + d^2 + e^2 + f^2 - g^2\right)^2 &= 0 \\ \left(2(q-\lambda)g - 2af + 2be - 2cd\right)^2 &= 0 \end{aligned}$$

From the second equation, we learn

$$(q - \lambda) = \frac{af - be + cd}{g}$$

From the first equation, we have the restriction

$$\left(+\left(\frac{af - be + cd}{g}\right)^2 + d^2 + e^2 + f^2 - a^2 - b^2 - c^2 - g^2\right) = 0$$

or

$$\left(\frac{af - be + cd}{g}\right)^2 + d^2 + e^2 + f^2 = a^2 + b^2 + c^2 + g^2$$

We see that for cherry-picking example purposes, we can choose most of our parameters, and solve a simple quadratic (or biquadratic, for g) to satisfy our selection.

Revisit The Complex 2D Example

In general, the pseudoscalar e_{xyz} mimics $i = \sqrt{-1}$. This pseudoscalar squares to negative one, and commutes with all elements of 3D Euclidean geometric algebra. We now look at the complex example from our 2D case, now with a third dimension.

$$M = 5 + 3e_x + 4e_y + 13e_{xy}$$

We calculate our eigenvalue equation.

$$\det = \lambda^4 - 20\lambda^3 + 438\lambda^2 - 3380\lambda + 28561$$

We find our four roots.

$$\begin{aligned}
\lambda_1 &= \lambda_3 = 5 + 12i = 5 + 12e_{xyz} \\
\lambda_2 &= \lambda_4 = 5 - 12i = 5 - 12e_{xyz}
\end{aligned}$$

We find our eigenfactors

$$\begin{aligned}
E_1 &= +3e_x + 4e_y + 13e_{xy} - 12e_{xyz} \\
E_2 &= +3e_x + 4e_y + 13e_{xy} + 12e_{xyz}
\end{aligned}$$

We expect these terms have zero determinant, and null product. Simple calculation proves this to be the case.

Both E_1 and E_2 have eigenvalues of 0 and $\pm 24e_{xyz}$. Our squares and products are

$$\begin{aligned}
E_1^2 &= -288 + 312e_z + 96e_{xz} - 72e_{yz} = (-24e_{xyz})E_1 \\
E_2^2 &= -288 - 312e_z - 96e_{xz} + 72e_{yz} = (+24e_{xyz})E_2 \\
E_1M &= ME_1 = \lambda_2 E_1 \\
&= -144 + 15e_x + 20e_y + 156e_z + 65e_{xy} + 48e_{xz} - 36e_{yz} - 60e_{xyz} \\
E_2M &= ME_2 = \lambda_1 E_2 \\
&= -144 + 15e_x + 20e_y - 156e_z + 65e_{xy} - 48e_{xz} + 36e_{yz} + 60e_{xyz}
\end{aligned}$$

Look at a Generic 3D Example

We now look at a generic multivector.

$$M = 5 + 3e_x + 4e_y + 2e_z + 7e_{xy} + 9e_{xz} + 13e_{yz} + 11e_{xyz}$$

Our determinant equation is

$$\det(M) = \lambda^4 - 20 * \lambda^3 + 932 * \lambda^2 - 6824 * \lambda + 36052$$

Our four verified roots are

$$\begin{aligned}
\lambda_1 &= 3.9674495760703445413 + i(-5.4640869888966907553)\\
\lambda_2 &= 3.9674495760703454295 + i(\ 5.4640869888966907553)\\
\lambda_3 &= 6.0325504239296545705 + i(\ 27.464086988896696084)\\
\lambda_4 &= 6.0325504239296554587 + i(-27.464086988896696084)
\end{aligned}$$

Our expected eigenfactors are

```
E_1 = (  1.0325504239296554587, 3,4,2, 7,9,13,  16.464086988896690755)
E_2 = (  1.0325504239296554587, 3,4,2, 7,9,13,   5.535913011103309245)
E_3 = ( -1.0325504239296545705, 3,4,2, 7,9,13, -16.464086988896696084)
E_4 = ( -1.0325504239296545705, 3,4,2, 7,9,13,  38.464086988896696084)
```

However, only two of the four solutions (E_1 and E_3) have a zero determinant and zero product.

```
det(E_1) = 1.25930588680653752E-28
det(E_2) = 58310.998215567783944
det(E_3) = 2.820497676462303673E-26
det(E_4) = 1473145.3477513345606
```

Expecting more than two valid eigenvalues, I want to re-examine our equations for λ, from another point of view.

Another Re-factoring

The eigenvalue equation is

$$\begin{aligned}
\det(M - \lambda) &= \left(+(q-\lambda)^2 - a^2 - b^2 - c^2 + d^2 + e^2 + f^2 - g^2\right)^2\\
&\quad + \left(2\,(q-\lambda)\,g - 2af + 2be - 2cd\right)^2 = 0
\end{aligned}$$

Re-arranging,

$$\left(+(q-\lambda)^2 - a^2 - b^2 - c^2 + d^2 + e^2 + f^2 - g^2\right)^2 = -\left(2\,(q-\lambda)\,g - 2af + 2be - 2cd\right)^2$$

Taking the square root of both sides, we have two equations

$$\left(+(q-\lambda)^2 - a^2 - b^2 - c^2 + d^2 + e^2 + f^2 - g^2\right) = \pm i\left(2\,(q-\lambda)\,g - 2af + 2be - 2cd\right)$$

Putting all terms on the left, and organizing by powers of $(q-\lambda)$, we have

$$\begin{aligned}(q-\lambda)^2 + (2ig)\,(q-\lambda) - (a^2+b^2+c^2-d^2-e^2-f^2+g^2) - i(2af-2be+2cd) &= 0\\ (q-\lambda)^2 - (2ig)\,(q-\lambda) - (a^2+b^2+c^2-d^2-e^2-f^2+g^2) + i(2af-2be+2cd) &= 0\end{aligned}$$

Substituting our numerical values into these equations, we find

$$\begin{aligned}(q-\lambda)^2 + (22i)\,(q-\lambda) - (149) - i(34) &= 0\\ (q-\lambda)^2 - (22i)\,(q-\lambda) - (149) + i(34) &= 0\end{aligned}$$

The solution to the top equation works, while the second does not. These four roots match those from the quartic, in a different order.

$$(q-\lambda)^2 + (22i)\,(q-\lambda) - (149) - i(34) = 0$$

$$\begin{aligned}\lambda_1 &= 6.0325504239296554587 + i(27.464086988896692532)\\ \lambda_2 &= 3.9674495760703445413 + i(-5.464086988896692532)\end{aligned}$$

```
M   = ( 5,  3,4,2,  7,9,13,  11)
E_1 = ( -1.0325504239296545705, 3,4,2, 7,9,13, -16.464086988896696084)
E_2 = (  1.0325504239296554587, 3,4,2, 7,9,13,  16.464086988896690755)
```

So far, I have only seen two eigenvalues, both complex. Let me try to find a case with four real eigenvalues.

Real Eigenvalues

The generic three dimensional Euclidean multivector is

$$M = q + ae_x + be_y + ce_z + de_{xy} + ee_{xz} + fe_{yz} + ge_{xyz}$$

The eigenvalue equation is

$$\begin{aligned}\det(M-\lambda) &= \left(+(q-\lambda)^2 - a^2 - b^2 - c^2 + d^2 + e^2 + f^2 - g^2\right)^2\\ &\quad + \left(2\,(q-\lambda)\,g - 2af + 2be - 2cd\right)^2 = 0\end{aligned}$$

This expression is the sum of two squares, and for real values of λ, each term must independently be zero.

$$
\begin{aligned}
\left(+(q-\lambda)^2 - a^2 - b^2 - c^2 + d^2 + e^2 + f^2 - g^2\right)^2 &= 0 \\
\left(2(q-\lambda)g - 2af + 2be - 2cd\right)^2 &= 0
\end{aligned}
$$

From the second equation, we learn

$$
(q-\lambda) = \frac{af - be + cd}{g}
$$

We see that if $g = 0$, our eigen equation will have double roots. Keep g non-zero.

Substitute for $(q-\lambda)$ in the first equation.

$$
\left(+\left(\frac{af - be + cd}{g}\right)^2 + d^2 + e^2 + f^2 - a^2 - b^2 - c^2 - g^2\right) = 0
$$

Organize as a quadratic in g^2.

$$
g^4 - (d^2 + e^2 + f^2 - a^2 - b^2 - c^2)g^2 - (af - be + cd)^2 = 0
$$

From the discriminant, we see that q^2 will always be real. From the constant term, we see we will have a positive and negative root.

We see by inspection (of a computer printout) that `a = 1, b = 2, c = 3, d = 4, e = 5, f = 6, g = 8` forms a nice example, and that `a = 1, b = 2, c = 6, d = 9, e = 7, f = 4, g = 11` forms another.

For our first example, we have

$$
\begin{aligned}
M &= 7 + e_x + 2e_y + 3e_z + 4e_{xy} + 5e_{xz} + 6e_{yz} + 8e_{xyz} \\
\lambda_1 &= 6 \\
\lambda_2 &= 8 + 16e_{xyz} \\
E_1 &= +1 + e_x + 2e_y + 3e_z + 4e_{xy} + 5e_{xz} + 6e_{yz} + 8e_{xyz} \\
E_2 &= -1 + e_x + 2e_y + 3e_z + 4e_{xy} + 5e_{xz} + 6e_{yz} - 8e_{xyz}
\end{aligned}
$$

For our second example, we have

$$
\begin{aligned}
M &= 5 + e_x + 2e_y + 6e_z + 9e_{xy} + 7e_{xz} + 4e_{yz} + 11e_{xyz} \\
\lambda_1 &= 1 \\
\lambda_2 &= 9 + 22e_{xyz} \\
E_1 &= +4 + e_x + 2e_y + 6e_z + 0e_{xy} + 7e_{xz} + 4e_{yz} + 11e_{xyz} \\
E_2 &= -4 + e_x + 2e_y + 6e_z + 0e_{xy} + 7e_{xz} + 4e_{yz} - 11e_{xyz}
\end{aligned}
$$

For a third example, we have

$$\begin{aligned}
M &= 7+3e_x+4e_y+5e_z+8e_{xy}+9e_{xz}+2e_{yz}+10e_{xyz}\\
\lambda_1 &= 6\\
\lambda_2 &= 8+20e_{xyz}\\
E_1 &= +1+3e_x+4e_y+5e_z+8e_{xy}+9e_{xz}+2e_{yz}+10e_{xyz}\\
E_2 &= -1+3e_x+4e_y+5e_z+8e_{xy}+9e_{xz}+2e_{yz}-10e_{xyz}
\end{aligned}$$

So, at this point, I believe there are only two eigenvalues per three dimensional multivector, and that usually, these are complex values. For the case where we have a real root, we only have a single real root.

The fact that there are only two eigenvalues, and that the two eigenfactors are nilfactors, teaches me that the simplest determinant for two and three dimensional systems is the quadratic factor referenced above. A matrix satisfies its own characteristic (determinant) equation. As each eigenfactor is the original multivector minus the eigenvalue symbol, the cascaded product of eigenfactors re-creates the determinant polynomial, which being satisfied by the original matrix, yields a zero product.

Minkowski Spacetime

The defining feature of Minkowski spacetime, is that the time basis squares to negative one, rather than positive one for the spatial basis. The generic Minkowski Spacetime multivector is

$$\begin{aligned}
M = {} & +a\\
& +be_x+ce_y+de_z+ee_t\\
& +fe_{xy}+ge_{xz}+he_{xt}+je_{yz}+ke_{yt}+le_{zt}\\
& +me_{xyz}+ne_{xyt}+pe_{xzt}+re_{yzt}\\
& +se_{xyzt}
\end{aligned}$$

I have skipped the symbols i and o to reduce confusion. Because of the large number of terms, I will usually write the multivector in a multiline format, each grade on its own line, as above. Alternatively, I will use a computer function format, with extra space separating the grade components, as in

```
M = Mink(q, x,y,z,t, xy,xz,xt,yz,yt,zt, xyz,xyt,xzt,yzt, xyzt)
```

The determinant formula, expanded, occupies a page of text and is not very informative. However, the algorithm for calculating the determinant is informative. Multiply a Mink by its reverse.

```
V = Mink(A,  B, C, D, E,  F, G, H, J, K, L,  M, N, P, R,  S)
U = Mink(A,  B, C, D, E, -F,-G,-H,-J,-K,-L, -M,-N,-P,-R,  S)
W = U*V

a = W.q = A^2+B^2+C^2+D^2-E^2+F^2+G^2+H^2-J^2-K^2-L^2+M^2-N^2-P^2-R^2-S^2

b = W.x =  + 2*A*B + 2*C*F + 2*D*G - 2*E*J + 2*H*M - 2*K*N - 2*L*P - 2*R*S
c = W.y =  + 2*A*C - 2*B*F + 2*D*H - 2*E*K - 2*G*M + 2*J*N - 2*L*R + 2*P*S
d = W.z =  + 2*A*D - 2*B*G - 2*C*H - 2*E*L + 2*J*P + 2*K*R + 2*M*F - 2*N*S
e = W.t =  + 2*A*E - 2*B*J - 2*C*K - 2*D*L + 2*F*N + 2*G*P + 2*H*R - 2*M*S

s = W.xyzt =  + 2*A*S - 2*B*R + 2*C*P - 2*D*N + 2*E*M - 2*F*L + 2*G*K - 2*H*J

det(W) = det(V)*det(U) = (det(V))^2 = (a*a - b*b - c*c - d*d + e*e + s*s)^2
det(V) = a*a - b*b - c*c - d*d + e*e + s*s (sign verified)
```

We take the product of our Mink and its reverse. The resulting product has zeroes in the bivector and trivector components. The remaining six terms, sum of squares with signs shown above, yields the determinant. Using the notation of the code fragment above, a zero determinant requires $a^2 + e^2 + s^2 = b^2 + c^2 + d^2$.

Minkowski Spacetime Numerical Example

We now gain some experience with Minkowski multivector eigenvalues.

Begin with the arbitrary Mink

$$
\begin{aligned}
M \;=\; & +1 \\
& +2e_x + 3e_y + 4e_z + 5e_t \\
& +6e_{xy} + 7e_{xz} + 8e_{xt} + 9e_{yz} + 10e_{yt} + 11e_{zt} \\
& +12e_{xyz} + 13e_{xyt} + 14e_{xzt} + 15e_{yzt} \\
& +16e_{xyzt}
\end{aligned}
$$

In computer code format, this multivector is

```
MV = Mink( 1, 2,3,4,5, 6,7,8,9,10,11, 12,13,14,15, 16)
```

We find our characteristic equation from $\det(M - \lambda) = 0$

$$\det = +L^4 - 4 * L^3 - 688 * L^2 + 3208 * L - 22052$$

We find four roots, two real, two complex. The complex roots require a fifth dimension for a geometrical interpretation. For the moment, I just accept the complex math, and not worry.

```
r1 = -27.000997368653746025
r2 = 2.228135842519931009+5.079640695255969765*I
r3 = 2.228135842519931009-5.079640695255969765*I
r4 = 26.544725683613884005
```

My four eigenvalues are these roots.

```
Lambda_1 = ( -27.000997368653746025, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)
Lambda_2 = ( 2.2281+5.07964*I, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)
Lambda_3 = ( 2.2281-5.07964*I, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)
Lambda_4 = ( 26.544725683613884005, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)
```

My four eigenfactors, which all have zero determinant, are

```
E_1 = ( 28.000997368653746025, 2,3,4,5, 6,7,8,9,10,11, 12,13,14,15, 16)
E_2 = ( -1.2281-5.07964*I, 2,3,4,5, 6,7,8,9,10,11, 12,13,14,15, 16)
E_3 = ( -1.2281+5.07964*I, 2,3,4,5, 6,7,8,9,10,11, 12,13,14,15, 16)
E_4 = ( -25.544725683613884005, 2,3,4,5, 6,7,8,9,10,11, 12,13,14,15, 16)
```

Pairwise, none of eigenfactor pairs is a zero result. However, the cascased product of all four in any order is zero. This teaches me that the minimum characteristic equation is the fourth order determinant equation used above.

Cascaded Eigenfactors

Each of the four eigenfactors above has its own set of eigenvalues and eigenfactors. Because the determinant of these eigenfactors is zero, one of these eigenvalues will be zero. I notice that the other eigenvalues are simple differences between the original set, and the eigenvalue of interest.

```
Lambda_1 = ( -27.001, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)
Lambda_2 = ( 2.2281+5.079641*I, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)
Lambda_3 = ( 2.2281-5.079641*I, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)
Lambda_4 = ( 26.545, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)
```

```
L11 = ( 0, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)
L12 = ( 29.229+5.079641*I, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)
L13 = ( 29.229-5.079641*I, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)
L14 = ( 53.546, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)

L21 = ( -29.229-5.079641*I, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)
L22 = ( 0, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)
L23 = (     0 -  10.159*I, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)
L24 = ( 24.317-5.079641*I, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)

L31 = ( -29.229+5.079641*I, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)
L32 = ( 0, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)
L33 = (      0 + 10.159*I, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)
L34 = ( 24.317+5.079641*I, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)

L41 = ( -53.546, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)
L42 = ( -24.317+5.079641*I, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)
L43 = ( -24.317-5.079641*I, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)
L44 = ( 0, 0,0,0,0, 0,0,0,0,0,0, 0,0,0,0, 0)
```

Minkowski Nilpotent and Idempotent Examples

Our first example is the classic nilpotent $M = (e_x + e_t)/2$, which squares to zero. The eigenvalue equation is $\lambda^4 = 0$, which has all roots zero. The eigenfactors are just four copies of the nilpotent above.

Next is the classic nilpotent offset by three, $M = (6 + e_x + e_t)/2$. This has eigenvalue equation $(\lambda - 3)^4 = 0$, which has all roots equal three.

Idempotents square to themselves. Using the example of $M = (1+e_x)/2$, we find an eigenvalue equation of $\lambda^4 - 2\lambda^3 + \lambda^2$, which has double roots of zero and one. Our eigenfactors are two copies each of $E_1 = E_2 = (1 + e_x)/2$ and $E_3 = E_4 = (-1 + e_x)/2$.

Next is an offset idempotent, $M = (5 + e_x)/2$. Our eigenvalue equation is $\lambda^4 - 10\lambda^3 + 37\lambda^2 - 60\lambda + 36 = 0$, which has double roots at two and three. The eigenfactors match those of the parent idempotent above.

A more complicated idempotent example is $M = (1 + e_x + e_{yz} + e_{yt} + e_{zt} + e_{xyz} + e_{xyt} + e_{xzt})/4$. Despite its apparent complexity, the eigenvalue equation is $\lambda^4 - \lambda^3 = 0$, with a triple root at zero, and single root at one. For our eigenfactors, we have $E_1 = E_2 = E_3 = (1 + e_x + e_{yz} + e_{yt} + e_{zt} + e_{xyz} + e_{xyt} + e_{xzt})/4$ and $E_4 = (-3 + e_x + e_{yz} + e_{yt} + e_{zt} + e_{xyz} + e_{xyt} + e_{xzt})/4$

GA5_4_1 Spacetime

GA5_4_1 is a five dimensional geometric algebra with four space (w, x, y, and z), and one time (t) dimension. The Dirac matrices of quantum mechanics can be mapped to this algebra. The highest grade element, e_{wxyzt}, squares to negative one, commutes with all other elements, and so mimics $i = \sqrt{-1}$. When calculating determinants or roots of polynomials, as complex numbers are found, we map the real portion to the scalar of the multivector (e_q component), and the imaginary porton to the pseudoscalar portion (e_{wxyzt} component).

Numeric Example

```
The data structure, with each grade on its own line, is
MV = GA5_4_1(  q,
w,x,y,z,t,
wx,wy,wz,wt,xy,xz,xt,yz,yt,zt,
wxy,wxz,wxt,wyz,wyt,wzt,xyz,xyt,xzt,yzt,
wxyz,wxyt,wxzt,wyzt,xyzt,
wxyzt)

Look at the specific prime sequence example

MV = ( 3, 5,7,11,13,17,
19,23,29,31,37,41,43,47,53,59,
61,67,71,73,79,83,89,97,101,103,
07,109,113,127,131, 137)

Our determinant has complex values from the complexification of
Minkowski space, or from use of Dirac matrices (your choice of interpretation).

The eigenvalue quartic is

 L^4 + (-12-548*I) L^3    + (-77808-24888*I) L^2
     + (449648-65264*I) L + (-98748240-54398496*I) = 0

The four roots, in mutivector format, are

Lambda_1 = (-125.633761392544956545,
0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0, 152.70837345144692412)
```

```
Lambda_2 = (1.2311675184366665976,
0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0, 40.279304977732559585)

Lambda_3 = (5.816920903217879333,
0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0, -33.98458175894367425)

Lambda_4 = (130.58567297089041062,
0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0, 388.99690332976419055)

Our four eigenfactors are

E_1 = (128.63376139254495654,
5,7,11,13,17,
19,23,29,31,37,41,43,47,53,59,
61,67,71,73,79,83,89,97,101,103,
107,109,113,127,131, -15.708373451446924124)

E_2 = (1.7688324815633334024,
5,7,11,13,17,
19,23,29,31,37,41,43,47,53,59,
61,67,71,73,79,83,89,97,101,103,
107,109,113,127,131, 96.720695022267440415)

E_3 = (-2.816920903217879333,
5,7,11,13,17,
19,23,29,31,37,41,43,47,53,59,
61,67,71,73,79,83,89,97,101,103,
107,109,113,127,131, 170.98458175894367425)

E_4 = (-127.58567297089041062,
5,7,11,13,17,
19,23,29,31,37,41,43,47,53,59,
61,67,71,73,79,83,89,97,101,103,
107,109,113,127,131, -251.99690332976419055)
```

```
These eigenfactors have zero determinants:

det(E_1) = 8.29459168016910553E-10
det(E_2) = 1.0913936421275138855E-9+3.2014213502407073975E-10*I
det(E_3) = -5.5297277867794036865E-10+4.1791281546466052532E-10*I
det(E_4) = -2.3283064365386962890E-10+4.6566128730773925780E-10*I
```

These four eigenfactors have a cascaded product of zero, as expected, while each pair and triple is non-zero, as expected.

Revisit Frustrated Minkowski Example

Now that we have a fifth dimension which geometrically supports complex numbers, we can redo the frustrated Minkowski examples.

`MV = Mink( 1, 2,3,4,5, 6,7,8,9,10,11, 12,13,14,15, 16)` becomes
`MV = GA5_4_1( 1, 0,2,3,4,5, 0,0,0,0,6,7,8,9,10,11, 0,0,0,0,0,0,12,13,14,15, 0,0,0,0,16, 0)`

We find the same characteristic equation, as in Minkowski spacetime using $\det(M - \lambda) = 0$

$$\det = +L^4 - 4 * L^3 - 688 * L^2 + 3208 * L - 22052$$

Our four eigenvalues are

$$\begin{aligned}
\lambda_1 &= -27.001 \\
\lambda_2 &= 2.2281 + 5.07964\ e_{wxyzt} \\
\lambda_3 &= 2.2281 - 5.07964\ e_{wxyzt} \\
\lambda_4 &= 26.545
\end{aligned}$$

In Minkowski fivespace, everything fits.

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