Exercises with Magnetic Monopoles

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Abstract

This note presents some classical exercises with magnetic monopoles. I am guided by notes and homework problems from Julian Schwinger's "Classical Electrodynamics" and Jackson's "Classical Electrodynamics". I start with the extended Maxwell equations for monopoles and the extended Lorentz force for monopoles. I then work with the intrinsic angular momentum due to charge/monopole interactions. Next, I examine the duality transforms for charge/monopole mixing.

Maxwell Equations and Monopole Extensions

Conventional Maxwell equations in SI units are

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \frac{\rho_e}{\epsilon} \\
\vec{\nabla} \cdot \vec{B} &= 0 \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\vec{\nabla} \times \vec{B} &= \mu\epsilon \frac{\partial \vec{E}}{\partial t} + \mu \vec{j}_e
\end{aligned}$$

Extending these equations to include magnetic monopoles, we have

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \frac{\rho_e}{\epsilon} \\
\vec{\nabla} \cdot \vec{B} &= \mu \rho_m \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} - \mu \vec{j}_m \\
\vec{\nabla} \times \vec{B} &= \mu \epsilon \frac{\partial \vec{E}}{\partial t} + \mu \vec{j}_e
\end{aligned}$$

Extended Lorentz Force

The standard Lorentz force is

$$\vec{F} = q_0 \left(\vec{E} + \vec{v} \times \vec{B} \right)$$

Extending this to include monopoles, we have

$$\vec{F} = q_0 \left(\vec{E} + \vec{v} \times \vec{B} \right) + q_m \left(\vec{B} - \vec{v} \times \frac{\vec{E}}{c^2} \right)$$

Electric Field from a Point Charge

The standard electric field due to a point charge q_e at (x_s, y_s, z_s) seen at point (x, y, z) is

$$\vec{E}(x, y, z) = \frac{q_e}{4\pi\epsilon} \frac{(x - x_s)\vec{a}_x + (y - y_s)\vec{a}_y + (z - z_s)\vec{a}_z}{\left((x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2\right)^{3/2}}$$

Magnetic Field from a Point Monopole

The magnetic field due to a point charge q_m at (x_s, y_s, z_s) seen at point (x, y, z) is

$$\vec{B}(x, y, z) = \frac{\mu q_m}{4\pi} \frac{(x - x_s)\vec{a}_x + (y - y_s)\vec{a}_y + (z - z_s)\vec{a}_z}{\left((x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2\right)^{3/2}}$$

Field Angular Momentum due to Charge/Monopole Interaction

This section follows homework problem 8, chapter 3 of Julian Schwinger's Classical Electrodynamics. Place an electric charge at $(0,0,R/2)$. Place a magnetic charge at $(0,0,-R/2)$.

The electric field is

$$\begin{aligned}
\vec{E}(x,y,z) &= \frac{q_e}{4\pi\epsilon}\left[-\vec{\nabla}\left(\frac{1}{\sqrt{x^2+y^2+(z-R/2)^2}}\right)\right] \\
&= \frac{q_e}{4\pi\epsilon}\frac{x\vec{a}_x+y\vec{a}_y+(z-R/2)\vec{a}_z}{\left(x^2+y^2+(z-R/2)^2\right)^{3/2}}
\end{aligned}$$

The magnetic field is

$$\begin{aligned}
\vec{B}(x,y,z) &= \frac{\mu q_m}{4\pi}\left[-\vec{\nabla}\left(\frac{1}{\sqrt{x^2+y^2+(z+R/2)^2}}\right)\right] \\
&= \frac{\mu q_m}{4\pi}\frac{x\vec{a}_x+y\vec{a}_y+(z+R/2)\vec{a}_z}{\left(x^2+y^2+(z+R/2)^2\right)^{3/2}}
\end{aligned}$$

The power flux Poynting vector is

$$\begin{aligned}
\vec{S} &= \frac{1}{\mu}(\vec{E}\times\vec{B}) \\
&= \frac{1}{\mu}\frac{q_e}{4\pi\epsilon}\frac{x\vec{a}_x+y\vec{a}_y+(z-R/2)\vec{a}_z}{\left(x^2+y^2+(z-R/2)^2\right)^{3/2}}\times\frac{\mu q_m}{4\pi}\frac{x\vec{a}_x+y\vec{a}_y+(z+R/2)\vec{a}_z}{\left(x^2+y^2+(z+R/2)^2\right)^{3/2}} \\
&= \frac{q_e q_m}{16\pi^2}\frac{1}{\epsilon}\frac{yR\vec{a}_x-xR\vec{a}_y}{\left(x^2+y^2+(z-R/2)^2\right)^{3/2}\left(x^2+y^2+(z+R/2)^2\right)^{3/2}}
\end{aligned}$$

The momentum density is

$$\epsilon\mu\vec{S} = \frac{\mu q_e q_m}{16\pi^2}\frac{yR\vec{a}_x-xR\vec{a}_y}{\left(x^2+y^2+(z-R/2)^2\right)^{3/2}\left(x^2+y^2+(z+R/2)^2\right)^{3/2}}$$

This expression has zero divergence, away from the sources. We show this most easily using the gradient definitions for the electric and magnetic fields.

$$\begin{aligned}
\rho_e &= \sqrt{x^2+y^2+(z-R/2)^2} \\
\rho_m &= \sqrt{x^2+y^2+(z+R/2)^2} \\
\vec{E} &= -\frac{q_e}{4\pi\epsilon}\vec{\nabla}\left(\frac{1}{\rho_e}\right) \\
\vec{B} &= -\frac{\mu q_m}{4\pi}\vec{\nabla}\left(\frac{1}{\rho_m}\right) \\
\epsilon\mu\vec{S} &= \frac{\mu q_e q_m}{16\pi^2}\left[\vec{\nabla}\left(\frac{1}{\rho_e}\right)\times\vec{\nabla}\left(\frac{1}{\rho_m}\right)\right] \\
\vec{\nabla}\cdot\left(\epsilon\mu\vec{S}\right) &= \frac{\mu q_e q_m}{16\pi^2}\vec{\nabla}\cdot\left[\vec{\nabla}\left(\frac{1}{\rho_e}\right)\times\vec{\nabla}\left(\frac{1}{\rho_m}\right)\right] \\
\vec{\nabla}\cdot\left(\epsilon\mu\vec{S}\right) &= \frac{\mu q_e q_m}{16\pi^2}\left[\vec{\nabla}\left(\frac{1}{\rho_m}\right)\cdot\vec{\nabla}\times\left(\vec{\nabla}\left(\frac{1}{\rho_e}\right)\right)-\vec{\nabla}\left(\frac{1}{\rho_e}\right)\cdot\vec{\nabla}\times\left(\vec{\nabla}\left(\frac{1}{\rho_m}\right)\right)\right] \\
&= 0 \quad \text{since curl of gradient is zero}
\end{aligned}$$

Our next challenge, having shown zero divergence for this expression, is to recast this as the curl of some other function. (This is motivated by the identity that the divergence of a curl is zero.) Using the relationship

$$\vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A}) - \vec{A}\times(\vec{\nabla}f)$$

We see that if $\vec{A}=\vec{\nabla}g$,

$$\begin{aligned}
\vec{\nabla}\times(f\vec{\nabla}g) &= f(\vec{\nabla}\times\vec{\nabla}g) - \vec{\nabla}g\times(\vec{\nabla}f) \\
&= \vec{\nabla}f\times\vec{\nabla}g
\end{aligned}$$

We easily see two nice answers.

$$\begin{aligned}
\vec{\nabla}f\times\vec{\nabla}g &= \vec{\nabla}\times(f\vec{\nabla}g) \\
&= -\vec{\nabla}\times(g\vec{\nabla}f)
\end{aligned}$$

Given that the curl of a gradient is zero, we have a family of answers, just like gauge transformations. The two answers shown above differ by

$\vec{\nabla}(\rho_e\rho_m)^{-1}$. Each form favors one of the two charge types. I prefer to use a version which is more symmetric (actually anti-symmetric) with respect to the charges. Thus, I introduce

$$\begin{aligned}
\vec{\alpha} &= \frac{1}{2}\left(\frac{1}{\rho_e}\vec{\nabla}\left(\frac{1}{\rho_m}\right) - \frac{1}{\rho_m}\vec{\nabla}\left(\frac{1}{\rho_e}\right)\right) \\
&= \frac{\left(x^2+y^2+z^2+\left(\frac{R}{2}\right)^2\right)\frac{R}{2}\vec{a}_z - zR\left(x\vec{a}_x + y\vec{a}_y + z\vec{a}_z\right)}{\rho_e^3\rho_m^3} \\
\vec{r}\times\vec{\alpha} &= -\frac{\left(x^2+y^2+z^2+\left(\frac{R}{2}\right)^2\right)\frac{R}{2}\rho\vec{a}_\theta}{\rho_e^3\rho_m^3} \\
\vec{r}\cdot\vec{\alpha} &= \frac{\left(\left(\frac{R}{2}\right)^2 - x^2 - y^2 - z^2\right)\frac{Rz}{2}}{\rho_e^3\rho_m^3} \\
&= \frac{\left(\left(\frac{R}{2}\right)^2 - r^2\right)\frac{Rz}{2}}{\rho_e^3\rho_m^3} \\
\vec{\nabla}\times\vec{\alpha} &= \vec{\nabla}\left(\frac{1}{\rho_e}\right)\times\vec{\nabla}\left(\frac{1}{\rho_m}\right)
\end{aligned}$$

We see that $\vec{\alpha}$ is to momentum density as $\vec{A}$ is to magnetic field density $\vec{B}$. Consequently, I'll call $\vec{\alpha}$ the momentum density vector potential.

So, going back to our momentum density, we have

$$\epsilon\mu\vec{S} = \frac{\mu q_e q_m}{16\pi^2}\frac{yR\vec{a}_x - xR\vec{a}_y}{\left(x^2+y^2+(z-R/2)^2\right)^{3/2}\left(x^2+y^2+(z+R/2)^2\right)^{3/2}}$$

The momentum density circulates around the z axis, so we expect our total linear momentum to be zero. Our integral for total momentum is

$$\begin{aligned}
\vec{M} &= \iiint \epsilon\mu\vec{S}\,dxdydz \\
&= \frac{\mu q_e q_m}{16\pi^2}\iiint \frac{yR\vec{a}_x - xR\vec{a}_y}{\left(x^2+y^2+(z-R/2)^2\right)^{3/2}\left(x^2+y^2+(z+R/2)^2\right)^{3/2}}dxdydz
\end{aligned}$$

The denominator is invariant under sign inversion for x, y and z. To see the z invariance, notice that the left and right halves of the denominator swap, and $(-z - R/2)^2 = (z + R/2)^2$. The numerator is odd in x and y. By symmetry, away from the two sources, everything zeroes out.

The expression has two poles, at $(0, 0, \pm R/2)$. On the z axis, away from the sources, the contribution to the integral is zero. As we hit the actual pole, the numerator goes to zero linearly while the denominator goes to zero as z^3. I assert that the three integrals infinitesimals dx, dy and dz compensate for the cubic nature of the denominator, and that the linear numerator drives this contribution to zero.

Total Angular Momentum

As a reference, here is the approach found in Jackson, translated into SI units.

Goldhaber and Jackson Derivation

This is the approach in Jackson, Second Edition, pp. 254-256.

Place the monopole charge q_m at $z = R$. Place the electric charge q_e at the origin. The fields are

$$\begin{aligned}
\rho_e &= r \\
\vec{E} &= -\frac{q_e}{4\pi\epsilon}\vec{\nabla}\left(\frac{1}{r}\right) = q_e\frac{\vec{a}_r}{4\pi\epsilon r^2} = q_e\frac{\vec{r}}{4\pi\epsilon r^3} \\
\vec{\rho}_m &= \vec{r} - R\vec{a}_z \\
\vec{B} &= q_m\frac{\mu\vec{\rho}_m}{4\pi\rho_m^3}
\end{aligned}$$

We now write the field angular momentum as

$$\begin{aligned}
\vec{L} &= \epsilon \iiint \vec{r} \times \left(\vec{E} \times \vec{B}\right) dxdydz \\
&= \epsilon \iiint \left(\vec{E}\left(\vec{r} \cdot \vec{B}\right) - \vec{B}\left(\vec{r} \cdot \vec{E}\right)\right) dxdydz \\
&= \frac{q_e}{4\pi} \iiint \frac{1}{r^3} \left(\vec{r}\left(\vec{r} \cdot \vec{B}\right) - \vec{B}\left(\vec{r} \cdot \vec{r}\right)\right) dxdydz \\
&= \frac{q_e}{4\pi} \iiint \frac{1}{r} \left(\vec{a}_r\left(\vec{a}_r \cdot \vec{B}\right) - \vec{B}\right) dxdydz \\
&= -\frac{q_e}{4\pi} \iiint \left[\left(\vec{B} \cdot \vec{\nabla}\right) \vec{a}_r\right] dxdydz
\end{aligned}$$

I am good to this point. The step, I haven't verified to my satisfaction. The claim is to integrate by parts to achieve

$$\vec{L} = -\frac{q_e}{4\pi} \iint \vec{a}_r \left(\vec{B} \cdot d\vec{S}\right) + \frac{q_e}{4\pi} \iiint \vec{a}_r \left(\vec{\nabla} \cdot \vec{B}\right) dxdydz$$

Given this step, the next step is to assert that the surface integral goes to zero at infinity. We then identify

$$\begin{aligned}
\vec{B} &= -\frac{\mu q_m}{4\pi} \vec{\nabla}\left(\frac{1}{\rho_m}\right) \\
\vec{\nabla} \cdot \vec{B} &= -\frac{\mu q_m}{4\pi} \left(-4\pi\delta\left(\vec{r} - R\vec{a}_z\right)\right) = \mu q_m \delta\left(\vec{r} - R\vec{a}_z\right) \\
\vec{L} &= \frac{q_e}{4\pi} \iiint \vec{a}_r \left(\vec{\nabla} \cdot \vec{B}\right) dxdydz \\
&= \frac{q_e}{4\pi} \iiint \vec{a}_r \left(\mu q_m \delta\left(\vec{r} - R\vec{a}_z\right)\right) dxdydz \\
&= \frac{\mu q_e q_m}{4\pi} \vec{a}_z
\end{aligned}$$

We have postive electric charge, positive magnetic charge, and $\vec{L}$ points from electric to magnetic charge.

Evaluation In Cylindrical Coordinates

The arrangement of charges on the z axis naturally encourages the use of cylindrical coordinates. Place an electric charge at $(0,0,R/2)$. Place a magnetic charge at $(0,0,-R/2)$. I will use ρ as the distance from the z axis and θ as the angle from the x axis. We have

$$\begin{aligned}
r^2 &= x^2+y^2+z^2 \\
&= \rho^2+z^2 \\
x &= \rho\cos\theta \\
y &= \rho\sin\theta \\
\rho\vec{a}_\rho &= x\vec{a}_x + y\vec{a}_y \\
\rho\vec{a}_\theta &= -y\vec{a}_x + x\vec{a}_y \\
\vec{a}_\rho\times\vec{a}_\theta &= \vec{a}_z
\end{aligned}$$

Our angular momentum density is

$$\begin{aligned}
\vec{r}\times\left(\epsilon\mu\vec{S}\right) &= \vec{r}\times\left(\frac{\mu q_e q_m}{16\pi^2}\left[\vec{\nabla}\left(\frac{1}{\rho_e}\right)\times\vec{\nabla}\left(\frac{1}{\rho_m}\right)\right]\right) \\
&= \frac{\mu q_e q_m}{16\pi^2}\left(\vec{r}\times\left[\vec{\nabla}\left(\frac{1}{\rho_e}\right)\times\vec{\nabla}\left(\frac{1}{\rho_m}\right)\right]\right) \\
&= \frac{\mu q_e q_m}{16\pi^2}\frac{(x\vec{a}_x+y\vec{a}_y+z\vec{a}_z)\times(yR\vec{a}_x - xR\vec{a}_y)}{\left(x^2+y^2+(z-R/2)^2\right)^{3/2}\left(x^2+y^2+(z+R/2)^2\right)^{3/2}} \\
&= \frac{\mu q_e q_m}{16\pi^2}\frac{(\rho\vec{a}_\rho+z\vec{a}_z)\times(-R\rho\vec{a}_\theta)}{\left(\rho^2+(z-R/2)^2\right)^{3/2}\left(\rho^2+(z+R/2)^2\right)^{3/2}} \\
&= \frac{\mu q_e q_m}{16\pi^2}\frac{Rz\rho\vec{a}_\rho - R\rho^2\vec{a}_z}{\left(\rho^2+(z-R/2)^2\right)^{3/2}\left(\rho^2+(z+R/2)^2\right)^{3/2}}
\end{aligned}$$

The total angular momentum is

$$\begin{aligned}
\vec{L} &= \frac{\mu q_e q_m R}{16\pi^2}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi}\frac{z\rho\vec{a}_\rho - \rho^2\vec{a}_z}{\left(\rho^2+(z-R/2)^2\right)^{3/2}\left(\rho^2+(z+R/2)^2\right)^{3/2}}\rho d\theta\ d\rho\ dz \\
&= \frac{\mu q_e q_m R}{8\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{z\rho^2\vec{a}_\rho - \rho^3\vec{a}_z}{\left(\rho^2+(z-R/2)^2\right)^{3/2}\left(\rho^2+(z+R/2)^2\right)^{3/2}}\ d\rho\ dz
\end{aligned}$$

Looking at the radial component in the integral, we see positive outflows in the northern hemisphere, and negative outflows in the southern hemisphere. Looking at the denominator, we see that this term is symmetric with regard to the sign of z. Every ring in the northern hemisphere has a cancelling ring in the southern hemisphere, and the total radial flux will be zero. I am very interested in seeing the local effects of this flux at a later time.

Consequently, our integral simplifies to

$$\vec{L} = -\frac{\mu q_e q_m R\vec{a}_z}{8\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{\rho^3}{\left(\rho^2+(z-R/2)^2\right)^{3/2}\left(\rho^2+(z+R/2)^2\right)^{3/2}}\,d\rho\,dz$$

We can make a number of statements about this expression. First, we only have a z component of spin. Second, at every slice in z, the interior integral is positive. Third, we have azimuthal symmetry about the z axis. Finally, this expression has two poles, at $(0,0,\pm R/2)$.

Using Wolfram Alpha, we find an expression for the inner integral.

$$\int_r \frac{r^3}{\left(r^2+a^2\right)^{3/2}\left(r^2+b^2\right)^{3/2}}dr = \frac{2a^2b^2+r^2\left(a^2+b^2\right)}{\left(a^2-b^2\right)^2\sqrt{r^2+a^2}\sqrt{r^2+b^2}}$$

$$\begin{aligned}
Inner &= \int_0^{\infty}\frac{\rho^3}{\left(\rho^2+(z-R/2)^2\right)^{3/2}\left(\rho^2+(z+R/2)^2\right)^{3/2}}\,d\rho \\
&= \left[\frac{2(z+R/2)^2(z-R/2)^2+\rho^2((z+R/2)^2+(z-R/2)^2))}{((z+R/2)^2-(z-R/2)^2)^2\sqrt{\rho^2+(z-R/2)^2}\sqrt{\rho^2+(z+R/2)^2}}\right]_0^{\infty} \\
&= \left[\frac{2(z^2-R^2/4)^2+2\rho^2(z^2+R^2/4))}{(2Rz)^2\sqrt{\rho^2+(z-R/2)^2}\sqrt{\rho^2+(z+R/2)^2}}\right]_0^{\infty} \\
&= \left[\frac{(z^2-R^2/4)^2+\rho^2(z^2+R^2/4))}{2R^2z^2\sqrt{\rho^2+(z-R/2)^2}\sqrt{\rho^2+(z+R/2)^2}}\right]_0^{\infty} \\
&= \frac{(z^2+R^2/4)}{2R^2z^2}-\frac{(z^2-R^2/4)^2}{2R^2z^2\sqrt{(z-R/2)^2}\sqrt{(z+R/2)^2}} \\
&= \frac{(z^2+R^2/4)}{2R^2z^2}-\frac{(z^2-R^2/4)^2}{2R^2z^2\sqrt{(z^2-R^2/4)^2}}
\end{aligned}$$

Before we do the next step, I want to point out that the right hand portion after the minus sign is alway positive definite. Thus, I put some absolute value signs in the expression that follows.

$$\begin{aligned} Inner &= \frac{(z^2 + R^2/4)}{2R^2z^2} - \frac{|z^2 - R^2/4|}{2R^2z^2} \\ &= \frac{(z^2 + R^2/4 - |z^2 - R^2/4|)}{2R^2z^2} \\ &= \frac{(4z^2 + R^2 - |4z^2 - R^2|)}{8R^2z^2} \end{aligned}$$

Five Zones for the Integral

The inner radial integral really has five zones along the z axis. The first is from positive infinity to just outside our positive pole. The second is at the positive pole. The third spans between the positive and negative poles. The fourth is at the negative pole, while the fifth extends from the negative pole to negative infinity.

In zones one, three and five, we have an integral of the form above. Let's evaluate this inner integral.

For zones one and five, we have $z^2 > R^2/4$, and the inner integral is

$$Inner = \frac{4z^2 + R^2 - (4z^2 - R^2)}{8R^2z^2} = \frac{2R^2}{8R^2z^2} = \frac{1}{4z^2}$$

For zone three, we have $z^2 < R^2/4$, and the integral is

$$Inner = \frac{4z^2 + R^2 - (R^2 - 4z^2)}{8R^2z^2} = \frac{8z^2}{8R^2z^2} = \frac{1}{R^2}$$

We see we have continuity at the poles with this expression, but we should really evaluate at points two and four for thoroughness.

For this point, I choose $z = R/2$ and the integral specializes to

$$\int_{\rho=0}^{\rho=\infty} \frac{\rho^2}{\left(\rho^2+(z-R/2)^2\right)^{3/2}\left(\rho^2+(z+R/2)^2\right)^{3/2}}\rho d\rho = \int_{\rho=0}^{\rho=\infty} \frac{\rho^2}{\left(\rho^2\right)^{3/2}\left(\rho^2+(R)^2\right)^{3/2}}\rho d\rho$$
$$= \int_{\rho=0}^{\rho=\infty} \frac{d\rho}{\left(\rho^2+R^2\right)^{3/2}}$$

Now

$$\int_{\rho=0}^{\rho=\infty} \frac{d\rho}{\left(\rho^2+R^2\right)^{3/2}} = \left[\frac{\rho}{R^2\sqrt{\rho^2+R^2}}\right]_{\rho=0}^{\rho=\infty}$$
$$= \frac{1}{R^2}$$

In a similar fashion, at the other pole, we get the same value. This is satisfying, as our values are continuous across the poles. We are now able to finish evaluating our integral for total field angular momentum.

$$\vec{L} = -\vec{a}_z \frac{\mu q_e q_m R}{8\pi} \int_z \left[\int_{\rho=0}^{\rho=\infty} \frac{\rho^2}{\left(\rho^2+(z+R/2)^2\right)^{3/2}\left(\rho^2+(z-R/2)^2\right)^{3/2}} d(\rho^2)\right] dz$$

$$\vec{L} = -\vec{a}_z \frac{\mu q_e q_m R}{8\pi} \left[\int_{R/2}^{\infty} \frac{dz}{4z^2} + \int_{-R/2}^{R/2} \frac{dz}{R^2} + \int_{-\infty}^{-R/2} \frac{dz}{4z^2}\right]$$
$$= -\vec{a}_z \frac{\mu q_e q_m R}{8\pi} \left[\left[-\frac{1}{4z}\right]_{R/2}^{\infty} + \left[\frac{z}{R^2}\right]_{-R/2}^{R/2} + \left[-\frac{1}{4z}\right]_{-\infty}^{-R/2}\right]$$
$$= -\vec{a}_z \frac{\mu q_e q_m R}{8\pi} \left[\left(0+\frac{2}{4R}\right) + \left(\frac{R}{R^2}\right) + \left(\frac{2}{4R}+0\right)\right]$$

$$\vec{L} = -\vec{a}_z \frac{\mu q_e q_m}{4\pi}$$

This results agrees with Goldhaber and Jackson. We have the same magnitude, irregardless of distance, and the momentum points from the positive electric to the positive magnetic charge. My minus sign on the spin is due to my placement of the electric charge at $z = R/2$ and the magnetic charge at $z = -R/2$, which is upside down from the Goldhaber model.

However, as we further examine this expression, we see a few items of interest. Half the field contribution occurs in between the two charges. As our charges approach each other, this planar contribution spikes. For the case where the charges coincide, we will see a planar delta function. This might be the basis for Fermi Exclusion.

Repeat Integration Using Spherical Coordinates

In the limit as $R \to 0$, I see half the angular momentum compressed into a plane. This result should show in Cartesian and cylindrical integrations, but *not* show in spherical coordinates, due to mismatch between the surfaces of integration versus surface of interest. Let us repeat the integration using spherical, rather than cylindrical coordinates.

My integral is

$$\begin{aligned}
\vec{L} &= \frac{\mu q_e q_m}{16\pi^2} \iiint \vec{r} \times \left[\vec{\nabla}\left(\frac{1}{\rho_e}\right) \times \vec{\nabla}\left(\frac{1}{\rho_m}\right)\right] r d\theta \; r\sin\phi \; d\phi \; dr \\
\vec{L} &= \frac{\mu q_e q_m}{16\pi^2} \iiint \vec{r} \times \left[\vec{\nabla} \times \vec{\alpha}\right] r d\theta \; r\sin\phi \; d\phi \; dr \\
\vec{L} &= -\frac{\mu q_e q_m}{16\pi^2} \iiint \left[\vec{\nabla} \times \vec{\alpha}\right] \times (r d\theta \vec{a}_\theta \times \; r\sin\phi \; d\phi \vec{a}_\phi) \; r dr \\
\vec{L} &= -\frac{\mu q_e q_m}{16\pi^2} \int_r \left[\int_\theta \int_\phi \left[\vec{\nabla} \times \vec{\alpha}\right] \times d\vec{S} \right] r dr
\end{aligned}$$

This last expression, looking at angular momentum as a series of integrations over spherical shells, invites a Stokes' theorem variation using curl rather than divergence for evaluation. This is on my 'to do' list.

So,

$$
\begin{aligned}
x &= r\cos\theta\sin\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\phi \\
\rho &= \sqrt{x^2+y^2} = r\sin\phi \\
r^2 &= x^2+y^2+z^2 \\
y\vec{a}_x - x\vec{a}_y &= \rho\vec{a}_\theta = r\sin\phi\vec{a}_\theta \\
\vec{\nabla}\times\vec{\alpha} &= \frac{Ry\vec{a}_x - Rx\vec{a}_y}{\left(x^2+y^2+(z-R/2)^2\right)^{3/2}\left(x^2+y^2+(z+R/2)^2\right)^{3/2}} \\
&= \frac{Rr\sin\phi\vec{a}_\theta}{(r^2+(R/2)^2-rR\cos\phi)^{3/2}(r^2+(R/2)^2+rR\cos\phi)^{3/2}}
\end{aligned}
$$

Our interior integrals from above are

$$
\int_\phi\int_\theta \frac{Rr\sin\phi\vec{a}_\theta\times\vec{a}_r(rd\theta r\sin\phi d\phi)}{(r^2+(R/2)^2-rR\cos\phi)^{3/2}(r^2+(R/2)^2+rR\cos\phi)^{3/2}} =
$$

$$
-\int_\phi\int_\theta \frac{Rr\sin\phi\vec{a}_\phi(rd\theta r\sin\phi d\phi)}{(r^2+(R/2)^2-rR\cos\phi)^{3/2}(r^2+(R/2)^2+rR\cos\phi)^{3/2}} =
$$

$$
-\int_\phi\int_\theta \frac{Rr\sin\phi(\cos\phi\vec{a}_\rho - \sin\phi\vec{a}_z)(rd\theta r\sin\phi d\phi)}{(r^2+(R/2)^2-rR\cos\phi)^{3/2}(r^2+(R/2)^2+rR\cos\phi)^{3/2}}
$$

This is a good time to take the θ integration. The $\vec{a}_\rho$ terms disappear by symmetry, as $\vec{a}_\rho(\theta) = -\vec{a}_\rho(\theta+\pi)$, and the integral arguments have cylindrical symmetry.

$$
2\pi R\int_\phi \frac{(\sin\phi\vec{a}_z)(r^3\sin^2\phi d\phi)}{(r^2+(R/2)^2-rR\cos\phi)^{3/2}(r^2+(R/2)^2+rR\cos\phi)^{3/2}} =
$$

Our integral is now

$$
2\pi Rr^3\vec{a}_z\int_\phi \frac{\sin^3\phi d\phi}{(r^2+(R/2)^2-rR\cos\phi)^{3/2}(r^2+(R/2)^2+rR\cos\phi)^{3/2}} =
$$

$$
2\pi Rr^3\vec{a}_z\int_\phi \frac{(-\sin^2\phi)d(\cos\phi)}{(r^2+(R/2)^2-rR\cos\phi)^{3/2}(r^2+(R/2)^2+rR\cos\phi)^{3/2}} =
$$

$$
-2\pi r^3R\vec{a}_z\int_{\phi=0}^{\phi=\pi} \frac{(1-\cos^2\phi)d(\cos\phi)}{(r^2+(R/2)^2-rR\cos\phi)^{3/2}(r^2+(R/2)^2+rR\cos\phi)^{3/2}}
$$

We now substitute $m = \cos\phi$.

$$-2\pi r^3 R\vec{a}_z \int_{\phi=0}^{\phi=\pi} \frac{(1-\cos^2\phi)d(\cos\phi)}{(r^2+(R/2)^2 - rR\cos\phi)^{3/2}(r^2+(R/2)^2+rR\cos\phi)^{3/2}} =$$

$$-2\pi r^3 R\vec{a}_z \int_{m=1}^{m=-1} \frac{(1-m^2)dm}{(r^2+(R/2)^2 - mrR)^{3/2}(r^2+(R/2)^2+mrR)^{3/2}} =$$

$$2\pi r^3 R\vec{a}_z \int_{m=-1}^{m=1} \frac{(1-m^2)dm}{(r^2+(R/2)^2 - mrR)^{3/2}(r^2+(R/2)^2+mrR)^{3/2}}$$

Let

$$\begin{aligned} a &= r^2 + (R/2)^2 \\ b &= rR \end{aligned}$$

Our integral template becomes

$$\int_{m=-1}^{m=1} \frac{(1-m^2)dm}{(a-mb)^{3/2}(a+mb)^{3/2}} = \int_{m=-1}^{m=1} \frac{(1-m^2)dm}{(a^2-b^2m^2)^{3/2}}$$

Courtesy of Sage and Maxsyma, we have

$$\int \frac{1-m^2}{(a^2-b^2m^2)^{3/2}}dm = \frac{1}{b^3}\tan^{-1}\left(\frac{bm}{\sqrt{a^2-b^2m^2}}\right) - \frac{m}{a^2b^2}\frac{a^2-b^2}{\sqrt{a^2-b^2m^2}}$$

We note $a^2 - b^2$ is always positive. Applying our limits, we have

$$\begin{aligned} \int_{m=-1}^{m=1} \frac{(1-m^2)dm}{(a^2-b^2m^2)^{3/2}} &= \frac{2}{b^3}\tan^{-1}\left(\frac{b}{\sqrt{a^2-b^2}}\right) - \frac{2}{a^2b^2}\frac{a^2-b^2}{\sqrt{a^2-b^2}} \\ &= \frac{2}{b^3}\tan^{-1}\left(\frac{b}{\sqrt{a^2-b^2}}\right) - 2\frac{\sqrt{a^2-b^2}}{a^2b^2} \end{aligned}$$

We now back substitute.

$$\begin{aligned} a &= r^2 + (R/2)^2 \\ b &= rR \\ a^2 - b^2 &= r^4 + (R/2)^4 + 2r^2(R/2)^2 - r^2R^2 \\ &= \left(r^2 - (R/2)^2\right)^2 \\ \sqrt{a^2-b^2} &= |r^2 - (R/2)^2| \end{aligned}$$

For the zone $R/2 < r < \infty$, we have the expression

$$\frac{2}{r^3R^3}\tan^{-1}\left(\frac{rR}{r^2-(R/2)^2}\right) - \frac{2\left(r^2-(R/2)^2\right)}{\left(r^2+(R/2)^2\right)^2 r^2R^2}$$

For the zone $0 < r < R/2$, we have the expression

$$\frac{2}{r^3R^3}\tan^{-1}\left(\frac{rR}{(R/2)^2-r^2}\right) - \frac{2\left((R/2)^2-r^2\right)}{\left(r^2+(R/2)^2\right)^2 r^2R^2}$$

We are now ready to finish our integrations.

$$\begin{aligned}
\vec{L} &= -\frac{\mu q_e q_m}{16\pi^2}\int_{R/2}^{\infty} 2\pi r^3 R\vec{a}_z\left[\frac{2}{r^3R^3}\tan^{-1}\left(\frac{rR}{r^2-(R/2)^2}\right) - \frac{2\left(r^2-(R/2)^2\right)}{\left(r^2+(R/2)^2\right)^2 r^2R^2}\right] rdr \\
&\quad -\frac{\mu q_e q_m}{16\pi^2}\int_{0}^{R/2} 2\pi r^3 R\vec{a}_z\left[\frac{2}{r^3R^3}\tan^{-1}\left(\frac{rR}{(R/2)^2-r^2}\right) - \frac{2\left((R/2)^2-r^2\right)}{\left(r^2+(R/2)^2\right)^2 r^2R^2}\right] rdr \\
\vec{L} &= -\vec{a}_z\frac{\mu q_e q_m}{8\pi}\int_{R/2}^{\infty}\left[\frac{2r}{R^2}\tan^{-1}\left(\frac{rR}{r^2-(R/2)^2}\right) - \frac{2r^2\left(r^2-(R/2)^2\right)}{\left(r^2+(R/2)^2\right)^2 R}\right] dr \\
&\quad -\vec{a}_z\frac{\mu q_e q_m}{8\pi}\int_{0}^{R/2}\left[\frac{2r}{R^2}\tan^{-1}\left(\frac{rR}{(R/2)^2-r^2}\right) - \frac{2r^2\left((R/2)^2-r^2\right)}{\left(r^2+(R/2)^2\right)^2 R}\right] dr
\end{aligned}$$

Our first integral is

$$\begin{aligned}
&-\vec{a}_z\frac{\mu q_e q_m}{8\pi}\int_{R/2}^{\infty}\left[\frac{2r}{R^2}\tan^{-1}\left(\frac{rR}{r^2-(R/2)^2}\right) - \frac{2r^2\left(r^2-(R/2)^2\right)}{\left(r^2+(R/2)^2\right)^2 R}\right] dr \\
= &-\vec{a}_z\frac{\mu q_e q_m}{8\pi}\left[-\frac{2Rr}{R^2+4r^2} + \frac{r^2}{R^2}\tan^{-1}\left(\frac{4Rr}{4r^2-R^2}\right) - \frac{r}{R} + \frac{3}{2}\tan^{-1}\left(\frac{2r}{R}\right)\right]_{R/2}^{\infty}
\end{aligned}$$

As we approach infinity, the first term in the bracket disappears, the second and third terms cancel each other, and we have a residual value from the last term of $3\pi/4$. For the lower limit, we have a term in brackets of

$$-\frac{1}{2} + \frac{1}{4}\frac{\pi}{2} - \frac{1}{2} + \frac{3}{2}\frac{\pi}{4} = -1 + \frac{\pi}{2}$$

Taking our difference, our first integral becomes

$$-\vec{a}_z \frac{\mu q_e q_m}{8\pi}\left[1+\frac{\pi}{4}\right]$$

Our second integral evaluates much like first, with an overall change in sign.

$$\begin{aligned}
& -\vec{a}_z \frac{\mu q_e q_m}{8\pi}\int_0^{R/2}\left[\frac{2r}{R^2}\tan^{-1}\left(\frac{rR}{(R/2)^2-r^2}\right)-\frac{2r^2\left((R/2)^2-r^2\right)}{\left(r^2+(R/2)^2\right)^2 R}\right]dr \\
& \vec{a}_z \frac{\mu q_e q_m}{8\pi}\int_0^{R/2}\left[\frac{2r}{R^2}\tan^{-1}\left(\frac{rR}{r^2-(R/2)^2}\right)-\frac{2r^2\left(r^2-(R/2)^2\right)}{\left(r^2+(R/2)^2\right)^2 R}\right]dr \\
= \ & \vec{a}_z \frac{\mu q_e q_m}{8\pi}\left[-\frac{2Rr}{R^2+4r^2}+\frac{r^2}{R^2}\tan^{-1}\left(\frac{4Rr}{4r^2-R^2}\right)-\frac{r}{R}+\frac{3}{2}\tan^{-1}\left(\frac{2r}{R}\right)\right]_0^{R/2}
\end{aligned}$$

With this integral, we easily see the lower limit is zero. For the upper limit, we need to be a bit careful. Figure 1 shows a plot of the first $\tan^{-1}$ term of this integral. We have a discontinuity at $r = R/2$. In our zone, we are negative.

We will need to take the first inverse tangent as the negative value here. We have terms in the brackets of

$$-\frac{1}{2}-\frac{1}{4}\frac{\pi}{2}-\frac{1}{2}+\frac{3}{2}\frac{\pi}{4}=-1+\frac{\pi}{4}$$

Our second integral is

$$\vec{a}_z \frac{\mu q_e q_m}{8\pi}\left[-1+\frac{\pi}{4}\right]=-\vec{a}_z \frac{\mu q_e q_m}{8\pi}\left[1-\frac{\pi}{4}\right]$$

Our total integral becomes

$$-\vec{a}_z \frac{\mu q_e q_m}{8\pi}2=-\vec{a}_z \frac{\mu q_e q_m}{4\pi}$$

This overall result agrees with the previous cylindrical integration, and totally misses the concentration of angular momentum between the charges, as shown previously.

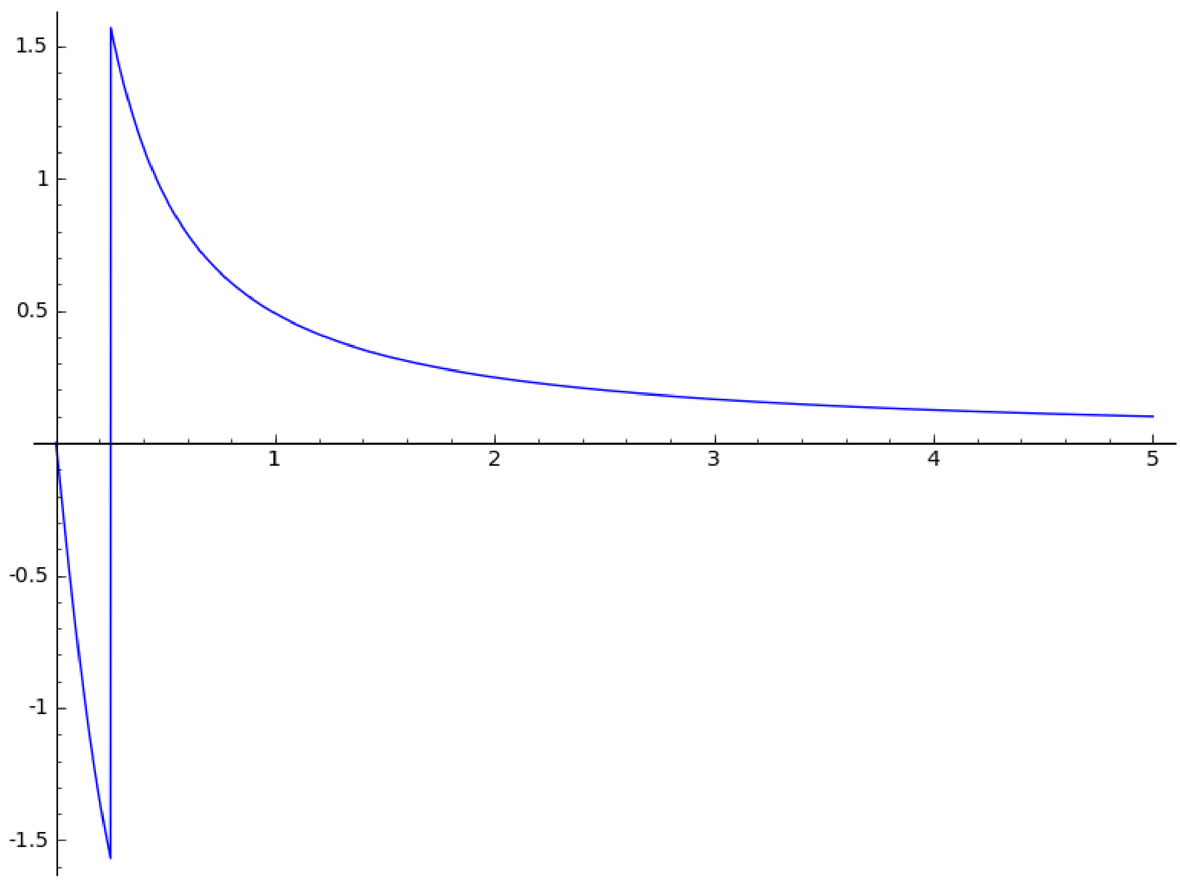

Figure 1: $\tan^{-1}(4rR/(4r^2 - R^2))$

Electromagnetic Duality and Colocation

So, I want to examine the 'dyon' model where magnetic charge exists, but is colocated, with electric charge. This brings us to the discussion of duality in Maxwell's equations.

While duality is covered in both Jackson and Schwinger using CGS units, I will be following the notes from Professor Steven Errede, who uses SI units, in his UIUC Physics 435 Lecture Notes 18, from Fall 2007.

The generalized Maxwell equations have an internal symmetry regarding charges and fields. We can simultaneously rotate electric and magnetic charges, concurrent with rotating electric and magnetic fields, and satisfy the same Maxwell Equations.

Items being mixed need to have the same units. Consequently, formulas in CGS versus SI will appear different. In SI, we will see factors of c being applied to charge, currents and B fields.

Using SI units, this duality transformation with mixing angle ψ for charges, currents and fields, is

$$\begin{aligned} cq_e &= cq'_e \cos\psi + q'_m \sin\psi \\ q_m &= -cq'_e \sin\psi + q'_m \cos\psi \end{aligned}$$

$$\begin{aligned} c\vec{J}_e &= c\vec{J}'_e \cos\psi + \vec{J}'_m \sin\psi \\ \vec{J}_m &= -c\vec{J}'_e \sin\psi + \vec{J}'_m \cos\psi \end{aligned}$$

$$\begin{aligned} \vec{E} &= \vec{E}' \cos\psi + c\vec{B}' \sin\psi \\ c\vec{B} &= -\vec{E}' \sin\psi + c\vec{B}' \cos\psi \end{aligned}$$

The force on a particle possessing both electric and magnetic charge is

$$\vec{F} = q_e \left(\vec{E} + \vec{v} \times \vec{B}\right) + q_m \left(\frac{\vec{B}}{\mu} - \epsilon\vec{v} \times \vec{E}\right)$$

The usual interpretation of this duality transformation is if all particles have the same ratio of electric to magnetic charge, we can choose a convention (meaning angle ψ) which eliminates magnetic monopole contributions, resulting in the standard Maxwell equations. Repeating, slightly differently, if the ratio of magnetic to electric charge is constant across all charged particles, the claim is that we cannot ascertain the ratio, and might as well choose our standard Maxwell equations.

The presense of inherent electron spin provides an additional restraint on the above equations, which allows the determination of the possibility of colocation, and if colocated, the mixing angle ψ.

Let's examine the charge mixing equation from above. Our standard measurements of electric fields, magnetic fields and charges are based upon the force law without magnetic charge terms. This set of equations conserves the quantity $c^2 q_e^2 + q_m^2$. Our measurements assume all charge is electric, and none magnetic, so the magnitude above is simply $c^2 q_e^2$ in our SI units. If I set the x axis to be c times the electric charge, and the y axis to be magnetic charge, the mixing formula simply plots a nice circle. Now, we know the electron has an inherent spin, and we know that a magnetic charge interacting with an electric charge also has an inherent spin, independent of distance, *which may be zero.* This momentum,

$$L = \frac{\mu q_e q_m}{4\pi}$$

has a hyperbolic relationship between q_e and q_m, and if plotted on the same axis as the duality relationship above, can identify the mixing angle. Being pendantic, the hyperbola might not intersect, might intersect at a single point, or might intersect at a pair of points in the first quadrant depending upon the equation constants.

For a colocated magnetic and electric charge, the maximum angular momentum will occur at the 45 degree tangent criteria. For the electron, assuming our measured charge is the radius of the duality circle, we can calculate the maximum internal spin momentum as

$$\begin{aligned}
q_{e45} &= q_e \frac{\sqrt{2}}{2} \\
q_{m45} &= c q_e \frac{\sqrt{2}}{2} \\
L_{max} &= \frac{\mu q_{e45} q_{m45}}{4\pi} \\
&= \frac{\mu c q_e^2}{8\pi} = 3.84 \cdot 10^{-37} \text{ Js}
\end{aligned}$$

This number is much smaller than Planck's constant, by a factor of 274 or so, effectively ruling out colocated charge as the origin of electron spin at the large scale.

Let's look at these numbers from another point of view. From the conserved magnitude of the duality rotated charge, we have

$$c^2 q_e^2 + q_m^2 \;=\; \left[\left(2.9989 \cdot 10^8 \text{ m/s}\right) \cdot \left(1.6 \cdot 10^{-19} C\right)\right]^2 = 2.3 \cdot 10^{-21} \text{ Amp}^2 \text{ m}^2$$

For the conserved angular momentum, we have

$$\begin{aligned}
L &= \frac{\mu q_e q_m}{4\pi} = \frac{1}{2}\hbar \\
\mu q_e q_m &= 4\pi \left(\frac{1}{2}\hbar\right) = 2h \\
q_e q_m &= \frac{2h}{\mu} \\
c q_e q_m &= \frac{2hc}{\mu} = \frac{2(1.325 \cdot 10^{-33}\text{J s})(2.9989 \cdot 10^8 \text{ m/s})}{4\pi 10^{-7} \text{ H/m}} \\
&= 3.16 \cdot 10^{-19} \text{ Amp}^2 \text{ m}^2
\end{aligned}$$

Before proceding, we note that the ratio $c q_e q_m/(c^2 q_e^2 + q_m^2) = 137.04$, the (inverse) fine structure constant.

Returning to the discussion, the point of closest approach for the hyperbola $xy = K^2$ to the origin is $K\sqrt{2}$. With our values above, the angular momentum hyperbola never intersects the duality circle by a factor of 137, effectively ruling out any model of the electron as a simple colocated electric and magnetic charge combo.

Magnetic Monopole and Charge Trajectories

While we can rule out a colocated electric and real magnetic charge model for the electron, we have not excluded composite, separated assemblies of electric and magnetic charge, nor have we addressed imaginary magnetic charge, which will be a separate note.

Electron Motion in a Radial Magnetic Field

The force law for a pure electron in an electromagnetic field is

$$\vec{F} = q_e \left(\vec{E} + \vec{v} \times \vec{B} \right)$$

A pure magnetic field can do no work. Force is at right angles to motion, and results only in a change of direction. Consequently, the speed is constant, and the velocity and acceleration are always orthogonal.

To describe the motion of the electron, I will use the Frenet-Serret approach, and identify the curvature and torsion formulas for the electron in the radial magnetic field. I assume that Newtonian mass and acceleration still apply in this scenario.

$$\begin{aligned}
\vec{B} &= \frac{\mu q_m}{4\pi} \frac{\vec{r}}{r^3} \\
\vec{F} &= q_e \left(\vec{v} \times \vec{B} \right) \\
&= \frac{\mu q_e q_m}{4\pi} \frac{\vec{v} \times \vec{r}}{r^3} \\
\vec{a} &= \frac{\vec{F}}{m} = \frac{\mu q_e q_m}{4\pi} \frac{1}{m} \frac{\vec{v} \times \vec{r}}{r^3}
\end{aligned}$$

We see above that $\vec{a}$ is normal to velocity, and see below that speed will be constant.

$$\begin{aligned}
\vec{a} \cdot \vec{v} &= 0 = \frac{1}{2} \frac{d}{dt} \left(\vec{v} \cdot \vec{v} \right) \\
&= \frac{1}{2} \frac{dv^2}{dt} \\
v^2 &= \text{const}
\end{aligned}$$

The magnitude of curvature in three dimensions is

$$\begin{aligned}
\vec{\kappa} &= \frac{\vec{a} \times \vec{v}}{v^3} \\
\kappa &= \frac{av \sin \theta}{v^3} \\
&= \frac{a}{v^2} \quad \text{for ṽ ⊥ ã}
\end{aligned}$$

The magnitude of torsion in three dimensions is

$$\begin{aligned}
\tau &= \frac{\vec{j}\cdot(\vec{a}\times\vec{v})}{(\vec{a}\times\vec{v})\cdot(\vec{a}\times\vec{v})} \\
\vec{j} &= \frac{d\vec{a}}{dt} \\
\vec{a} &= \frac{\vec{F}}{m} = \frac{\mu q_e q_m}{4\pi}\frac{1}{m}\frac{\vec{v}\times\vec{r}}{r^3} \\
\vec{j} &= \frac{\mu q_e q_m}{4\pi m}\left[\frac{\vec{a}\times\vec{r}}{r^3} - \frac{3}{r^4}\frac{dr}{dt}(\vec{v}\times\vec{r})\right] \\
&= \frac{\mu q_e q_m}{4\pi m}\left[\frac{\vec{a}\times\vec{r}}{r^3} - \frac{3}{r^4}\frac{\vec{r}\cdot\vec{v}}{r}(\vec{v}\times\vec{r})\right] \\
&= \frac{\mu q_e q_m}{4\pi m}\left[\frac{\vec{a}\times\vec{r}}{r^3} - \frac{3}{r^2}(\vec{r}\cdot\vec{v})\frac{(\vec{v}\times\vec{r})}{r^3}\right] \\
&= \frac{\mu q_e q_m}{4\pi m}\left[\frac{\vec{a}\times\vec{r}}{r^3} - \frac{3(\vec{r}\cdot\vec{v})}{r^2}\vec{a}\right]
\end{aligned}$$

Before proceding, I want to comment on these jerk terms. The left portion, $\vec{a}\times\vec{r}$, is a turning term, bring $\vec{J}$ perpendicular to $\vec{a}$ and resulting in constant acceleration for the electron. The right hand term, $-3\vec{a}(\vec{r}\cdot\vec{v})/r^2$, is a strong damping term for radial motion. I'll do some simulations shortly, but at first glance, the jerk should quickly circularize the electron in a flat orbit.

Simplifying a bit, since $\vec{a}\perp\vec{v}$,

$$\begin{aligned}
\tau &= \frac{\vec{j}\cdot(\vec{a}\times\vec{v})}{(\vec{a}\times\vec{v})\cdot(\vec{a}\times\vec{v})} \\
&= \frac{\vec{j}\cdot(\vec{a}\times\vec{v})}{a^2v^2}
\end{aligned}$$

Continuing with our development for the torsion term, $\vec{a}\times\vec{v}$ is perpendicular to $\vec{a}$, so the second term in the jerk formula will drop out.

$$
\begin{aligned}
\tau &= \frac{\mu q_e q_m}{4\pi m a^2 v^2}\left[\frac{\vec{a}\times\vec{r}}{r^3} - \frac{3(\vec{r}\cdot\vec{v})}{r^2}\vec{a}\right]\cdot(\vec{a}\times\vec{v}) \\
&= \frac{\mu q_e q_m}{4\pi m a^2 v^2}\left[\frac{(\vec{a}\times\vec{r})\cdot(\vec{a}\times\vec{v})}{r^3}\right] \\
&= \frac{\mu q_e q_m}{4\pi m a^2 v^2}\left[\frac{a^2(\vec{r}\cdot\vec{v}) - (\vec{a}\cdot\vec{v})(\vec{r}\cdot\vec{a})}{r^3}\right] \\
&= \frac{\mu q_e q_m}{4\pi m a^2 v^2}\left[\frac{a^2(\vec{v}\cdot\vec{r})}{r^3}\right]
\end{aligned}
$$

We have the results,

$$
\begin{aligned}
\kappa &= \frac{a}{v^2} \\
\vec{\kappa} &= \frac{\mu q_e q_m}{4\pi}\frac{1}{m}\frac{\vec{r}v^2 - \vec{v}(\vec{v}\cdot\vec{r})}{r^3 v^3} \\
\vec{\tau} &= \frac{\mu q_e q_m}{4\pi}\frac{1}{m}\left[\frac{\vec{v}(\vec{v}\cdot\vec{r})}{r^3 v^3}\right] \\
\vec{\kappa}+\vec{\tau} &= \frac{\mu q_e q_m}{4\pi}\frac{1}{mrv}\frac{\vec{r}}{r^2} \\
\sqrt{\kappa^2+\tau^2} &= \frac{\mu q_e q_m}{4\pi}\frac{1}{mrv}\frac{1}{r}
\end{aligned}
$$

Since the velocity is constant for this system, we see that curvature is proportional to acceleration. From the torsion formula, we see that the torsion drops as the inverse square of separation. We also see that if the velocity ever goes perpendicular to separation, planar orbiting will result. Finally, we see that the combined curvature skyrockets as $r \to 0$.

Planar Motion

To restrict motion to a plane, we need zero torsion, which implies $\vec{v} \perp \vec{r}$. This in turn, implies constant distance from monopole, and constant magnitude of B. We have already seen constant speed required, and a constant speed and radius leads to constant acceleration. All in all, a pretty boring circular orbit. Some novelties are found, however. The orbital radius is not always an equatorial radius. The electron can orbit at any arbitrary latitude.

The next observation is more interesting. A positive charge orbits a positive monopole in a clockwise direction seen from further out, in the lower density B region. If I magically reverse the direction of the electric charge at a point, the new orbit is in a plane at right angles to the previous orbit. A 180 degree change in initial conditions leads to a 90 degree change in the solution. We still orbit in a clockwise direction about the radial flux. I find this fascinating.

Here are the formulae for the orbital radius and acceleration of our planar solution.

$$\begin{aligned}
\rho &= \frac{1}{\kappa} \\
&= \frac{v^2}{a} \\
a &= \frac{\mu q_e q_m}{4\pi}\frac{1}{m}\frac{v}{r^2} \\
\rho &= mv\frac{4\pi r^2}{\mu q_e q_m}
\end{aligned}$$

We have a little gem hidden in this formula. Suggestively re-arranging our equation, we have

$$\begin{aligned}
\rho &= mv\frac{4\pi r^2}{\mu q_e q_m} \\
mvr &= \frac{\mu q_e q_m}{4\pi}\frac{\rho}{r}
\end{aligned}$$

This shows that for the case of equatorial orbits, where $\rho = r$ and $L = mvr$, the classical angular momentum is equal to the field momentum of the electric charge/magnetic monopole pair.

Minimum Radius for Equatorial Orbits

We have yet another gem hidden in this formula. The maximum value for ρ is $\rho = r$ for an equatorial orbit. Our speed for this system is constant. Rather than looking for ρ, let's find r as a function of speed and mass.

$$\rho = r = mv\frac{4\pi r^2}{\mu q_e q_m}$$
$$r = \frac{\mu q_e q_m}{4\pi}\frac{1}{mv}$$

Given the maximum speed is the speed of light, we see we have a minimum separation between electric and magnetic charges for equatorial planar orbits.

$$r_{min} = \frac{\mu q_e q_m}{4\pi}\frac{1}{mc}$$

Force Ratios

We earlier had to dismiss colocated electric and magnetic charges, as the known electron spin greatly exceeds the maximum Maxwell Equation duality spin. Can we go backwards, and estimate the magnetic charge from the known spin? The answer, is yes, of course.

$$L = \frac{\mu q_e q_m}{4\pi} = \hbar/2$$
$$q_m = \frac{\hbar 4\pi}{2q_e} = \frac{h}{\mu q_e} = 3.29 \cdot 10^{-9} \text{ A m}$$

So, we have an expression for the magnetic charge, can we figure out ratio of magnetic to electric forces?

The answer, is yes, of course.

$$F_e = \frac{q_e^2}{4\pi\epsilon r^2}$$
$$F_m = \frac{\mu q_m^2}{4\pi r^2}$$
$$F_m/F_e = \frac{\mu\epsilon q_m^2}{q_e^2} = \frac{q_m^2}{c^2 q_e^2} = 4720$$

This is in agreement with Professor Errede's notes.

We see that magnetic interaction component completely dominates the electric forces. This has the effect that magnetic field effects will be satisfied on a much smaller distance scale than the electric effects.

Fine Structure Constant

We have an expression for magnetic charge. How about calculating the charge ratio? We will need to have the same units for this ratio, so I will have a factor of c to apply to charge to get consistent units.

$$\begin{aligned}
q_m &= \frac{h}{\mu q_e} = 3.29102 \cdot 10^{-9} \text{ A m} \\
c q_e &= 4.80321 \cdot 10^{-11} \text{ A m} \\
\frac{q_m}{c q_e} &= 68.5171 = 0.5 * 137.034 \\
\frac{c q_e}{q_m} &= \frac{\mu c q_e^2}{h} \\
&= \frac{q_e^2}{\epsilon c h} \\
&= \frac{q_e^2}{2\pi \epsilon c \hbar} \\
&= 2\alpha
\end{aligned}$$

This very delightful result shows the inverse fine structure constant to be twice the ratio of magnetic to electric charge.

Magnetic Dipoles

We now get closer to a model for the electron. We know electron have spin, and we also know electrons have a magnetic dipole (not monopole) moment. Here, we look at models for a single charge interacting with two, opposite polarity monopoles.

A concern with dual monopole models has been why the opposite monopoles don't simply recombine. A clue comes from the minimum radial separation behavior between electrons and monopoles. In effect, the electron becomes the spacer between the oppositely charge monopoles, preventing recombination.

Without loss of generality, I can place the positive magnetic charge at $(0, 0, R/2)$ and the negative magnetic charge at $(0, 0, -R/2)$. The resulting magnetic dipole field is

$$\vec{B} = \frac{\mu q_m}{4\pi}\left[\frac{\vec{r} - (R/2)\vec{a}_z}{\left(x^2 + y^2 + (z - R/2)^2\right)^{3/2}} - \frac{\vec{r} + (R/2)\vec{a}_z}{\left(x^2 + y^2 + (z + R/2)^2\right)^{3/2}}\right]$$

For the special case of the midplane between the two charges, we have

$$\vec{B}_{z=0} = \frac{\mu q_m}{4\pi}\frac{-R\vec{a}_z}{(x^2 + y^2 + R^2/4)^{3/2}}$$

The force on an electric charge at $\vec{r}$ is

$$\begin{aligned}\vec{F} &= q_e(\vec{v} \times \vec{B}) \\ &= \frac{\mu q_e q_m}{4\pi}\left[\frac{\vec{v} \times (\vec{r} - (R/2)\vec{a}_z)}{\left(x^2 + y^2 + (z - R/2)^2\right)^{3/2}} - \frac{\vec{v} \times (\vec{r} + (R/2)\vec{a}_z)}{\left(x^2 + y^2 + (z + R/2)^2\right)^{3/2}}\right]\end{aligned}$$

The acceleration on an electric charge at $\vec{r}$ is

$$\begin{aligned}\vec{a} &= \frac{\vec{F}}{m} \\ &= \frac{\mu q_e q_m}{4\pi m}\left[\frac{\vec{v} \times (\vec{r} - (R/2)\vec{a}_z)}{\left(x^2 + y^2 + (z - R/2)^2\right)^{3/2}} - \frac{\vec{v} \times (\vec{r} + (R/2)\vec{a}_z)}{\left(x^2 + y^2 + (z + R/2)^2\right)^{3/2}}\right]\end{aligned}$$

The curvature is

$$\begin{aligned}\vec{\kappa} &= \frac{\vec{a} \times \vec{v}}{v^3} \\ &= \frac{\mu q_e q_m}{4\pi m}\left[\frac{[\vec{v} \times (\vec{r} - (R/2)\vec{a}_z)] \times \vec{v}}{v^3\left(x^2 + y^2 + (z - R/2)^2\right)^{3/2}} - \frac{[\vec{v} \times (\vec{r} + (R/2)\vec{a}_z)] \times \vec{v}}{v^3\left(x^2 + y^2 + (z + R/2)^2\right)^{3/2}}\right]\end{aligned}$$

For the special case of motion confined to the midplane, we have

$$\vec{\kappa}_{z=0} = \frac{\mu q_e q_m}{4\pi}\frac{1}{mv}\left[\frac{-R\vec{a}_z}{(x^2 + y^2 + R^2/4)^{3/2}}\right]$$

Now, the total field angular momentum is no longer constant, as we have the vector addition of the two monopoles with the same electron.

$$\begin{aligned}
\vec{L} &= \frac{\mu q_e q_m}{4\pi}\left[\frac{-\vec{r}-(R/2)\vec{a}_z}{|-\vec{r}-(R/2)\vec{a}_z|}+\frac{\vec{r}-(R/2)\vec{a}_z}{|\vec{r}-(R/2)\vec{a}_z|}\right] \\
&= \frac{\mu q_e q_m}{4\pi}\left[\frac{\vec{r}-(R/2)\vec{a}_z}{|\vec{r}-(R/2)\vec{a}_z|}-\frac{\vec{r}+(R/2)\vec{a}_z}{|\vec{r}+(R/2)\vec{a}_z|}\right] \\
&= \frac{\mu q_e q_m}{4\pi}\left[\frac{\vec{r}-(R/2)\vec{a}_z}{\sqrt{x^2+y^2+(z-(R/2))^2}}-\frac{\vec{r}+(R/2)\vec{a}_z}{\sqrt{x^2+y^2+(z+(R/2))^2}}\right]
\end{aligned}$$

Define

$$\begin{aligned}
r_1 &= \sqrt{x^2+y^2+(z+(R/2))^2} \\
r_2 &= \sqrt{x^2+y^2+(z-(R/2))^2}
\end{aligned}$$

Then

$$\vec{L} = \frac{\mu q_e q_m}{4\pi}\left[-\frac{R}{2}\vec{a}_z\left(\frac{1}{r_1}+\frac{1}{r_2}\right)-\vec{r}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]$$

For the special case of midplane motion, where $r_1 = r_2$, we have

$$\vec{L}_{z=0} = \frac{\mu q_e q_m}{4\pi}\left[\frac{-R\vec{a}_z}{\sqrt{x^2+y^2+R^2/4}}\right]$$

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