Matrix Operations in Minkowski Geometric Algebra

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Abstract

Geometric algebras are associative, non-commutative algebras which can be mapped to matrices. As such, the geometric algebras have matrix related operations, such as determinants, eigenvalues and eigenvectors. This note documents and provides formulas for matrix related operations for the specific case of Minkowski geometric algebra, using an east coast metric of (+,+,+,-).

Minkowski Geometric Algebra

Minkowski geometric algebra is a spacetime algebra using east coast metric $(+,+,+,-)$. Our components are one scalar, e_q, three space directions e_x, e_y, and e_z, a time dimension e_t, and higher order products of the previous, yielding three pure space bivectors $e_x e_y, e_x e_z$, and $e_y e_z$, three spacetime bivectors $e_x e_t, e_y e_t$, and $e_z e_t$, four trivectors $e_x e_y e_z, e_x e_y e_t, e_x e_z e_t$, and $e_y e_z e_t$, and a quadvector $e_x e_y e_z e_t$.

A generic multivector is the scaled sum of the previous elements, which I usually write as

$$
\begin{aligned}
&Ae_q + \\
&Be_x + Ce_y + De_z + Ee_t + \\
&Fe_{xy} + Ge_{xz} + He_{yz} + Je_{xt} + Ke_{yt} + Le_{zt} + \\
&Me_{xyz} + Ne_{xyt} + Pe_{xzt} + Re_{yzt} + \\
&Se_{xyzt}
\end{aligned}
$$

We can define this algebra by a multiplication table. In condensed format, to fit on one page, we have Table 1.

Dirac Matrices are Multivector Elements

In text format, one set of real 4x4 matrices implementing Minkowski geometric algebra are

```
     Unity                          xyzt
[  1  0  0  0]                [  0  0  0  1]
[  0  1  0  0]                [  0  0 -1  0]
[  0  0  1  0]                [  0  1  0  0]
[  0  0  0  1]                [ -1  0  0  0]

       x              y              z              t
[  1  0  0  0] [  0  1  0  0] [  0  0  0 -1] [  0 -1  0  0]
[  0 -1  0  0] [  1  0  0  0] [  0  0  1  0] [  1  0  0  0]
[  0  0  1  0] [  0  0  0  1] [  0  1  0  0] [  0  0  0  1]
[  0  0  0 -1] [  0  0  1  0] [ -1  0  0  0] [  0  0 -1  0]

      xy             xz             yz             xt             yt             zt
[  0  1  0  0] [  0  0  0 -1] [  0  0  1  0] [  0 -1  0  0] [  1  0  0  0] [  0  0  1  0]
[ -1  0  0  0] [  0  0 -1  0] [  0  0  0 -1] [ -1  0  0  0] [  0 -1  0  0] [  0  0  0  1]
[  0  0  0  1] [  0  1  0  0] [ -1  0  0  0] [  0  0  0  1] [  0  0 -1  0] [  1  0  0  0]
[  0  0 -1  0] [  1  0  0  0] [  0  1  0  0] [  0  0  1  0] [  0  0  0  1] [  0  1  0  0]

      xyz            xyt            xzt            yzt
[  0  0  1  0] [  1  0  0  0] [  0  0  1  0] [  0  0  0  1]
[  0  0  0  1] [  0  1  0  0] [  0  0  0 -1] [  0  0  1  0]
[ -1  0  0  0] [  0  0 -1  0] [  1  0  0  0] [  0  1  0  0]
[  0 -1  0  0] [  0  0  0 -1] [  0 -1  0  0] [  1  0  0  0]
```

For a generic Minkowski multivector,

$$
\begin{aligned}
&Ae_q + \\
&Be_x + Ce_y + De_z + Ee_t + \\
&Fe_{xy} + Ge_{xz} + He_{yz} + Je_{xt} + Ke_{yt} + Le_{zt} + \\
&Me_{xyz} + Ne_{xyt} + Pe_{xzt} + Re_{yzt} + \\
&Se_{xyzt}
\end{aligned}
$$

q	x	y	z	t	xy	xz	yz	xt	yt	zt	xyz	xyt	xzt	yzt	xyzt
x	q	xy	xz	xt	y	z	xyz	t	xyt	xzt	yz	yt	zt	xyzt	yzt
y	-xy	q	yz	yt	-x	-xyz	z	-xyt	t	yzt	-xz	-xt	-xyzt	zt	-xzt
z	-xz	-yz	q	zt	xyz	-x	-y	-xzt	-yzt	t	xy	xyzt	-xt	-yt	xyt
t	-xt	-yt	-zt	-q	xyt	xzt	yzt	x	y	z	-xyzt	-xy	-xz	-yz	xyz
xy	-y	x	xyz	xyt	-q	-yz	xz	-yt	xt	xyzt	-z	-t	-yzt	xzt	-zt
xz	-z	-xyz	x	xzt	yz	-q	-xy	-zt	-xyzt	xt	y	yzt	-t	-xyt	yt
yz	xyz	-z	y	yzt	-xz	xy	-q	xyzt	-zt	yt	-x	-xzt	xyt	-t	-xt
xt	-t	-xyt	-xzt	-x	yt	zt	xyzt	q	xy	xz	-yzt	-y	-z	-xyz	yz
yt	xyt	-t	-yzt	-y	-xt	-xyzt	zt	-xy	q	yz	xzt	x	xyz	-z	-xz
zt	xzt	yzt	-t	-z	xyzt	-xt	-yt	-xz	-yz	q	-xyt	-xyz	x	y	xy
xyz	yz	-xz	xy	xyzt	-z	y	-x	yzt	-xzt	xyt	-q	-zt	yt	-xt	-t
xyt	yt	-xt	-xyzt	-xy	-t	-yzt	xzt	-y	x	xyz	zt	q	yz	-xz	-z
xzt	zt	xyzt	-xt	-xz	yzt	-t	-xyt	-z	-xyz	x	-yt	-yz	q	xy	y
yzt	-xyzt	zt	-yt	-yz	-xzt	xyt	-t	xyz	-z	y	xt	xz	-xy	q	-x
xyzt	-yzt	xzt	-xyt	-xyz	-zt	yt	-xt	yz	-xz	xy	t	z	-y	x	-q

Table 1: Minkowski Geometric Algebra Multiplication Table

the associated matrix representation is

$$\begin{pmatrix} +A+B+K+N & +C-E+F-J & +H+L+M+P & -D-G+R+S \\ +C+E-F-J & +A-B-K+N & +D-G+R-S & -H+L+M-P \\ -H+L-M+P & +D+G+R+S & +A+B-K-N & +C+E+F+J \\ -D+G+R-S & +H+L-M-P & +C-E-F+J & +A-B+K-N \end{pmatrix}$$

To recover the multivector components from a generic 4x4 matrix $W[4][4]$, we have the trace-like formulas

$$\begin{aligned}
A &= (+W[0][0]+W[1][1]+W[2][2]+W[3][3])/4 \\
B &= (+W[0][0]-W[1][1]+W[2][2]-W[3][3])/4 \\
C &= (+W[0][1]+W[1][0]+W[2][3]+W[3][2])/4 \\
D &= (-W[0][3]+W[1][2]+W[2][1]-W[3][0])/4 \\
E &= (-W[0][1]+W[1][0]+W[2][3]-W[3][2])/4 \\
F &= (+W[0][1]-W[1][0]+W[2][3]-W[3][2])/4 \\
G &= (-W[0][3]-W[1][2]+W[2][1]+W[3][0])/4 \\
H &= (+W[0][2]-W[1][3]-W[2][0]+W[3][1])/4 \\
J &= (-W[0][1]-W[1][0]+W[2][3]+W[3][2])/4 \\
K &= (+W[0][0]-W[1][1]-W[2][2]+W[3][3])/4 \\
L &= (+W[0][2]+W[1][3]+W[2][0]+W[3][1])/4 \\
M &= (+W[0][2]+W[1][3]-W[2][0]-W[3][1])/4 \\
N &= (+W[0][0]+W[1][1]-W[2][2]-W[3][3])/4 \\
P &= (+W[0][2]-W[1][3]+W[2][0]-W[3][1])/4 \\
R &= (+W[0][3]+W[1][2]+W[2][1]+W[3][0])/4 \\
S &= (+W[0][3]-W[1][2]+W[2][1]-W[3][0])/4
\end{aligned}$$

In this fashion, we can convert from multivector to matrix, use matrix tools such as Sage, Mathematica, Sympy and Ginac, and convert the results back to multivector format. For this work, I have used Ginac.

Matrix Operations on Multivectors

To find our matrix operations for multivectors, we convert our multivector to matrix format, carry out the desired operation, then convert back to multivector format to see the result. We begin with the matrix transpose operation.

Transpose

Transposition pivots the matrix array around the descending diagonal. Here, I take a generic Mink to array, transpose the array, then convert back to Mink format.

For a generic Minkowski multivector,

$$\begin{aligned} V &= Ae_q + \\ &\quad Be_x + Ce_y + De_z + Ee_t + \\ &\quad Fe_{xy} + Ge_{xz} + He_{yz} + Je_{xt} + Ke_{yt} + Le_{zt} + \\ &\quad Me_{xyz} + Ne_{xyt} + Pe_{xzt} + Re_{yzt} + \\ &\quad Se_{xyzt} \end{aligned}$$

the associated matrix representation is

$$V = \begin{pmatrix} +A+B+K+N & +C-E+F-J & +H+L+M+P & -D-G+R+S \\ +C+E-F-J & +A-B-K+N & +D-G+R-S & -H+L+M-P \\ -H+L-M+P & +D+G+R+S & +A+B-K-N & +C+E+F+J \\ -D+G+R-S & +H+L-M-P & +C-E-F+J & +A-B+K-N \end{pmatrix}$$

The transposed array

$$V.T = \begin{pmatrix} +A+B+K+N & +C+E-F-J & -H+L-M+P & -D+G+R-S \\ +C-E+F-J & +A-B-K+N & +D+G+R+S & +H+L-M-P \\ +H+L-M+P & +D-G+R-S & +A+B-K-N & +C-E-F+J \\ -D-G+R-S & -H+L+M-P & +C+E+F+J & +A-B+K-N \end{pmatrix}$$

The transposed multivector is

$$\begin{aligned} V.T &= Ae_q + \\ &\quad Be_x + Ce_y + De_z - Ee_t + \\ &\quad -Fe_{xy} - Ge_{xz} - He_{yz} + Je_{xt} + Ke_{yt} + Le_{zt} + \\ &\quad -Me_{xyz} + Ne_{xyt} + Pe_{xzt} + Re_{yzt} + \\ &\quad -Se_{xyzt} \end{aligned}$$

We see the simple pattern that the sign of each transposed basis element is changed per the square of that basis element, while the magnitudes remain the same.

Hermitian Conjugate

In this section, the dagger notation indicates Hermitian conjugation, as used in matrix operations, rather than the reverse factor operator commonly used with multivectors.

In matrices, the Hermitian conjugate is the combination of matrix transposition and complex component conjugation. Applied to multivectors, if we let the multivector components become complex numbers, we can define the Hermitian conjugate of a multivector as

$$\begin{aligned} V^\dagger = \; & A^* e_q + \\ & B^* e_x + C^* e_y + D^* e_z - E^* e_t + \\ & -F^* e_{xy} - G^* e_{xz} - H^* e_{yz} + J^* e_{xt} + K^* e_{yt} + L^* e_{zt} + \\ & -M^* e_{xyz} + N^* e_{xyt} + P^* e_{xzt} + R^* e_{yzt} + \\ & -S^* e_{xyzt} \end{aligned}$$

For a multivector to be Hermitian, we require that the multivector be equal to its Hermitian conjugate. For the ten components whose basis square to one, such as e_x, we require purely real coefficients. Using e_x as an example, $B = B^*$ requires $B_r + iB_i = B_r - iB_i$, or $B_i = 0$. For the six components whose basis square to negative one, such as e_{xy}, we likewise require purely imaginary components. Using e_{xy} as an example, $F = -F^*$ requires $F_r + iF_i = -F_r + iF_i$, or $F_r = 0$.

The imaginary basis elements of the Hermitian form compensate the negative squares for the six Minkowski multivector components, leading to positive definite forms akin to a four dimensional **Euclidean** space. I find this positive only format very significant.

Summarizing, the generic Hermitian multivector format is

$$\begin{aligned} V = V^\dagger = \; & A_r e_q + \\ & B_r e_x + C_r e_y + D_r e_z + iE_i e_t + \\ & +iF_i e_{xy} + iG_i e_{xz} + iH_i e_{yz} + J_r e_{xt} + K_r e_{yt} + L_r e_{zt} + \\ & +iM_i e_{xyz} + N_r e_{xyt} + P_r e_{xzt} + R_r e_{yzt} + \\ & +iS_i e_{xyzt} \end{aligned}$$

Determinant

The determinant of the multivector is easily found from the matrix form using Sympy or Ginac (or patiently by hand).

$$V = \begin{pmatrix} +A+B+K+N & +C-E+F-J & +H+L+M+P & -D-G+R+S \\ +C+E-F-J & +A-B-K+N & +D-G+R-S & -H+L+M-P \\ -H+L-M+P & +D+G+R+S & +A+B-K-N & +C+E+F+J \\ -D+G+R-S & +H+L-M-P & +C-E-F+J & +A-B+K-N \end{pmatrix}$$

The sum of products formula for the determinant is broken into three chunks for typesetting purposes. The first chunk is the sum of the fourth power of the components.

$$\begin{aligned} \det_1 &= +A^4+B^4+C^4+D^4+E^4+F^4+G^4+H^4 \\ &\quad +J^4+K^4+L^4+M^4+N^4+P^4+R^4+S^4 \end{aligned}$$

The second chunk consists of the signed products of squares, with associated scale factor of two.

$$
\begin{aligned}
\det_2 \;=\; & 2*(-A^2B^2 - A^2C^2 - A^2D^2 + A^2E^2 + A^2F^2 + A^2G^2 + A^2H^2 \\
& -A^2J^2 - A^2K^2 - A^2L^2 + A^2M^2 - A^2N^2 - A^2P^2 - A^2R^2 + A^2S^2 \\
& +B^2C^2 + B^2D^2 - B^2E^2 - B^2F^2 - B^2G^2 + B^2H^2 \\
& +B^2J^2 - B^2K^2 - B^2L^2 + B^2M^2 - B^2N^2 - B^2P^2 + B^2R^2 - B^2S^2 \\
& +C^2D^2 - C^2E^2 - C^2F^2 + C^2G^2 - C^2H^2 \\
& -C^2J^2 + C^2K^2 - C^2L^2 + C^2M^2 - C^2N^2 + C^2P^2 - C^2R^2 - C^2S^2 \\
& -D^2E^2 + D^2F^2 - D^2G^2 - D^2H^2 \\
& -D^2J^2 - D^2K^2 + D^2L^2 + D^2M^2 + D^2N^2 - D^2P^2 - D^2R^2 - D^2S^2 \\
& -E^2F^2 - E^2G^2 - E^2H^2 \\
& -E^2J^2 - E^2K^2 - E^2L^2 + E^2M^2 + E^2N^2 + E^2P^2 + E^2R^2 + E^2S^2 \\
& +F^2G^2 + F^2H^2 \\
& -F^2J^2 - F^2K^2 + F^2L^2 - F^2M^2 + F^2N^2 - F^2P^2 - F^2R^2 - F^2S^2 \\
& +G^2H^2 \\
& -G^2J^2 + G^2K^2 - G^2L^2 - G^2M^2 - G^2N^2 + G^2P^2 - G^2R^2 - G^2S^2 \\
& +H^2J^2 - H^2K^2 - H^2L^2 - H^2M^2 - H^2N^2 - H^2P^2 + H^2R^2 - H^2S^2 \\
& +J^2K^2 + J^2L^2 - J^2M^2 - J^2N^2 - J^2P^2 + J^2R^2 + J^2S^2 \\
& +K^2L^2 - K^2M^2 - K^2N^2 + K^2P^2 - K^2R^2 + K^2S^2 \\
& -L^2M^2 + L^2N^2 - L^2P^2 - L^2R^2 + L^2S^2 \\
& -M^2N^2 - M^2P^2 - M^2R^2 + M^2S^2 \\
& +N^2P^2 + N^2R^2 - N^2S^2 + P^2R^2 - P^2S^2 - R^2S^2)
\end{aligned}
$$

The third chunk consists of four products, with scale factor of eight.

$$\begin{aligned}
\det_3 \;=\; & 8 * (-ABHM + ABKN + ABLP + ACGM \\
& -ACJN + ACLR - ADFM - ADJP \\
& -ADKR + AEFN + AEGP + AEHR \\
& -AFLS + AGKS - AHJS \\
& -BCGH + BCJK - BCPR + BDFH \\
& +BDJL + BDNR \\
& -BEFK - BEGL - BEMR + BFPS \\
& -BGNS + BJMS \\
& -CDFG + CDKL - CDNP + CEFJ - CEHL \\
& +CEMP + CFRS - CHNS + CKMS \\
& +DEGJ + DEHK - DEMN + DGRS - DHPS + DLMS \\
& -EJRS + EKPS - ELNS \\
& -FGKL + FGNP + FHJL + FHNR - FJMP - FKMR \\
& -GHJK + GHPR + GJMN - GLMR \\
& +HKMN + HLMP \\
& -JKPR + JLNR - KLNP)
\end{aligned}$$

The full determinant is then $\det = \det_1 + \det_2 + \det_3$.

Preferred Factoring for the Determinant

The unwieldy expressions above can be grouped and factored into a much simpler form, found from determinant preserving unary operators.

$$\begin{aligned}
a &= +A^2 + B^2 + C^2 + D^2 - E^2 + F^2 + G^2 + H^2 \\
&\quad -J^2 - K^2 - L^2 + M^2 - N^2 - P^2 - R^2 - S^2 \\
b &= +2(AB + CF + DG - EJ + HM - KN - LP - RS) \\
c &= +2(AC - BF + DH - EK - GM + JN - LR + PS) \\
d &= +2(AD - BG - CH - EL + JP + KR + MF - NS) \\
e &= +2(AE - BJ - CK - DL + FN + GP + HR - MS) \\
s &= +2(AS - BR + CP - DN + EM - FL + GK - HJ) \\
\det &= a^2 - b^2 - c^2 - d^2 + e^2 + s^2
\end{aligned}$$

The Determinant is the Multivector Magnitude

I use the determinant as the measure of the oriented magnitude for the multivector. In matrices, the cascaded product of several matrices has a determinant which is also the cascade product of the individual determinants. In geometric terms, if we treat the rows (or columns) of square matrices as spatial vectors, the cascaded wedge product of the rows (or columns) is the determinant times the pseudo scalar. For rotations and other transforms, unit vectors, unit bivectors, and the like also have unit determinants. In my Minkowski implementation, each multivector basis element has positive unit determinant. For example, $\det(e_x) = \det(e_{yzt}) = 1$.

Determinant for Embedded Three Dimensional Subspaces

Two common three dimensional subspaces in Minkowsi spacetime are the conventional space implementation

```
V = Mink(a, b,c,d,0, e,f,g, 0,0,0, h,0,0,0, 0);
```

with matrix representation

$$V = \begin{pmatrix} +a+b & +c+e & +g+h & -d-f \\ +c-e & +a-b & +d-f & -g+h \\ -g-h & +d+f & +a+b & +c+e \\ -d+f & +g-h & +c-e & +a-b \end{pmatrix}$$

and the even subspace implementation

```
W = Mink(a, 0,0,0,0, e,f,g, b,c,d, 0,0,0,0, h);
```

with matrix representation

$$W = \begin{pmatrix} +a+c & -b+e & +d+g & -f+h \\ -b-e & +a-c & -f-h & -g+d \\ +d-g & +f+h & +a-c & +b+e \\ +f-h & +d+g & +b-e & +a-c \end{pmatrix}$$

Both have the same determinant

$$
\begin{aligned}
&+a^4+b^4+c^4+d^4+e^4+f^4+g^4+h^4\\
&-2a^2b^2-2a^2c^2-2a^2d^2+2a^2e^2+2a^2f^2+2a^2g^2+2a^2h^2\\
&+2b^2c^2+2b^2d^2-2b^2e^2-2b^2f^2+2b^2g^2+2b^2h^2\\
&+2c^2d^2-2c^2e^2+2c^2f^2-2c^2g^2+2c^2h^2\\
&+2d^2e^2-2d^2f^2-2d^2g^2+2d^2h^2\\
&+2e^2f^2+2e^2g^2-2e^2h^2\\
&+2f^2g^2-2f^2h^2\\
&-2g^2h^2\\
&-8abgh+8acfh-8adeh-8bcfg+8bdeg-8cdef
\end{aligned}
$$

This determinant can be written in the form

$$
\begin{aligned}
\det_{3\mathrm{D}}(W) \;=\; & (a^2-b^2-c^2-d^2+e^2+f^2+g^2-h^2)^2\\
& +4(ah-bg-cf-de)^2
\end{aligned}
$$

If we allow the use of complex numbers, this expression can be further factored to the simple form

$$
\begin{aligned}
\det_{3\mathrm{D}}(W) \;=\; & \left[(a+ih)^2-(b+ig)^2-(c-if)^2-(d+ie)^2\right] *\\
& \left[(a-ih)^2-(b-ig)^2-(c+if)^2-(d-ie)^2\right]
\end{aligned}
$$

This expression is the complex planar determinant of the following section, times its complex conjugate. Because these three dimensional spaces are embedded in a four dimensional representation, this determinant is a polynomial of fourth order.

3D Determinant Using 2D Complex Matrices

I want to contrast 3D Euclidean implementations in complexified 2D matrices versus the embedded Minkowski above. We will find, no surprise, that the determinants are different. The differences are due to the inherent dimensionality (quartic versus quadratic determinants), as well as the geometric orientation of the basis elements, as deduced from the sign of the determinant of the basis elements.

We can implement three dimensional Euclidean geometric algebra as a complexified two dimensional matrix algebra. Our (Pauli) matrices are

```
 Unity       e_x        e_y        e_z
[ 1  0]    [ 0  1]    [ 0 -I]    [ 1  0]
[ 0  1]    [ 1  0]    [ I  0]    [ 0 -1]

 e_xy       e_xz       e_yz       e_xyz
[ I  0]    [ 0 -1]    [ 0  I]    [ I  0]
[ 0 -I]    [ 1  0]    [ I  0]    [ 0  I]
```

The determinant of the space vectors above is -1, which differs from the Minkowksi implementation, where all basis elements have positive unit determinant. For the three dimensional multivector

$$A + Be_x + Ce_y + De_z + Ee_{xy} + Fe_{xz} + Ge_{yz} + He_{xyz}$$

we can write the equivalent matrix and determinant

```
ThreeD = [[ + A + D + I*E + I*H, + B - F - I*C + I*G],
          [ + B + F + I*C + I*G, + A - D - I*E + I*H]]
```

$$\det_{3\mathrm{D}} = (A + iH)^2 - (B + iG)^2 - (C - iF)^2 - (D + iE)^2$$

The Minkowski format determinant is the product of this planar determinant times its complex conjugate.

$$\det_{4\mathrm{D}} = (\det_{3\mathrm{D}}) * (\det_{3\mathrm{D}})^*$$

Determinant for Embedded Two Dimensional Subspaces

A two dimensional subspace in Minkowski spacetime is

```
V = Mink(a, b,c,0,0, d,0,0, 0,0,0, 0,0,0,0, 0);
```

This corresponds to the matrix

$$V = \begin{pmatrix} a+b & c+d & 0 & 0 \\ c-d & a-b & 0 & 0 \\ 0 & 0 & a+b & c+d \\ 0 & 0 & c-d & a-b \end{pmatrix}$$

This block diagonal form has a determinant equal the product of the determinants of the two blocks.

$$\det_{2\mathrm{D}} = \left(+a^2 - b^2 - c^2 + d^2\right)^2$$

Determinants for Special Cases

We now look at the determinants for various subsets of multivectors.

Scalar Only

The determinant of the scalar portion only is the scalar raised to the fourth power.

$$\det(Ae_q) = A^4$$

Vector Only

The determinant of a spatial vector is the sum of the squares of the components squared.

$$\det(Be_x + Ce_y + De_z) = (B^2 + C^2 + D^2)^2$$

Space-time Vector

The determinant of a spatial vector is the the sum of the squares of the components squared, taking into account the metric for the time component.

$$\det(Be_x + Ce_y + De_z + Ee_t) = (B^2 + C^2 + D^2 - E^2)^2$$

Space Bivector

The determinant of a spatial bivector is the sum of the square of the components squared.

$$\det(Fe_{xy} + Ge_{xz} + He_{yz}) = (F^2 + G^2 + H^2)^2$$

Spacetime Bivector

The determinant of a spacetime bivector is the sum of the square of the components squared.

$$\det(Je_{xt} + Ke_{yt} + Le_{zt}) = (J^2 + K^2 + L^2)^2$$

Combined Bivectors

The determinant of the combined bivectors is

$$\det(Fe_{xy}+Ge_{xz}+He_{yz}+Je_{xt}+Ke_{yt}+Le_{zt}) = (F^2+G^2+H^2-J^2-K^2-L^2)^2+4(-FL+GK-HJ)^2$$

All Trivectors

The determinant of the combined bivectors is

$$\det(Me_{xyz} + Ne_{xyt} + Pe_{xzt} + Re_{yzt}) = (-M^2 + N^2 + P^2 + R^2)^2$$

Unitary Multivectors

As the determinant scales as the fourth power of linear dimensions, to normalize or unitize a multivector, divide all components by the fourth root of the absolute value of the determinant, assuming that the determinant is not zero. In this regard, the fourth root of the determinant behaves as a multivector norm, similar to the square root of the sum of squares of components of a vector in conventional vector arithmetic.

Inverse Multivectors

If the determinant of a multivector is not zero, by analogy with matrices, we can solve for inverse multivector, which multiplying the original yields unity. Symbolically, these expression are awkward. Numerically, these are straightforward. Schematically, the process is

1. Convert Mink to Matrix.
2. Check determinant for non-zero.
3. Invert Matrix.
4. Convert Matrix back to Mink.

Interesting Special Cases

For the special case of a pure vector, we have

$$\begin{aligned} M &= be_x \\ M^{-1} &= \frac{1}{b}e_x \end{aligned}$$

For the special case of a scalar plus vector, we have

$$\begin{aligned} M &= a + be_x \\ M^{-1} &= \left[\frac{a}{a^2-b^2} - \frac{b}{a^2-b^2}e_x\right] \end{aligned}$$

For the special case of a scalar plus vector plus space bivector, we have

$$\begin{aligned} M &= a + be_x + fe_{xy} \\ \det(M) &= (a^2 - b^2 + f^2)^2 \\ M^{-1} &= \frac{a - be_x - fe_{xy}}{(a^2-b^2+f^2)} \end{aligned}$$

When we go to more terms, additional components appear in the inverse multivector, and the expressions become significantly more complex.

Adjugate Matrix

Given the matrix

$$V = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$$

the inverse matrix can be written as

$$\frac{1}{\det(V)}\begin{pmatrix} \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} & -\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} & \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} & -\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix} \\ -\begin{vmatrix} b & c & d \\ j & k & l \\ n & o & p \end{vmatrix} & \begin{vmatrix} a & c & d \\ i & k & l \\ m & o & p \end{vmatrix} & -\begin{vmatrix} a & b & d \\ i & j & l \\ m & n & p \end{vmatrix} & \begin{vmatrix} a & b & c \\ i & j & k \\ m & n & o \end{vmatrix} \\ \begin{vmatrix} b & c & d \\ f & g & h \\ n & o & p \end{vmatrix} & -\begin{vmatrix} a & c & d \\ e & g & h \\ m & o & p \end{vmatrix} & \begin{vmatrix} a & b & d \\ e & f & h \\ m & n & p \end{vmatrix} & -\begin{vmatrix} a & b & c \\ e & f & g \\ m & n & o \end{vmatrix} \\ -\begin{vmatrix} b & c & d \\ f & g & h \\ j & k & l \end{vmatrix} & \begin{vmatrix} a & c & d \\ e & g & h \\ i & k & l \end{vmatrix} & -\begin{vmatrix} a & b & d \\ e & f & h \\ i & j & l \end{vmatrix} & \begin{vmatrix} a & b & c \\ e & f & g \\ i & j & k \end{vmatrix} \end{pmatrix}^T$$

The adjugate matrix is the right hand matrix above. The product of a matrix and its adjugate equals the determinant times the unit matrix. Using the technique where a multivector is converted to matrix format, the matrix converted to adjugate form, and the result converted back to a multivector, we have the following formulas for the adjugate of a multivector in Minkowski spacetime.

```
V is the input multivector. Local variables used are

a = V.q;
b = V.x;   c = V.y;   d = V.z;   e = V.t;
f = V.xy;  g = V.xz;  h = V.yz;  j = V.xt;  k = V.yt;  l = V.zt;
m = V.xyz; n = V.xyt; p = V.xzt; r = V.yzt;
s = V.xyzt;


W.q =
 + a*( + a*a - b*b - c*c - d*d + e*e + f*f + g*g + h*h - j*j - k*k - l*l
                                        + m*m - n*n - p*p - r*r + s*s)
 + 2*( - b*h*m + b*k*n + b*l*p + c*g*m - c*j*n + c*l*r - d*f*m - d*j*p
               - d*k*r + e*f*n + e*g*p + e*h*r - f*l*s + g*k*s - h*j*s);

W.x =
 + b*( - a*a + b*b + c*c + d*d - e*e - f*f - g*g + h*h + j*j - k*k - l*l
```

```
                                        + m*m - n*n - p*p + r*r - s*s)
 + 2*( - a*h*m + a*k*n + a*l*p - c*g*h + c*j*k - c*p*r + d*f*h + d*j*l
               + d*n*r - e*f*k - e*g*l - e*m*r + f*p*s - g*n*s + m*j*s);

W.y =
 + c*( - a*a + b*b + c*c + d*d - e*e - f*f + g*g - h*h - j*j + k*k - l*l
                                        + m*m - n*n + p*p - r*r - s*s)
 + 2*( + a*g*m - a*j*n + a*r*l - b*g*h + b*j*k - b*p*r - d*f*g + d*k*l
              - d*n*p + e*f*j - e*h*l + e*m*p + f*r*s - h*n*s + m*k*s);

W.z =
 + d*( - a*a + b*b + c*c + d*d - e*e + f*f - g*g - h*h - j*j - k*k + l*l
                                        + m*m + n*n - p*p - r*r - s*s)
 + 2*( - a*f*m - a*j*p - a*k*r + b*f*h + b*j*l + b*n*r - c*f*g + c*k*l
              - c*n*p + e*g*j + e*h*k - e*m*n + g*r*s - h*p*s + l*m*s);

W.t =
 + e*( - a*a + b*b + c*c + d*d - e*e + f*f + g*g + h*h + j*j + k*k + l*l
                                        - m*m - n*n - p*p - r*r - s*s)
 + 2*( - a*f*n - a*g*p - a*h*r + b*f*k + b*g*l + b*m*r - c*f*j + c*h*l
              - c*m*p - d*g*j - d*h*k + d*m*n + j*r*s - k*p*s + l*n*s);




W.xy =
 + f*( - a*a + b*b + c*c - d*d + e*e - f*f - g*g - h*h + j*j + k*k - l*l
                                        + m*m - n*n + p*p + r*r + s*s)
 + 2*( + a*d*m - a*e*n + a*l*s - b*d*h + b*e*k - b*p*s + c*d*g - c*r*s
              - e*c*j + g*k*l - g*n*p - h*j*l - h*n*r + j*m*p + k*m*r);

W.xz =
 + g*( - a*a + b*b - c*c + d*d + e*e - f*f - g*g - h*h + j*j - k*k + l*l
                                        + m*m + n*n - p*p + r*r + s*s)
 + 2*( - a*c*m - a*e*p - a*k*s + b*c*h + b*e*l + b*n*s + c*d*f - d*e*j
              - d*r*s + f*k*l - f*n*p + h*j*k - h*p*r - j*m*n + l*m*r);

W.yz =
 + h*( - a*a - b*b + c*c + d*d + e*e - f*f - g*g - h*h - j*j + k*k + l*l
                                        + m*m + n*n + p*p - r*r + s*s)
 + 2*( + a*b*m - a*e*r + a*j*s + b*c*g - b*d*f + c*e*l + c*n*s - d*e*k
              + d*p*s + g*j*k - g*p*r - f*j*l - f*n*r - k*m*n - m*l*p);

W.xt =
```

```
 + j*( - a*a + b*b - c*c - d*d - e*e - f*f - g*g + h*h + j*j + k*k + l*l
                                         - m*m - n*n - p*p + r*r + s*s)
 + 2*( - a*c*n - a*d*p - a*h*s + b*c*k + b*d*l + b*m*s + c*e*f + d*e*g
               - e*r*s + f*h*l - f*m*p - g*h*k + g*m*n - k*p*r + l*n*r);

W.yt =
 + k*( - a*a - b*b + c*c - d*d - e*e - f*f + g*g - h*h + j*j + k*k + l*l
                                         - m*m - n*n + p*p - r*r + s*s)
 + 2*( + a*b*n - a*d*r + a*g*s + b*c*j - b*e*f + c*d*l + c*m*s + d*e*h
                + e*p*s - f*g*l - f*m*r - g*h*j + h*m*n - j*p*r - l*n*p);

W.zt =
 + l*( - a*a - b*b - c*c + d*d - e*e + f*f - g*g - h*h + j*j + k*k + l*l
                                         - m*m + n*n - p*p - r*r + s*s)
 + 2*( + a*b*p + a*c*r - a*f*s + b*d*j - b*e*g + c*d*k - c*e*h + d*m*s
               - e*n*s - f*g*k + f*h*j - g*m*r + h*m*p + j*n*r - k*n*p);

W.xyz =
 + m*( - a*a - b*b - c*c - d*d - e*e + f*f + g*g + h*h + j*j + k*k + l*l
                                          - m*m + n*n + p*p + r*r - s*s)
 + 2*( + a*b*h - a*c*g + a*d*f + b*e*r - b*j*s - c*e*p - c*k*s + d*e*n
               - d*s*l + f*j*p + f*k*r - g*j*n + g*l*r - h*k*n - h*l*p);


W.xyt =
 + n*( - a*a - b*b - c*c + d*d + e*e + f*f - g*g - h*h - j*j - k*k + l*l
                                         - m*m + n*n + p*p + r*r - s*s)
 + 2*( + a*b*k - a*c*j + a*e*f + b*d*r - b*g*s - c*d*p - c*h*s - d*e*m
               - e*l*s + f*g*p + f*h*r + g*j*m + h*k*m + j*l*r - k*l*p);

W.xzt =
 + p*( - a*a - b*b + c*c - d*d + e*e - f*f + g*g - h*h - j*j + k*k - l*l
                                         - m*m + n*n + p*p + r*r - s*s)
 + 2*( + a*b*l - a*d*j + a*e*g - b*c*r + b*f*s - c*d*n + c*e*m - d*h*s
               + e*k*s + f*g*n - f*j*m + g*h*r + h*l*m - j*k*r - k*l*n);

W.yzt =
 + r*( - a*a + b*b - c*c - d*d + e*e - f*f - g*g + h*h + j*j - k*k - l*l
                                         - m*m + n*n + p*p + r*r - s*s)
 + 2*( + a*c*l - a*d*k + a*e*h - b*c*p + b*d*n - b*e*m + c*f*s + d*g*s
               - e*j*s + f*h*n - f*k*m + g*h*p - g*l*m - j*k*p + j*l*n);

W.xyzt =
 + s*( - a*a + b*b + c*c + d*d - e*e + f*f + g*g + h*h - j*j - k*k - l*l
```

```
                                         - m*m + n*n + p*p + r*r - s*s)
 + 2*( + a*f*l - a*g*k + a*h*j - b*f*p + b*g*n - b*j*m - c*f*r + c*h*n
              - c*k*m  - d*g*r + d*h*p - d*l*m + e*j*r - e*k*p + e*l*n);
```

The inverse of a multivector is the adjugate divided by the determinant.

Eigenvalues in General

Eigenvalue problems in linear algebra are systems of equations representing linear transformations, where the input vector and output vector are parallel. Using 4x4 matrices as the relevant example, an eigen problem is to find λ such that

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ j & k & m & n \\ o & p & r & s \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

This is equivalent to the problem

$$\begin{pmatrix} a-\lambda & b & c & d \\ e & f-\lambda & g & h \\ j & k & m-\lambda & n \\ o & p & r & s-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = 0$$

Non-zero solutions for the above exist when the determinant of the above matrix is zero. The expression for this determinant is the characteristic equation for λ. The solutions for λ are the eigenvalues.

Generic Multivector Eigenvalue Problem

The generic multivector is

$$\begin{aligned} &Ae_q + \\ &Be_x + Ce_y + De_z + Ee_t + \\ &Fe_{xy} + Ge_{xz} + He_{yz} + Je_{xt} + Ke_{yt} + Le_{zt} + \\ &Me_{xyz} + Ne_{xyt} + Pe_{xzt} + Re_{yzt} + \\ &Se_{xyzt} \end{aligned}$$

The associated array is

$$\begin{pmatrix} +A+B+K+N & +C-E+F-J & +H+L+M+P & -D-G+R+S \\ +C+E-F-J & +A-B-K+N & +D-G+R-S & -H+L+M-P \\ -H+L-M+P & +D+G+R+S & +A+B-K-N & +C+E+F+J \\ -D+G+R-S & +H+L-M-P & +C-E-F+J & +A-B+K-N \end{pmatrix}$$

We see that the eigenvalue problem applies a DC offset λ against the scalar component A, looking for values of λ where the multivector determinant (volume) goes to zero.

$$\begin{vmatrix} (A-\lambda)+B+K+N & +C-E+F-J & +H+L+M+P & -D-G+R+S \\ +C+E-F-J & (A-\lambda)-B-K+N & +D-G+R-S & -H+L+M-P \\ -H+L-M+P & +D+G+R+S & (A-\lambda)+B-K-N & +C+E+F+J \\ -D+G+R-S & +H+L-M-P & +C-E-F+J & (A-\lambda)-B+K-N \end{vmatrix}$$

As seen in the following examples, the eigenvalue seems related to measures of length.

Eigenvalues for Specialized Multivectors

We begin with a multivector being a single constant Ae_q. The associated (trivial) matrix is

$$Ae_q = \begin{pmatrix} +A & 0 & 0 & 0 \\ 0 & +A & 0 & 0 \\ 0 & 0 & +A & 0 \\ 0 & 0 & 0 & +A \end{pmatrix}$$

The associated eigenvalue determinant is

$$\begin{vmatrix} +A-\lambda & 0 & 0 & 0 \\ 0 & +A-\lambda & 0 & 0 \\ 0 & 0 & +A-\lambda & 0 \\ 0 & 0 & 0 & +A-\lambda \end{vmatrix} = (A-\lambda)^4 = 0$$

We see the four way degenerate result that $\lambda = A$. From the matrix representation for multivectors, we see that we will always be able to combine λ and the scalar term.

Diagonal Multivector

The next low hanging fruit is to look at the multivector $Ae_q + Be_x + Ke_{yt} + Ne_{xyt}$, a strange combination which happens to lie along the diagonal in matrix form. The associated eigenvalue determinant is

$$\begin{vmatrix} A-\lambda+B+K+N & 0 & 0 & 0 \\ 0 & A-\lambda-B-K+N & 0 & 0 \\ 0 & 0 & A-\lambda+B-K-N & 0 \\ 0 & 0 & 0 & A-\lambda-B+K-N \end{vmatrix}$$

The characteristic equation is

$$(A-\lambda+B+K+N)*(A-\lambda-B-K+N)*(A-\lambda+B-K-N)*(A-\lambda-B+K-N) = 0$$

The four eigenvalues are

$$\begin{aligned} \lambda_1 &= +A+B+K+N \\ \lambda_2 &= +A-B-K+N \\ \lambda_3 &= +A+B-K-N \\ \lambda_4 &= +A-B+K-N \end{aligned}$$

Fourspace Vector

Our next item of interest is the humble fourspace vector. Fock and Ivanenko [1] pointed out in 1929 that the eigenvalue of a vector is, within a sign, it's magnitude. Start with

$$Be_x + Ce_y + De_z + Ee_t$$

The matrix format is

$$\begin{pmatrix} +B & +C-E & 0 & -D \\ +C+E & -B & +D & 0 \\ 0 & +D & +B & +C+E \\ -D & 0 & +C-E & -B \end{pmatrix}$$

The characteristic equation, courtesy of GiNaC, is

$$(B^2 + C^2 + D^2 - E^2 - \lambda^2)^2 = 0$$

with solutions

$$\lambda = \pm\sqrt{B^2 + C^2 + D^2 - E^2}$$

Scalar Plus Fourspace Vector

Combining a scalar with the spacetime vector above, we start with

$$Ae_q + Be_x + Ce_y + De_z + Ee_t$$

The matrix format is

$$\begin{pmatrix} +A+B & +C-E & 0 & -D \\ +C+E & +A-B & +D & 0 \\ 0 & +D & +A+B & +C+E \\ -D & 0 & +C-E & +A-B \end{pmatrix}$$

Noting earlier that A can be grouped with λ, we are not surprised to find that the characteristic equation is

$$(B^2 + C^2 + D^2 - E^2 - (\lambda - A)^2)^2 = 0$$

with solutions

$$\lambda = A \pm \sqrt{B^2 + C^2 + D^2 - E^2}$$

Spatial Bivectors

Let's look at purely spatial bivectors.

$$Fe_{xy} + Ge_{xz} + He_{yz}$$

The associated matrix is

$$\begin{pmatrix} 0 & +F & +H & -G \\ -F & 0 & -G & -H \\ -H & +G & 0 & +F \\ +G & +H & -F & 0 \end{pmatrix}$$

The characteristic equation is

$$(F^2 + G^2 + H^2 + \lambda^2)^2 = 0$$

$$\lambda = \pm\sqrt{-F^2 - G^2 - H^2}$$

We notice that λ is now imaginary. This reflects the negative square nature of the spatial bivectors in a fashion similar to the negative sign associated with E^2 above associated with the negative metric assigned to time.

Spacetime Bivectors

The previous section looked at pure space bivectors, such as associated with planar rotations. Now look at spacetime bivectors, such as associated with velocities or time integrated distances.

$$Je_{xt} + Ke_{yt} + Le_{zt}$$

The associated matrix is

$$\begin{pmatrix} +K & -J & +L & 0 \\ -J & -K & 0 & +L \\ +L & 0 & -K & +J \\ 0 & +L & +J & +K \end{pmatrix}$$

The characteristic equation is

$$(J^2 + K^2 + L^2 - \lambda^2)^2 = 0$$
$$\lambda = \pm\sqrt{J^2 + K^2 + L^2}$$

Combined Space and Spacetime Bivectors

We now look all six bivectors combined.

$$Fe_{xy} + Ge_{xz} + He_{yz} + Je_{xt} + Ke_{yt} + Le_{zt}$$

The associated matrix is

$$\begin{pmatrix} +K & +F - J & +H + L & -G \\ -F - J & -K & -G & -H + L \\ -H + L & +G & -K & +F + J \\ +G & +H + L & -F + J & +K \end{pmatrix}$$

The characteristic equation now has some cross terms. Notice the GK term.

$$(-F^2 - G^2 - H^2 + J^2 + K^2 + L^2 - \lambda^2)^2 + (2FL - 2GK + 2HJ)^2 = 0$$

This can be simplified with imaginary numbers.

$$(-F^2 - G^2 - H^2 + J^2 + K^2 + L^2 - \lambda^2)^2 = -(2FL - 2GK + 2HJ)^2$$
$$(-F^2 - G^2 - H^2 + J^2 + K^2 + L^2 - \lambda^2) = \mp i(2FL - 2GK + 2HJ)$$
$$-\lambda^2 = +F^2 + G^2 + H^2 - J^2 - K^2 - L^2 \mp i(2FL - 2GK + 2HJ)$$
$$\lambda^2 = -F^2 - G^2 - H^2 + J^2 + K^2 + L^2 \pm i(2FL - 2GK + 2HJ)$$
$$\lambda^2 = (L \pm iF)^2 + (K \mp iG)^2 + (J \pm iH)^2$$

Even Grade Multivectors

The even grade multivectors can mimic a three dimensional Euclidean space embedded in Minkowski space, distinct from the standard three dimensional subset.

$$Ae_q + Fe_{xy} + Ge_{xz} + He_{yz} + Je_{xt} + Ke_{yt} + Le_{zt} + Se_{xyzt}$$

The associated matrix is

$$\begin{pmatrix} +A+K & +F-J & +H+L & -G+S \\ -F-J & +A-K & -G-S & -H+L \\ -H+L & +G+S & +A-K & +F+J \\ +G-S & +H+L & -F+J & +A+K \end{pmatrix}$$

The characteristic equation is

$$((\lambda-A)^2+F^2+G^2+H^2-J^2-K^2-L^2-S^2)^2+(2*(\lambda-A)*S+2FL-2GK+2HJ)^2 = 0$$

Using our imaginary numbers, as before, we find

$$((\lambda-A)^2+F^2+G^2+H^2-J^2-K^2-L^2-S^2)^2 = -(2*(\lambda-A)*S+2FL-2GK+2HJ)^2$$

$$((\lambda-A)^2+F^2+G^2+H^2-J^2-K^2-L^2-S^2) = \pm i(2*(\lambda-A)*S+2FL-2GK+2HJ)$$

$$((\lambda-A)^2\mp i2*(\lambda-A)*S-S^2) = (L^2\pm i2FL-F^2)+(K^2\mp i2GK-G^2)+(J^2\pm i2HJ-H^2)$$

$$(\lambda - A \mp iS)^2 = (L \pm iF)^2 + (K \mp iG)^2 + (J \pm iH)^2$$

Three Dimensional Space

Three dimensional embedded space is the multivector

$$Ae_q + Be_x + Ce_y + De_z + Fe_{xy} + Ge_{xz} + He_{yz} + Me_{xyz}$$

The associated matrix is

$$\begin{pmatrix} +A+B & +C+F & +H+M & -D-G \\ +C-F & +A-B & +D-G & -H+M \\ -H-M & +D+G & +A+B & +C+F \\ -D+G & +H-M & +C-F & +A-B \end{pmatrix}$$

This is similar to the previous matrix. The characteristic equation is

$$((\lambda-A)^2+F^2+G^2+H^2-B^2-C^2-D^2-M^2)^2+(2*(\lambda-A)*M+2DF-2CG+2BH)^2 = 0$$

$$(\lambda - A \mp iM)^2 = (D \pm iF)^2 + (C \mp iG)^2 + (B \pm iH)^2$$

Trivectors and Quadvector

Analogous to scalar and vector, we have the trivectors and pseudoscalar (quadvector)

$$Me_{xyz} + Ne_{xyt} + Pe_{xzt} + Re_{yzt} + Se_{xyzt}$$

The associated matrix is

$$\begin{pmatrix} +N & 0 & +M+P & +R+S \\ 0 & +N & +R-S & +M-P \\ -M+P & +R+S & -N & 0 \\ +R-S & -M-P & 0 & -N \end{pmatrix}$$

The characteristic equation is

$$(\lambda^2 + (M^2 - N^2 - P^2 - R^2 + S^2))^2 = 0$$

$$(\lambda^2 + (M^2 - N^2 - P^2 - R^2 + S^2)) = 0$$
$$\lambda^2 = -M^2 + N^2 + P^2 + R^2 - S^2$$

Minkowski Fourspace

The ultimate goal has been to factor the full fourspace eigenvalue equation. The generic multivector is

$$\begin{aligned} &Ae_q + \\ &Be_x + Ce_y + De_z + Ee_t + \\ &Fe_{xy} + Ge_{xz} + He_{yz} + Je_{xt} + Ke_{yt} + Le_{zt} + \\ &Me_{xyz} + Ne_{xyt} + Pe_{xzt} + Re_{yzt} + \\ &Se_{xyzt} \end{aligned}$$

The associated array is

$$\begin{pmatrix} +A+B+K+N & +C-E+F-J & +H+L+M+P & -D-G+R+S \\ +C+E-F-J & +A-B-K+N & +D-G+R-S & -H+L+M-P \\ -H+L-M+P & +D+G+R+S & +A+B-K-N & +C+E+F+J \\ -D+G+R-S & +H+L-M-P & +C-E-F+J & +A-B+K-N \end{pmatrix}$$

The characteristic equation, organized in a format which illustrates some interesting internal structure, is

$$
\begin{aligned}
&+((A-\lambda)^2+B^2+C^2+D^2-E^2-F^2-G^2-H^2 \\
&+J^2+K^2+L^2-M^2+N^2+P^2+R^2-S^2))^2 \\
&+4*(\\
&-((A-\lambda)*B+(-HM+KN+LP))^2 \\
&-((A-\lambda)*C+(-GM+JN-LR))^2 \\
&-((A-\lambda)*D+(+FM+JP+KR))^2 \\
&+((A-\lambda)*E+(+FN+GP+HR))^2 \\
&+((A-\lambda)*F+(+DM-EN+LS))^2 \\
\\
&+((A-\lambda)*G+(-CM-EP-KS))^2 \\
&+((A-\lambda)*H+(-BM+ER-JS))^2 \\
&-((A-\lambda)*J+(-CN-DP-HS))^2 \\
&-((A-\lambda)*K+(-BN+DR-GS))^2 \\
&-((A-\lambda)*L+(-BP-CR+FS))^2 \\
\\
&+((A-\lambda)*M+(-BH+CG-DF))^2 \\
&-((A-\lambda)*N+(-BK+CJ-EF))^2 \\
&-((A-\lambda)*P+(-BL+DJ-EG))^2 \\
&-((A-\lambda)*R+(+CL-DK+EH))^2 \\
&+((A-\lambda)*S+(-FL+GK-HJ))^2)=0
\end{aligned}
$$

The signs associated with the square terms is the negative of the square of the multivector base elements.

Another format for the same equation is a quartic polynomial in $(A-\lambda)$

is

$$
\begin{aligned}
&(A-\lambda)^4\\
&-2*(A-\lambda)^2*(+B^2+C^2+D^2-E^2-F^2-G^2-H^2\\
&+J^2+K^2+L^2-M^2+N^2+P^2+R^2-S^2)\\
\\
&-8*(A-\lambda)*(+B*(H*M-K*N-L*P)+C*(J*N-G*M-L*R)\\
&+D*(F*M+J*P+K*R)-E*(F*N+G*P+H*R)\\
&+S*(F*L-G*K+H*J)\\
\\
&+(+B^2+C^2+D^2-E^2-F^2-G^2-H^2\\
&+J^2+K^2+L^2-M^2+N^2+P^2+R^2-S^2)^2\\
\\
&+4*(\\
&+(-HM+KN+LP)^2\\
&+(-GM+JN-LR)^2\\
&+(+FM+JP+KR)^2\\
&+(+FN+GP+HR)^2\\
&+(+DM-EN+LS)^2\\
\\
&+(-CM-EP-KS)^2\\
&+(-BM+ER-JS)^2\\
&+(-CN-DP-HS)^2\\
&+(-BN+DR-GS)^2\\
&+(-BP-CR+FS)^2\\
\\
&+(-BH+CG-DF)^2\\
&+(-BK+CJ-EF)^2\\
&+(-BL+DJ-EG)^2\\
&+(+CL-DK+EH)^2\\
&+(-FL+GK-HJ)^2)=0
\end{aligned}
$$

This rather large equation is already in reduced quartic format, meaning that the sum of the roots is zero. As a reduced quartic, it is solvable in closed form. Numerically, the evaluation of roots is straight forward. However, the symbolic evaluation of the roots, while feasible, has not yet provided much insight.

References

[1] "Über eine mögliche geometrische Deutung der relativistischen Quantentheorie", V. Fock and D. Ivanenko, Zeit. f. Phys. 54 (1929), 798-802.