Anti-commutating Basis Elements in Minkowski Geometric Algebra

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Abstract

This note documents the anti-commutating sets in Minkowski Geometric algebra. I find 60 pairs, 80 triads, 30 tetrads and 6 pentads.

Minkowski Geometric Algebra

Minkowski geometric algebra has one scalar (q), four basis vectors (x, y, z, t), six bivectors (xy, xz, xt, yz, yt, zt), four trivectors (xyz, xyt, xzy, yzt) and one quadvector $(xyzt)$. Minkowski geometric algebra has a signature of $(+, +, +, -)$, meaning that the unit basis vectors x, y, z, and t have squares $x^2 = 1$, $y^2 = 1$, $z^2 = 1$ but $t^2 = -1$. Multiplication between the basis vectors is anti-commutative, meaning $xy = -yz$. Given that the scalar multiplication is commutative with all elements, that the basis vectors are anti-commutative among themselves, and that the squares of the basis vectors are given by the signature, we can define the Minkowski geometric product without ambiguity.

Table 1 lists the Minkowski geometric product in symbolic format.

The anti-commutating pairs are found by testing each possible pair for $AB + BA = 0$ behavior. The sixty pairs found are stored in an array for the later steps of finding triads, quads and pentads.

q	x	y	z	t	xy	xz	yz	xt	yt	zt	xyz	xyt	xzt	yzt	xyzt
x	q	xy	xz	xt	y	z	xyz	t	xyt	xzt	yz	yt	zt	xyzt	yzt
y	-xy	q	yz	yt	-x	-xyz	z	-xyt	t	yzt	-xz	-xt	-xyzt	zt	-xzt
z	-xz	-yz	q	zt	xyz	-x	-y	-xzt	-yzt	t	xy	xyzt	-xt	-yt	xyt
t	-xt	-yt	-zt	-q	xyt	xzt	yzt	x	y	z	-xyzt	-xy	-xz	-yz	xyz
xy	-y	x	xyz	xyt	-q	-yz	xz	-yt	xt	xyzt	-z	-t	-yzt	xzt	-zt
xz	-z	-xyz	x	xzt	yz	-q	-xy	-zt	-xyzt	xt	y	yzt	-t	-xyt	yt
yz	xyz	-z	y	yzt	-xz	xy	-q	xyzt	-zt	yt	-x	-xzt	xyt	-t	-xt
xt	-t	-xyt	-xzt	-x	yt	zt	xyzt	q	xy	xz	-yzt	-y	-z	-xyz	yz
yt	xyt	-t	-yzt	-y	-xt	-xyzt	zt	-xy	q	yz	xzt	x	xyz	-z	-xz
zt	xzt	yzt	-t	-z	xyzt	-xt	-yt	-xz	-yz	q	-xyt	-xyz	x	y	xy
xyz	yz	-xz	xy	xyzt	-z	y	-x	yzt	-xzt	xyt	-q	-zt	yt	-xt	-t
xyt	yt	-xt	-xyzt	-xy	-t	-yzt	xzt	-y	x	xyz	zt	q	yz	-xz	-z
xzt	zt	xyzt	-xt	-xz	yzt	-t	-xyt	-z	-xyz	x	-yt	-yz	q	xy	y
yzt	-xyzt	zt	-yt	-yz	-xzt	xyt	-t	xyz	-z	y	xt	xz	-xy	q	-x
xyzt	-yzt	xzt	-xyt	-xyz	-zt	yt	-xt	yz	-xz	xy	t	z	-y	x	-q

Table 1: Minkowski Geometric Algebra Multiplication Table

The sixty pairs in text format are

```
{ x  , y   } { x  , z   } { x  , t   } { x  , xy  } { x  , xz  } { x  , xt  }
{ x  , yzt } { x  , xyzt} { y  , z   } { y  , t   } { y  , xy  } { y  , yz  }
{ y  , yt  } { y  , xzt } { y  , xyzt} { z  , t   } { z  , xz  } { z  , yz  }
{ z  , zt  } { z  , xyt } { z  , xyzt} { t  , xt  } { t  , yt  } { t  , zt  }
{ t  , xyz } { t  , xyzt} { xy , xz  } { xy , xt  } { xy , yz  } { xy , yt  }
{ xy , xzt } { xy , yzt } { xz , xt  } { xz , yz  } { xz , zt  } { xz , xyt }
{ xz , yzt } { xt , yt  } { xt , zt  } { xt , xyz } { xt , yzt } { yz , yt  }
{ yz , zt  } { yz , xyt } { yz , xzt } { yt , zt  } { yt , xyz } { yt , xzt }
{ zt , xyz } { zt , xyt } { xyz, xyt } { xyz, xzt } { xyz, yzt } { xyz, xyzt}
{ xyt, xzt } { xyt, yzt } { xyt, xyzt} { xzt, yzt } { xzt, xyzt} { yzt, xyzt}
```

The eighty triads are found by a simple process. For each of the sixty pairs found previously, we test each of the sixteen basis as a candidate third member. If this candidate third member anti-commutates with the first two members, we add this triad to a list of anti-commutating triples.

Here are the eighty anti-commutating triples.

```
{x  ,y  ,z   } {x  ,y  ,t   } {x  ,y  ,xy  } {x  ,y  ,xyzt} {x  ,z  ,t   }
{x  ,z  ,xz  } {x  ,z  ,xyzt} {x  ,t  ,xt  } {x  ,t  ,xyzt} {x  ,xy ,xz  }
{x  ,xy ,xt  } {x  ,xy ,yzt } {x  ,xz ,xt  } {x  ,xz ,yzt } {x  ,xt ,yzt }
{x  ,yzt,xyzt} {y  ,z  ,t   } {y  ,z  ,yz  } {y  ,z  ,xyzt} {y  ,t  ,yt  }
{y  ,t  ,xyzt} {y  ,xy ,yz  } {y  ,xy ,yt  } {y  ,xy ,xzt } {y  ,yz ,yt  }
{y  ,yz ,xzt } {y  ,yt ,xzt } {y  ,xzt,xyzt} {z  ,t  ,zt  } {z  ,t  ,xyzt}
{z  ,xz ,yz  } {z  ,xz ,zt  } {z  ,xz ,xyt } {z  ,yz ,zt  } {z  ,yz ,xyt }
{z  ,zt ,xyt } {z  ,xyt,xyzt} {t  ,xt ,yt  } {t  ,xt ,zt  } {t  ,xt ,xyz }
{t  ,yt ,zt  } {t  ,yt ,xyz } {t  ,zt ,xyz } {t  ,xyz,xyzt} {xy ,xz ,xt  }
{xy ,xz ,yz  } {xy ,xz ,yzt } {xy ,xt ,yt  } {xy ,xt ,yzt } {xy ,yz ,yt  }
{xy ,yz ,xzt } {xy ,yt ,xzt } {xy ,xzt,yzt } {xz ,xt ,zt  } {xz ,xt ,yzt }
{xz ,yz ,zt  } {xz ,yz ,xyt } {xz ,zt ,xyt } {xz ,xyt,yzt } {xt ,yt ,zt  }
{xt ,yt ,xyz } {xt ,zt ,xyz } {xt ,xyz,yzt } {yz ,yt ,zt  } {yz ,yt ,xzt }
{yz ,zt ,xyt } {yz ,xyt,xzt } {yt ,zt ,xyz } {yt ,xyz,xzt } {zt ,xyz,xyt }
{xyz,xyt,xzt } {xyz,xyt,yzt } {xyz,xyt,xyzt} {xyz,xzt,yzt } {xyz,xzt,xyzt}
{xyz,yzt,xyzt} {xyt,xzt,yzt } {xyt,xzt,xyzt} {xyt,yzt,xyzt} {xzt,yzt,xyzt}
```

We now find our list of quads by the same method just used for the triads. For each triad, we test each of the sixteen basis as a candidate fourth element. If the basis anti-commutates with each of the previous three members, we add this quad to the list of quads. We find only thirty quads.

Here are the thirty quads.

```
{x  ,y  ,z  ,t   } {x  ,y  ,z  ,xyzt} {x  ,y  ,t  ,xyzt}
{x  ,z  ,t  ,xyzt} {x  ,xy ,xz ,xt  } {x  ,xy ,xz ,yzt }
{x  ,xy ,xt ,yzt } {x  ,xz ,xt ,yzt } {y  ,z  ,t  ,xyzt}
{y  ,xy ,yz ,yt  } {y  ,xy ,yz ,xzt } {y  ,xy ,yt ,xzt }
{y  ,yz ,yt ,xzt } {z  ,xz ,yz ,zt  } {z  ,xz ,yz ,xyt }
{z  ,xz ,zt ,xyt } {z  ,yz ,zt ,xyt } {t  ,xt ,yt ,zt  }
{t  ,xt ,yt ,xyz } {t  ,xt ,zt ,xyz } {t  ,yt ,zt ,xyz }
{xy ,xz ,xt ,yzt } {xy ,yz ,yt ,xzt } {xz ,yz ,zt ,xyt }
{xt ,yt ,zt ,xyz } {xyz,xyt,xzt,yzt } {xyz,xyt,xzt,xyzt}
{xyz,xyt,yzt,xyzt} {xyz,xzt,yzt,xyzt} {xyt,xzt,yzt,xyzt}
```

The process of finding pentads is just the same as previous steps. For each quad, we test the sixteen basis. If the new basis anti-commutates with the four previous elements, we have a pentad, which we add to our collection. Only six pentads are found.

Here are the six pentads.

```
{x  ,y  ,z  ,t  ,xyzt} {x  ,xy ,xz ,xt ,yzt } {y  ,xy ,yz ,yt ,xzt }
{z  ,xz ,yz ,zt ,xyt } {t  ,xt ,yt ,zt ,xyz } {xyz,xyt,xzt,yzt,xyzt}
```

Extending the process for sets of six anti-commutors, none are found.