The Magic Square at the Heart of Spacetime

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Abstract

Four two-by-two matrices can be used to implement two dimensional Euclidean geometric algebra. The pair-wise direct product of these matrices provides sixteen real four-by-four matrices implementing four dimensional Minkowski geometric algebra with (+,+,+,-) signature. These sixteen geometric algebra elements can be arranged in a four-by-four grid sharing many of the symmetries of magic squares, with the substitution of the geometric product in place of addition.

Two Dimensional Geometric Algebra Matrices

Two dimensional Euclidean geometric algebra can be implemented using regular matrix multiplication, and the following four basis matrices.

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad a_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad a_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad a_{xy} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The unit matrix is a_1. Both a_x and a_y square to a_1. The product of a_x and a_y is an antisymmetric bivector a_{xy}.

$$a_x a_y = -a_y a_x = a_{xy}$$

Any two dimensional multivector can be represented by a two by two matrix.

$$Aa_1 + Ba_x + Ca_y + Da_{xy} \quad = \quad \begin{pmatrix} (A+B) & (C+D) \\ (C-D) & (A-B) \end{pmatrix}$$

Likewise, from any two by two matrix, we can recover our multivector components. Given

$$
\begin{aligned}
m &= \begin{pmatrix} E & F \\ G & H \end{pmatrix} \\
A &= (E+H)/2 \\
B &= (E-H)/2 \\
C &= (F+G)/2 \\
D &= (F-G)/2
\end{aligned}
$$

Minkowski Basis Matrices

The sixteen outer products of the four 2x2 matrices in Table 1 provide a real valued basis set for Minkowski spacetime. The red highlighted terms square to negative one, while the remaining terms square to unity. This convention is used throughout this note.

On inspection, these matrices are seen to be the direct or Kronecker product of the four different 2x2 matrices which define geometric algebra in the plane. I will use a_x to define the 2x2 matrix set, and e_x for the 4x4 matrix.

Minkowski Spacetime Geometric Algebra

Minkowski spacetime is a four dimensional spacetime where space axii square to one, while the time axis squares to negative one.

We have a scalar element e_1 which commutes with all other terms. We have four directional axii, e_x, e_y, e_z, and e_t, which anticommute amoung themselves, with square properties noted above. We have six bivectors. The pure space bivectors e_{xy}, e_{yz}, and e_{zx} square to negative one, while the spacetime bivectors e_{tx}, e_{ty}, and e_{tz} square to unity. We have four trivectors. The pure space trivector e_{xyz} square to negative one, while the spacetime trivectors e_{tyz}, e_{tzx}, and e_{txy} square to unity. For Minkowski spacetime, ten elements square to unity, and six square to negative one.

$\otimes$	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}$	$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}$
$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix}$	$\begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{matrix}$
$\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{matrix}$
$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{matrix}$
$\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}$

Table 1: Minkowski Basis Via Direct Product

q	x	y	z	t	xy	xz	yz	xt	yt	zt	xyz	xyt	xzt	yzt	xyzt
x	q	xy	xz	xt	y	z	xyz	t	xyt	xzt	yz	yt	zt	xyzt	yzt
y	-xy	q	yz	yt	-x	-xyz	z	-xyt	t	yzt	-xz	-xt	-xyzt	zt	-xzt
z	-xz	-yz	q	zt	xyz	-x	-y	-xzt	-yzt	t	xy	xyzt	-xt	-yt	xyt
t	-xt	-yt	-zt	-q	xyt	xzt	yzt	x	y	z	-xyzt	-xy	-xz	-yz	xyz
xy	-y	x	xyz	xyt	-q	-yz	xz	-yt	xt	xyzt	-z	-t	-yzt	xzt	-zt
xz	-z	-xyz	x	xzt	yz	-q	-xy	-zt	-xyzt	xt	y	yzt	-t	-xyt	yt
yz	xyz	-z	y	yzt	-xz	xy	-q	xyzt	-zt	yt	-x	-xzt	xyt	-t	-xt
xt	-t	-xyt	-xzt	-x	yt	zt	xyzt	q	xy	xz	-yzt	-y	-z	-xyz	yz
yt	xyt	-t	-yzt	-y	-xt	-xyzt	zt	-xy	q	yz	xzt	x	xyz	-z	-xz
zt	xzt	yzt	-t	-z	xyzt	-xt	-yt	-xz	-yz	q	-xyt	-xyz	x	y	xy
xyz	yz	-xz	xy	xyzt	-z	y	-x	yzt	-xzt	xyt	-q	-zt	yt	-xt	-t
xyt	yt	-xt	-xyzt	-xy	-t	-yzt	xzt	-y	x	xyz	zt	q	yz	-xz	-z
xzt	zt	xyzt	-xt	-xz	yzt	-t	-xyt	-z	-xyz	x	-yt	-yz	q	xy	y
yzt	-xyzt	zt	-yt	-yz	-xzt	xyt	-t	xyz	-z	y	xt	xz	-xy	q	-x
xyzt	-yzt	xzt	-xyt	-xyz	-zt	yt	-xt	yz	-xz	xy	t	z	-y	x	-q

Table 2: Minkowski Geometric Algebra Multiplication Table

$\otimes$	a_1	a_x	a_y	a_{xy}
a_1	$a_1 \otimes a_1$	$a_1 \otimes a_x$	$a_1 \otimes a_y$	$a_1 \otimes a_{xy}$
a_x	$a_x \otimes a_1$	$a_x \otimes a_x$	$a_x \otimes a_y$	$a_x \otimes a_{xy}$
a_y	$a_y \otimes a_1$	$a_y \otimes a_x$	$a_y \otimes a_y$	$a_y \otimes a_{xy}$
a_{xy}	$a_{xy} \otimes a_1$	$a_{xy} \otimes a_x$	$a_{xy} \otimes a_y$	$a_{xy} \otimes a_{xy}$

Table 3: Symbolic Minkowski Basis

Direct Product Examples

Table 1 generates 16 basis 4x4 matrices from the direct product of the 2x2 geometric algebra basis. Prefactors are on the left, and postfactors are on top.

Table 2 is a multiplication table for the multivector basis for Minkowski spacetime.

Table 3 is the same direct product multiplication table as Table 1, but uses symbols rather than matrices.

As an example of the direct product process, here are two sample calculations used in the table.

$$a_1 \otimes a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$a_1 \otimes a_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Matrix and Direct Product Distribution

If $A, B, C,$ and D are square matrices,

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

If A, B, C, D, E and F are square matrices,

$$(A \otimes B)(C \otimes D)(E \otimes F) = (ACE) \otimes (BDF)$$

In cascaded matrix multiplication, we can collect the order sensitive pre-products and post-products, and evaluate those terms before doing the direct product.

Pandiagonal Four-by-Four Magic Squares

Three sample 4x4 magic squares are shown below [5].

```
 0   7   9  14     0   7  10  13     0   7  12  11
11  12   2   5    11  12   1   6    13  10   1   6
 6   1  15   8     5   2  15   8     3   4  15   8
13  10   4   3    14   9   4   3    14   9   2   5
```

The magic constant for these squares is thirty. Along each row (4), each column (4), ascending wrapped diagonal (4) and descending wrapped diagonal (4), we find the same magic sum of 30 in sixteen relationships. In a similar fashion, each block of four, such as (0, 7, 11, 12) in the top left square sums to 30. These foursome squares wrap around edges as in a toroid mapping, so the four corners also constitute a square. Corners of 3x3 squares, such as (0,9,6,15) sum to 30. Half diagonals, such as (0,15) and (7,8) sum to half the magic constant (15 in this case). In total, 52 different sums to 30 are found in pandiagonal 4x4 magic squares.

Minkowski Magic Square

Repeating Table 3, we have a structure similar to a pandiagonal magic square inside Minkowski geometric algebra, where instead of addition, we use matrix multiplication for the geometric product.

$\otimes$	a_1	a_x	a_y	a_{xy}
a_1	$a_1 \otimes a_1$	$a_1 \otimes a_x$	$a_1 \otimes a_y$	$a_1 \otimes a_{xy}$
a_x	$a_x \otimes a_1$	$a_x \otimes a_x$	$a_x \otimes a_y$	$a_x \otimes a_{xy}$
a_y	$a_y \otimes a_1$	$a_y \otimes a_x$	$a_y \otimes a_y$	$a_y \otimes a_{xy}$
a_{xy}	$a_{xy} \otimes a_1$	$a_{xy} \otimes a_x$	$a_{xy} \otimes a_y$	$a_{xy} \otimes a_{xy}$

Across each row and column, the extended product is negative unity. For example, the top row has

$$(a_1 \otimes a_1)(a_1 \otimes a_x)(a_1 \otimes a_y)(a_1 \otimes a_{xy}) = (a_1 a_1 a_1 a_1) \otimes (a_1 a_x a_y a_{xy}) = -a_1 \otimes a_1$$

The second row has

$$(a_x \otimes a_1)(a_x \otimes a_x)(a_x \otimes a_y)(a_x \otimes a_{xy}) = (a_x a_x a_x a_x) \otimes (a_1 a_x a_y a_{xy}) = -a_1 \otimes a_1$$

The left column is

$$(a_1 \otimes a_1)(a_x \otimes a_1)(a_y \otimes a_1)(a_{xy} \otimes a_1) = (a_1 a_x a_y a_{xy}) \otimes (a_1 a_1 a_1 a_1) = -a_1 \otimes a_1$$

We find descending diagonals product to $a_1 \otimes a_1$, while all other products lead to $-a_1 \otimes a_1$. Similarly, all 2x2 cells, such as the central $a_x \otimes a_x$, $a_x \otimes a_y$, $a_y \otimes a_x$, $a_y \otimes a_y$ quad product to plus or minus unity, depending upon order of multiplication chosen. We note the half diagonal product is $a_y \otimes a_y$.

Mapping to Minkowski Spacetime Units

Having this fine grid, we now want to map these elements to our Minkowski geometric algebra.

$\otimes$	a_1	a_x	a_y	a_{xy}
a_1	$a_1 \otimes a_1$	$a_1 \otimes a_x$	$a_1 \otimes a_y$	$a_1 \otimes a_{xy}$
a_x	$a_x \otimes a_1$	$a_x \otimes a_x$	$a_x \otimes a_y$	$a_x \otimes a_{xy}$
a_y	$a_y \otimes a_1$	$a_y \otimes a_x$	$a_y \otimes a_y$	$a_y \otimes a_{xy}$
a_{xy}	$a_{xy} \otimes a_1$	$a_{xy} \otimes a_x$	$a_{xy} \otimes a_y$	$a_{xy} \otimes a_{xy}$

It turns out that we have a number of ways this could be done.

The identity element is $e_1 = a_1 \otimes a_1$, with no choice possible. The red items square to negative one, and must to be members of e_t, e_{xyz}, e_{txyz}, e_{xy}, e_{yz}, and e_{zx}. The descending diagonal has a cascade product of unity, with the top left element being unity. This forces the lower three elements to be all positive square terms with a cascaded product of unity. Furthermore, the bottom right hand unit needs to be the product of two negative square terms, given that the four corners product must be plus or minus unity.

Using a trailing t notation, I can list my choices for the bottom corner, other corners, and descending diagonal.

Bottom Corner	Other Corners	Diagonal Terms
e_x	e_{yz} e_{xyz}	e_{yt} e_{xyt} e_{zt} e_{xzt}
e_y	e_{xz} e_{xyz}	e_{xt} e_{xyt} e_{zt} e_{zyt}
e_z	e_{xy} e_{xyz}	e_{xt} e_{xzt} e_{yt} e_{yzt}
e_{xt}	e_{yz} e_{xyzt}	e_z e_{xzt} e_y e_{xyt}
e_{yt}	e_{xz} e_{xyzt}	e_x e_{xyt} e_z e_{yzt}
e_{zt}	e_{xy} e_{xyzt}	e_x e_{xzt} e_y e_{yzt}
e_{xyt}	e_{xy} e_t	e_x e_{yt} e_y e_{xt}
e_{xzt}	e_{xz} e_t	e_x e_{zt} e_z e_{xt}
e_{yzt}	e_{yz} e_t	e_y e_{zt} e_z e_{yt}

I find 72 mappings of Minkowski spacetime to this Magic Square format. The full list is at the end of this document. The program 72_Mappings.c `http://www.kurtnalty.com/72_Mappings.c` explicitly verifies the mappings. There is a nomenclature shift in the simple text tables, where $q = e_1$, $x = e_x$, and so forth. Three of the 72 mappings are listed below.

```
+q      +x      +y      +xy    | +q      +xyt    +zt     +xyz | +q      +y      +z      +yz
+xyt    +yt     -xt     -t     | +x      +yt     +xzt    +yz  | +yzt    +zt     -yt     -t
+zt     +xzt    +yzt    +xyzt  | +y      -xt     +yzt    -xz  | +xt     -xyt    -xzt    +xyzt
+xyz    +yz     -xz     -z     | +xy     -t      +xyzt   -z   | +xyz    -xz     +xy     -x
```

Unexpected Symmetries

I had really expected an alignment between planar directions and the initial directions of spacetime. Using as an example, the first mapping from above, comparing to our planar direct products, we see this isn't necessarily so.

$\otimes$	a_1	a_x	a_y	a_{xy}
a_1	$a_1 \otimes a_1 = e_1$	$a_1 \otimes a_x = e_x$	$a_1 \otimes a_y = e_y$	$a_1 \otimes a_{xy} = e_{xy}$
a_x	$a_x \otimes a_1 = e_{xyt}$	$a_x \otimes a_x = e_{yt}$	$a_x \otimes a_y = -e_{xt}$	$a_x \otimes a_{xy} = -e_t$
a_y	$a_y \otimes a_1 = e_{zt}$	$a_y \otimes a_x = e_{xzt}$	$a_y \otimes a_y = e_{yzt}$	$a_y \otimes a_{xy} = e_{xyzt}$
a_{xy}	$a_{xy} \otimes a_1 = e_{xyz}$	$a_{xy} \otimes a_x = e_{yz}$	$a_{xy} \otimes a_y = -e_{xz}$	$a_{xy} \otimes a_{xy} = -e_z$

So what patterns do we see in the example above?

We see the negative square terms involve the planar bivector in a single term, while the bottom right corner involves the planar bivector twice, masquarading as a vector. Our perception of z as a space axis similar to x and y is based upon the symmetries z possesses, such as squaring to positive one, anti-commuting with x and y, and so forth. However, from this table, we can interpret the *origin* of z as fundamentally different from x and y. Likewise, the origin of the time axis as the outer product of vector and bivector is fascinating.

We see that this table implements a commutating product. This is due to the layout of the table, where the top rows and left hand column have the planar unity element in complementary positions. The real result is that this table uses the symmetric geometric product. A list of the commutating products is at the end of this note. Using the above example, here is an illustration of the commutating nature of this product.

Multiplication by e_1 is inherently commutative, as is the square operation, so to reduce table spread I have not filled in these squares.

```
+q    +x    +y    +xy
+xyt  +yt   -xt   -t
+zt   +xzt  +yzt  +xyzt
+xyz  +yz   -xz   -z
```

	e_1	e_x	e_y	e_{xy}
e_1	e_1	e_x	e_y	e_{xy}
e_{xyt}	e_{xyt}	e_{yt}	$-e_{xt} = e_{xyt}e_y = e_y e_{xyt}$	$-e_t = e_{xyt}e_{xy} = e_{xy}e_{xyt}$
e_{zt}	e_{zt}	$e_{xzt} = e_{zt}e_x = e_x e_{zt}$	e_{yzt}	$e_{xyzt} = e_{zt}e_{xy} = e_{xy}e_{zt}$
e_{xyz}	e_{xyz}	$e_{yz} = e_{xyz}e_x = e_x e_{xyz}$	$-e_{xz} = e_{xyz}e_y = e_y e_{xyz}$	e_z

We see the different mappings of the Minksowski spacetime elements to the planar direct product have a similarity to matrix row operations. Swapping the middle columns or rows creates another magic square, which is another remapping. The example below demonstrates the swapping of middle rows and columns, and associated sign changes on the right column or bottom row.

```
 +q     +x     +y     +xy   | +q     +x     +y     +xy
 +xyt   +yt    -xt    -t    | +zt    +xzt   +yzt   +xyzt
 +zt    +xzt   +yzt   +xyzt | +xyt   +yt    -xt    -t
 +xyz   +yz    -xz    -z    | -xyz   -yz    +xz    +z
---------------------------------------------------------
 +q     +y     +x     -xy   | +q     +y     +x     -xy
 +xyt   -xt    +yt    +t    | +zt    +yzt   +xzt   -xyzt
 +zt    +yzt   +xzt   -xyzt | +xyt   -xt    +yt    +t
 +xyz   -xz    +yz    +z    | -xyz   +xz    -yz    -z
```

Commutating Products in Minkowski Spacetime

The identity element e_q commutes with all elements, and is not listed below. Each row first lists an element, and then lists the other factors which commute with this element. For this table, the minus signs needed for a magic square format are not shown.

e_x	e_{yz}	e_{yt}	e_{zt}	e_{xyz}	e_{xyt}	e_{xzt}
e_y	e_{xz}	e_{xt}	e_{zt}	e_{xyz}	e_{xyt}	e_{yzt}
e_z	e_{xy}	e_{xt}	e_{yt}	e_{xyz}	e_{xzt}	e_{yzt}
e_t	e_{xy}	e_{yz}	e_{xz}	e_{xyt}	e_{yzt}	e_{xzt}
e_{xy}	e_z	e_t	e_{zt}	e_{xyz}	e_{xyt}	e_{xyzt}
e_{yz}	e_x	e_t	e_{xt}	e_{xyz}	e_{yzt}	e_{xyzt}
e_{xz}	e_y	e_t	e_{yt}	e_{xyz}	e_{xzt}	e_{xyzt}
e_{xt}	e_y	e_z	e_{yz}	e_{xyt}	e_{xzt}	e_{xyzt}
e_{yt}	e_x	e_z	e_{xz}	e_{xyt}	e_{yzt}	e_{xyzt}
e_{zt}	e_x	e_y	e_{xy}	e_{xzt}	e_{yzt}	e_{xyzt}
e_{xyz}	e_x	e_y	e_z	e_{yz}	e_{xz}	e_{xy}
e_{xyt}	e_x	e_y	e_t	e_{yt}	e_{xt}	e_{xy}
e_{xzt}	e_x	e_z	e_t	e_{zt}	e_{xt}	e_{xz}
e_{yzt}	e_y	e_z	e_t	e_{zt}	e_{yt}	e_{yz}
e_{xyzt}	e_{xy}	e_{yz}	e_{xz}	e_{zt}	e_{xt}	e_{yt}

How to Make a Magic Minkowski Square

To create a magic square format table, start by placing e_q in cell (0,0). Next, choose two positive square basis from above, which have a negative square product, for the (0,1) and (0,2) cells. The order sensitive geometric product of these two basis is placed in cell (0,3). Examine the list of commutating terms for your two choices. You will find three common terms, two of which square to one, the third which squares to negative one. Place the two terms which square to one in cells (1,0) and (2,0). Place their order sensitive geometric product in cell (3,0). To fill in the remaining cells, take the term in the left hand column, for example, cell (1,0). Use this cell as a prefactor applied to the terms of the top row to fill in the remaining cells of the current row. For example,

```
cell(1,1) = cell(1,0)*cell(0,1)
cell(1,2) = cell(1,0)*cell(0,2)
cell(1,3) = cell(1,0)*cell(0,3)
cell(2,1) = cell(2,0)*cell(0,1)
cell(2,2) = cell(2,0)*cell(0,2)
cell(2,3) = cell(2,0)*cell(0,3)
cell(3,1) = cell(3,0)*cell(0,1)
cell(3,2) = cell(3,0)*cell(0,2)
cell(3,3) = cell(3,0)*cell(0,3)
```

The 72 Variations in Ordered Basis Format

The text format table is available at `http://www.kurtnalty.com/72_Items_Standard_Listing.txt`. The three tables starting on page 13 list basis substitutions which maintain magic square symmetry in a tabular format of scalar, then vectors, then space bivectors, then time bivectors, trivectors and finally quadvector.

As an illustration of the remappings, a fragment of the table is reproduced below.

Canonical Form

	+q	+x	+y	+z	+t
1	+q	+x	+y	+z	+t
2	+q	+x	+y	-z	-xyzt
3	+q	+x	+z	-y	+xyzt
4	+q	+x	+z	+y	+t
5	+q	+x	+xt	-yzt	+xy
6	+q	+x	+xt	+yzt	+xz
7	+q	+x	+yzt	+xt	+xz
8	+q	+x	+yzt	-xt	-xy

In line 4, y and z are swapped, which is a standard symmetry. However, in line 5, the role of y is being played by xt, the role of z is played by -yzt, and the role of t is played by +xy. This is an example of the novel symmetries presented by the magic square relationship, which is not part of the standard physics toolkit.

Opportunities

Engineers routinely use the symmetries in space for coordinate transformations. Physicists routinely use the symmetries in spacetime for electromagnetism and Lorentz transformations. This magic square set of symmetries offers the opportunity for additional invariants beyond Cartesian space and Lorentz transforms.

Simple remappings, interchanging x and y, for example, lead to a change in sign of some of the Minkowski basis terms, but are totally compatible with existing physics.

More radical remappings, for example, interchanging the bivector zt with trivector xyt are also possible, using the interpretation that the Minkowski spacetime is an overlay on the outer product of planar forms. I believe these radical remappings will lead to a large number of previously overlooked invariants.

I hope the seventy-two mappings shown here lead to a rich set of transformations beyond the simple Lorentz transforms, ultimately leading to practical sigma deritrinitation and legenic acceleration systems.

Canonical Form

+q	+x	+y	+z	+t	+xy	+yz	+xz	+yt	+xt	+zt	+xyz	+xyt	+xzt	+yzt	+xyzt

The 72 Variations with Magic Square Symmetry

+q	+x	+y	+z	+t	+xy	+yz	+xz	+yt	+xt	+zt	+xyz	+xyt	+xzt	+yzt	+xyzt
+q	+x	+y	-z	-xyzt	+xy	-yz	-xz	+xzt	-yzt	+xyt	-xyz	+zt	+yt	-xt	-t
+q	+x	+z	-y	+xyzt	+xz	+yz	-xy	+xyt	+yzt	+xzt	+xyz	+yt	+zt	-xt	-t
+q	+x	+z	+y	+t	+xz	-yz	+xy	+zt	+xt	+yt	-xyz	+xzt	+xyt	-yzt	-xyzt
+q	+x	+xt	-yzt	+xy	+t	+xyz	-xyzt	+yt	+y	+xzt	+yz	+xyt	+zt	-z	-xz
+q	+x	+xt	+yzt	+xz	+t	-xyz	+xyzt	+zt	+z	+xyt	-yz	+xzt	+yt	-y	-xy
+q	+x	+yzt	+xt	+xz	+xyzt	+xyz	+t	+xyt	+z	+zt	+yz	+yt	+xzt	+y	+xy
+q	+x	+yzt	-xt	-xy	+xyzt	-xyz	-t	+xzt	-y	+yt	-yz	+zt	+xyt	-z	-xz
+q	+y	+x	-z	-t	-xy	-xz	-yz	-xt	-yt	+zt	+xyz	+xyt	+yzt	+xzt	-xyzt
+q	+y	+x	+z	+xyzt	-xy	+xz	+yz	+yzt	-xzt	+xyt	-xyz	+zt	-xt	+yt	+t
+q	+y	+z	-x	-xyzt	+yz	+xz	+xy	-xyt	+xzt	+yzt	-xyz	+xt	+zt	-yt	-t
+q	+y	+z	+x	+t	+yz	-xz	-xy	+zt	+yt	+xt	+xyz	+yzt	-xyt	-xzt	+xyzt
+q	+y	+yt	+xzt	+xy	+t	+xyz	-xyzt	-xt	-x	+yzt	-xz	+xyt	+zt	-z	-yz
+q	+y	+yt	-xzt	+yz	+t	-xyz	+xyzt	+zt	+z	+xyt	+xz	+yzt	-xt	+x	-xy
+q	+y	+xzt	+yt	+yz	-xyzt	-xyz	+t	-xyt	+z	+zt	+xz	+xt	+yzt	+x	-xy
+q	+y	+xzt	-yt	+xy	-xyzt	+xyz	-t	+yzt	-x	+xt	-xz	+zt	-xyt	-z	-yz
+q	+z	+x	+y	-xyzt	-xz	+xy	-yz	-yzt	-xyt	+xzt	+xyz	+yt	-xt	+zt	+t
+q	+z	+x	-y	-t	-xz	-xy	+yz	-xt	-zt	+yt	-xyz	+xzt	-yzt	+xyt	+xyzt
+q	+z	+y	+x	+xyzt	-yz	-xy	-xz	-xzt	+xyt	+yzt	-xyz	+xt	-yt	+zt	+t
+q	+z	+y	-x	-t	-yz	+xy	+xz	-yt	-zt	+xt	+xyz	+yzt	-xzt	-xyt	-xyzt
+q	+z	+xyt	+zt	-yz	+xyzt	+xyz	+t	-xzt	+y	+yt	+xy	+xt	-yzt	+x	-xz
+q	+z	+xyt	-zt	+xz	+xyzt	-xyz	-t	-yzt	-x	+xt	-xy	+yt	-xzt	-y	+yz
+q	+z	+zt	-xyt	+xz	+t	+xyz	-xyzt	-xt	-x	+yzt	+xy	+xzt	-yt	+y	-yz
+q	+z	+zt	+xyt	+yz	+t	-xyz	+xyzt	-yt	-y	+xzt	-xy	+yzt	-xt	+x	-xz

Table 4: Minkowski Geometric Algebra Substitution Table 1/3

Canonical Form

+q	+x	+y	+z	+t	+xy	+yz	+xz	+yt	+xt	+zt	+xyz	+xyt	+xzt	+yzt	+xyzt
Variations 28-54 with Magic Square Symmetry															
+q	+xt	+x	+yzt	-xy	-t	+xyzt	-xyz	-y	-yt	+xzt	+yz	+xyt	-z	+zt	+xz
+q	+xt	+x	-yzt	-xz	-t	-xyzt	+xyz	-z	-zt	+xyt	-yz	+xzt	-y	+yt	+xy
+q	+xt	+yt	+zt	-xyz	+xy	+yz	+xz	-xzt	+yzt	+xyt	+xyzt	+z	-y	+x	-t
+q	+xt	+yt	-zt	+t	+xy	-yz	-xz	-y	-x	+z	-xyzt	+xyt	-xzt	-yzt	+xyz
+q	+xt	+zt	+yt	+xyz	+xz	-yz	+xy	-xyt	-yzt	+xzt	-xyzt	+y	-z	+x	-t
+q	+xt	+zt	-yt	+t	+xz	+yz	-xy	-z	-x	+y	+xyzt	+xzt	-xyt	+yzt	-xyz
+q	+xt	+yzt	-x	-xz	-xyz	+xyzt	+t	-xyt	-zt	+z	+yz	+y	-xzt	-yt	-xy
+q	+xt	+yzt	+x	+xy	-xyz	-xyzt	-t	-xzt	+yt	+y	-yz	+z	-xyt	+zt	+xz
+q	+yt	+y	-xzt	-xy	-t	+xyzt	-xyz	+x	+xt	+yzt	-xz	+xyt	-z	+zt	+yz
+q	+yt	+y	+xzt	-yz	-t	-xyzt	+xyz	-z	-zt	+xyt	+xz	+yzt	+x	-xt	+xy
+q	+yt	+xt	-zt	+xyz	-xy	-xz	-yz	-yzt	+xzt	+xyt	+xyzt	+z	+x	-y	+t
+q	+yt	+xt	+zt	-t	-xy	+xz	+yz	+x	+y	+z	-xyzt	+xyt	-yzt	-xzt	-xyz
+q	+yt	+zt	+xt	-xyz	+yz	-xz	-xy	+xyt	-xzt	+yzt	+xyzt	+x	-z	+y	-t
+q	+yt	+zt	-xt	+t	+yz	+xz	+xy	-z	-y	+x	-xyzt	+yzt	+xyt	+xzt	+xyz
+q	+yt	+xzt	-y	-yz	+xyz	-xyzt	+t	+xyt	-zt	+z	+xz	+x	-yzt	-xt	+xy
+q	+yt	+xzt	+y	-xy	+xyz	+xyzt	-t	-yzt	+xt	+x	-xz	+z	+xyt	+zt	+yz
+q	+zt	+z	+xyt	-xz	-t	+xyzt	-xyz	+x	+xt	+yzt	+xy	+xzt	+y	-yt	+yz
+q	+zt	+z	-xyt	-yz	-t	-xyzt	+xyz	+y	+yt	+xzt	-xy	+yzt	+x	-xt	+xz
+q	+zt	+xt	-yt	-xyz	-xz	-xy	+yz	+yzt	+xyt	+xzt	-xyzt	+y	+x	-z	+t
+q	+zt	+xt	+yt	-t	-xz	+xy	-yz	+x	+z	+y	+xyzt	+xzt	+yzt	-xyt	+xyz
+q	+zt	+yt	-xt	+xyz	-yz	+xy	+xz	+xzt	-xyt	+yzt	+xyzt	+x	+y	-z	+t
+q	+zt	+yt	+xt	-t	-yz	-xy	-xz	+y	+z	+x	-xyzt	+yzt	+xzt	+xyt	-xyz
+q	+zt	+xyt	-z	+yz	-xyz	+xyzt	+t	+xzt	-yt	+y	+xy	+x	+yzt	-xt	+xz
+q	+zt	+xyt	+z	-xz	-xyz	-xyzt	-t	+yzt	+xt	+x	-xy	+y	+xzt	+yt	-yz

Table 5: Minkowski Geometric Algebra Substitution Table 2/3

Canonical Form

+q	+x	+y	+z	+t	+xy	+yz	+xz	+yt	+xt	+zt	+xyz	+xyt	+xzt	+yzt	+xyzt

Variations 55-72 with Magic Square Symmetry

+q	+xyt	+z	-zt	+yz	-xyzt	-t	-xyz	-y	+xzt	+yt	+xy	+xt	+x	-yzt	+xz
+q	+xyt	+z	+zt	-xz	-xyzt	+t	+xyz	+x	+yzt	+xt	-xy	+yt	-y	-xzt	-yz
+q	+xyt	+zt	+z	-yz	+xyz	-t	-xyzt	+yt	-xzt	+y	+xy	+x	-xt	+yzt	-xz
+q	+xyt	+zt	-z	+xz	+xyz	+t	+xyzt	-xt	-yzt	+x	-xy	+y	+yt	+xzt	+yz
+q	+xyt	+xzt	-yzt	-xyz	+yz	-xy	+xz	+yt	-zt	+xt	+t	+x	-y	-z	+xyzt
+q	+xyt	+xzt	+yzt	-xyzt	+yz	+xy	-xz	-y	+z	+x	-t	+xt	+yt	+zt	+xyz
+q	+xyt	+yzt	+xzt	-xyz	-xz	-xy	+yz	-xt	-zt	+yt	+t	+y	+x	-z	+xyzt
+q	+xyt	+yzt	-xzt	-xyzt	-xz	+xy	-yz	+x	+z	+y	-t	+yt	-xt	+zt	+xyz
+q	+xzt	+y	-yt	-yz	+xyzt	-t	+xyz	-z	+xyt	+zt	+xz	+xt	+x	+yzt	+xy
+q	+xzt	+y	+yt	-xy	+xyzt	+t	-xyz	+x	-yzt	+xt	-xz	+zt	-z	-xyt	+yz
+q	+xzt	+yt	+y	+yz	-xyz	-t	+xyzt	+zt	-xyt	+z	+xz	+x	-xt	-yzt	-xy
+q	+xzt	+yt	-y	+xy	-xyz	+t	-xyzt	-xt	+yzt	+x	-xz	+z	+zt	+xyt	-yz
+q	+xzt	+xyt	+yzt	+xyz	-yz	-xz	+xy	+zt	-yt	+xt	+t	+x	-z	-y	-xyzt
+q	+xzt	+xyt	-yzt	+xyzt	-yz	+xz	-xy	-z	+y	+x	-t	+xt	+zt	+yt	-xyz
+q	+xzt	+yzt	-xyt	-xyz	+xy	-xz	+yz	-xt	+yt	+zt	+t	+z	+x	+y	+xyzt
+q	+xzt	+yzt	+xyt	-xyzt	+xy	+xz	-yz	+x	-y	+z	-t	+zt	-xt	-yt	+xyz
+q	+yzt	+x	-xt	-xz	-xyzt	-t	-xyz	-z	-xyt	+zt	+yz	+yt	+y	+xzt	-xy
+q	+yzt	+x	+xt	+xy	-xyzt	+t	+xyz	+y	-xzt	+yt	-yz	+zt	-z	+xyt	+xz
+q	+yzt	+xt	+x	+xz	+xyz	-t	-xyzt	+zt	+xyt	+z	+yz	+y	-yt	-xzt	+xy
+q	+yzt	+xt	-x	-xy	+xyz	+t	+xyzt	-yt	+xzt	+y	-yz	+z	+zt	-xyt	-xz
+q	+yzt	+xyt	-xzt	+xyz	+xz	-yz	+xy	+zt	+xt	+yt	+t	+y	-z	+x	-xyzt
+q	+yzt	+xyt	+xzt	+xyzt	+xz	+yz	-xy	-z	-x	+y	-t	+yt	+zt	-xt	-xyz
+q	+yzt	+xzt	+xyt	+xyz	-xy	-yz	+xz	-yt	+xt	+zt	+t	+z	+y	+x	-xyzt
+q	+yzt	+xzt	-xyt	+xyzt	-xy	+yz	-xz	+y	-x	+z	-t	+zt	-yt	-xt	-xyz

Table 6: Minkowski Geometric Algebra Substitution Table 3/3

72 Mappings

Sorted 72 Mappings From Minkowski Spacetime to Magic Square Layout

```
 +q     +x     +xt    +t
 +xyt   +yt    -y     -xy
 +xzt   +zt    -z     -xz
 +yz    +xyz   +xyzt  +yzt

 +q     +x     +xt    +t
 +xzt   +zt    -z     -xz
 +xyt   +yt    -y     -xy
 -yz    -xyz   -xyzt  -yzt

 +q     +x     +y     +xy
 +xyt   +yt    -xt    -t
 +zt    +xzt   +yzt   +xyzt
 +xyz   +yz    -xz    -z

 +q     +x     +y     +xy
 +zt    +xzt   +yzt   +xyzt
 +xyt   +yt    -xt    -t
 -xyz   -yz    +xz    +z

 +q     +x     +yzt   +xyzt
 +yt    +xyt   -z     -xz
 +zt    +xzt   +y     +xy
 +yz    +xyz   -t     -xt

 +q     +x     +yzt   +xyzt
 +zt    +xzt   +y     +xy
 +yt    +xyt   -z     -xz
 -yz    -xyz   +t     +xt

 +q     +x     +z     +xz
 +xzt   +zt    -xt    -t
 +yt    +xyt   -yzt   -xyzt
 -xyz   -yz    -xy    -y

 +q     +x     +z     +xz
 +yt    +xyt   -yzt   -xyzt
 +xzt   +zt    -xt    -t
 +xyz   +yz    +xy    +y
```

+q	+xt	+x	-t
+xyt	-y	+yt	+xy
+xzt	-z	+zt	+xz
+yz	+xyzt	+xyz	-yzt

+q	+xt	+x	-t
+xzt	-z	+zt	+xz
+xyt	-y	+yt	+xy
-yz	-xyzt	-xyz	+yzt

+q	+xt	+yt	+xy
+xyt	-y	+x	-t
+z	-xzt	-yzt	+xyz
-xyzt	-yz	+xz	+zt

+q	+xt	+yt	+xy
+z	-xzt	-yzt	+xyz
+xyt	-y	+x	-t
+xyzt	+yz	-xz	-zt

+q	+xt	+yzt	-xyz
+y	-xyt	+zt	+xz
+z	-xzt	-yt	-xy
+yz	+xyzt	-t	+x

+q	+xt	+yzt	-xyz
+z	-xzt	-yt	-xy
+y	-xyt	+zt	+xz
-yz	-xyzt	+t	-x

+q	+xt	+zt	+xz
+xzt	-z	+x	-t
+y	-xyt	+yzt	-xyz
+xyzt	+yz	+xy	+yt

+q	+xt	+zt	+xz
+y	-xyt	+yzt	-xyz
+xzt	-z	+x	-t
-xyzt	-yz	-xy	-yt

+q	+xyt	+xzt	+yz
+x	+yt	+zt	+xyz
+xt	-y	-z	+xyzt
+t	-xy	-xz	+yzt

+q	+xyt	+xzt	+yz
+xt	-y	-z	+xyzt
+x	+yt	+zt	+xyz
-t	+xy	+xz	-yzt

+q	+xyt	+yzt	-xz
+yt	+x	-z	+xyzt
+y	-xt	+zt	+xyz
-t	+xy	+yz	+xzt

+q	+xyt	+yzt	-xz
+z	+xyzt	-yt	+x
+zt	-xyz	+y	+xt
+t	-xy	-yz	-xzt

+q	+xyt	+z	-xyzt
+xt	-y	-xzt	-yz
+yt	+x	-yzt	+xz
+xy	-t	+xyz	+zt

+q	+xyt	+z	-xyzt
+yt	+x	-yzt	+xz
+xt	-y	-xzt	-yz
-xy	+t	-xyz	-zt

+q	+xyt	+zt	+xyz
+x	+yt	+xzt	+yz
+y	-xt	+yzt	-xz
+xy	-t	+xyzt	-z

+q	+xyt	+zt	+xyz
+y	-xt	+yzt	-xz
+x	+yt	+xzt	+yz
-xy	+t	-xyzt	+z

+q	+xzt	+xyt	-yz
+x	+zt	+yt	-xyz
+xt	-z	-y	-xyzt
+t	-xz	-xy	-yzt

+q	+xzt	+xyt	-yz
+xt	-z	-y	-xyzt
+x	+zt	+yt	-xyz
-t	+xz	+xy	+yzt

```
+q      +xzt    +y      +xyzt
+xt     -z      -xyt    +yz
+zt     +x      +yzt    +xy
+xz     -t      -xyz    +yt

+q      +xzt    +y      +xyzt
+zt     +x      +yzt    +xy
+xt     -z      -xyt    +yz
-xz     +t      +xyz    -yt

+q      +xzt    +yt     -xyz
+x      +zt     +xyt    -yz
+z      -xt     -yzt    -xy
+xz     -t      -xyzt   -y

+q      +xzt    +yt     -xyz
+z      -xt     -yzt    -xy
+x      +zt     +xyt    -yz
-xz     +t      +xyzt   +y

+q      +xzt    +yzt    +xy
+z      -xt     -yt     +xyz
+zt     +x      +y      +xyzt
+t      -xz     -yz     +xyt

+q      +xzt    +yzt    +xy
+zt     +x      +y      +xyzt
+z      -xt     -yt     +xyz
-t      +xz     +yz     -xyt

+q      +y      +x      -xy
+xyt    -xt     +yt     +t
+zt     +yzt    +xzt    -xyzt
+xyz    -xz     +yz     +z

+q      +y      +x      -xy
+zt     +yzt    +xzt    -xyzt
+xyt    -xt     +yt     +t
-xyz    +xz     -yz     -z

+q      +y      +xzt    -xyzt
+xt     -xyt    -z      -yz
+zt     +yzt    +x      -xy
+xz     -xyz    -t      -yt
```

+q	+y	+xzt	-xyzt
+zt	+yzt	+x	-xy
+xt	-xyt	-z	-yz
-xz	+xyz	+t	+yt

+q	+y	+yt	+t
+yzt	+zt	-z	-yz
+xyt	-xt	+x	-xy
+xz	-xyz	-xyzt	+xzt

+q	+y	+z	+yz
+xt	-xyt	-xzt	+xyzt
+yzt	+zt	-yt	-t
-xyz	+xz	-xy	+x

+q	+y	+z	+yz
+yzt	+zt	-yt	-t
+xt	-xyt	-xzt	+xyzt
+xyz	-xz	+xy	-x

+q	+yt	+xt	-xy
+xyt	+x	-y	+t
+z	-yzt	-xzt	-xyz
-xyzt	+xz	-yz	-zt

+q	+yt	+xt	-xy
+z	-yzt	-xzt	-xyz
+xyt	+x	-y	+t
+xyzt	-xz	+yz	+zt

+q	+yt	+xzt	+xyz
+x	+xyt	+zt	+yz
+z	-yzt	-xt	+xy
+xz	-xyzt	-t	+y

+q	+yt	+xzt	+xyz
+z	-yzt	-xt	+xy
+x	+xyt	+zt	+yz
-xz	+xyzt	+t	-y

+q	+yt	+y	-t
+xyt	+x	-xt	+xy
+yzt	-z	+zt	+yz
-xz	+xyzt	+xyz	+xzt

```
+q     +yt    +y     -t
+yzt   -z     +zt    +yz
+xyt   +x     -xt    +xy
+xz    -xyzt  -xyz   -xzt

+q     +yt    +zt    +yz
+x     +xyt   +xzt   +xyz
+yzt   -z     +y     -t
+xyzt  -xz    +xy    -xt

+q     +yt    +zt    +yz
+yzt   -z     +y     -t
+x     +xyt   +xzt   +xyz
-xyzt  +xz    -xy    +xt

+q     +yzt   +x     -xyzt
+yt    -z     +xyt   +xz
+zt    +y     +xzt   -xy
+yz    -t     +xyz   +xt

+q     +yzt   +x     -xyzt
+zt    +y     +xzt   -xy
+yt    -z     +xyt   +xz
-yz    +t     -xyz   -xt

+q     +yzt   +xt    +xyz
+y     +zt    -xyt   -xz
+z     -yt    -xzt   +xy
+yz    -t     +xyzt  -x

+q     +yzt   +xt    +xyz
+z     -yt    -xzt   +xy
+y     +zt    -xyt   -xz
-yz    +t     -xyzt  +x

+q     +yzt   +xyt   +xz
+y     +zt    -xt    -xyz
+yt    -z     +x     -xyzt
+t     -yz    -xy    +xzt

+q     +yzt   +xyt   +xz
+yt    -z     +x     -xyzt
+y     +zt    -xt    -xyz
-t     +yz    +xy    -xzt
```

+q	+yzt	+xzt	-xy
+z	-yt	-xt	-xyz
+zt	+y	+x	-xyzt
+t	-yz	-xz	-xyt

+q	+yzt	+xzt	-xy
+zt	+y	+x	-xyzt
+z	-yt	-xt	-xyz
-t	+yz	+xz	+xyt

+q	+z	+x	-xz
+xzt	-xt	+zt	+t
+yt	-yzt	+xyt	+xyzt
-xyz	-xy	-yz	+y

+q	+z	+x	-xz
+yt	-yzt	+xyt	+xyzt
+xzt	-xt	+zt	+t
+xyz	+xy	+yz	-y

+q	+z	+xyt	+xyzt
+xt	-xzt	-y	+yz
+yt	-yzt	+x	-xz
+xy	+xyz	-t	-zt

+q	+z	+xyt	+xyzt
+yt	-yzt	+x	-xz
+xt	-xzt	-y	+yz
-xy	-xyz	+t	+zt

+q	+z	+y	-yz
+xt	-xzt	-xyt	-xyzt
+yzt	-yt	+zt	+t
-xyz	-xy	+xz	-x

+q	+z	+y	-yz
+yzt	-yt	+zt	+t
+xt	-xzt	-xyt	-xyzt
+xyz	+xy	-xz	+x

+q	+z	+zt	+t
+xyt	-xyzt	+xyz	-xy
+yzt	-yt	+y	-yz
-xz	-x	-xt	-xzt

+q	+z	+zt	+t
+xzt	-xt	+x	-xz
+yzt	-yt	+y	-yz
+xy	+xyz	+xyzt	+xyt

+q	+z	+zt	+t
+yzt	-yt	+y	-yz
+xzt	-xt	+x	-xz
-xy	-xyz	-xyzt	-xyt

+q	+zt	+xt	-xz
+xzt	+x	-z	+t
+y	+yzt	-xyt	+xyz
+xyzt	+xy	+yz	-yt

+q	+zt	+xt	-xz
+y	+yzt	-xyt	+xyz
+xzt	+x	-z	+t
-xyzt	-xy	-yz	+yt

+q	+zt	+xyt	-xyz
+x	+xzt	+yt	-yz
+y	+yzt	-xt	+xz
+xy	+xyzt	-t	+z

+q	+zt	+xyt	-xyz
+y	+yzt	-xt	+xz
+x	+xzt	+yt	-yz
-xy	-xyzt	+t	-z

+q	+zt	+yt	-yz
+x	+xzt	+xyt	-xyz
+yzt	+y	-z	+t
+xyzt	+xy	-xz	+xt

+q	+zt	+yt	-yz
+yzt	+y	-z	+t
+x	+xzt	+xyt	-xyz
-xyzt	-xy	+xz	-xt

+q	+zt	+z	-t
+xzt	+x	-xt	+xz
+yzt	+y	-yt	+yz
+xy	+xyzt	+xyz	-xyt

```
+q     +zt    +z     -t
+yzt   +y     -yt    +yz
+xzt   +x     -xt    +xz
-xy    -xyzt  -xyz   +xyt
```

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