# Electromagnetism and the Lorentz Field

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August 23, 2015

**Abstract**

In the geometric algebra presentation of classical electromagnetics, we have a multivector whose vector portion is the electric field, whose bivector portion is the magnetic field, and whose scalar portion is a measure of deviation from the Lorentz gauge. In this note, I propose viewing this Lorentz field $\Phi$ as of equal status as the electric and magnetic fields. I provide wave equations for **E**, **B** and $\Phi$, and point out the coalescent nature of the Lorentz field. This presentation is not done in geometric algebra, but rather conventional vector notation to allow easy comparison to standard texts, such as Griffiths [1].

## Maxwell Equations

The generic reference for this section is Griffiths, Introduction to Electrodynamics (1981) Chapter 7, section 7.4.

The Maxwell equations are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon} \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3}$$

$$\nabla \times \mathbf{B} = \mu\epsilon\frac{\partial \mathbf{E}}{\partial t} + \mu\mathbf{J} \tag{4}$$

Using a potential formulation, we define

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

In this work, I take the scalar portion of the electromagnetic multivector, which measures Lorentz gauge violation, and promote this to a scalar field similar to $\mathbf{B}$ and $\mathbf{E}$.

$$\Phi = \nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial \phi}{\partial t} \quad \text{Lorentz Field}$$

We will call this the Lorentz field. We deny the freedom of gauge transformations here. Instead, we say that there are approximation regimes, such as the Lorentz gauge, where $\Phi \to 0$, and the Coulomb gauge, where $\nabla \cdot \mathbf{A} \to 0$, but, in general, we leave the Lorentz terms intact.

So, our three fields are

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} && (5) \\ \mathbf{E} &= -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} && (6) \\ \Phi &= \nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial \phi}{\partial t} && (7) \end{aligned}$$

From the Lorentz field, we can isolate the divergence of $\mathbf{A}$.

$$\nabla \cdot \mathbf{A} = \Phi - \mu\epsilon \frac{\partial \phi}{\partial t} \tag{8}$$

We now do our gradient of the divergence of $\mathbf{E}$ to get the wave equation for $\phi$. This is equation (7.49) in Griffiths [1].

$$\nabla \cdot \mathbf{E} = \nabla \cdot \left( -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \right) = \frac{\rho}{\epsilon}$$

$$\begin{aligned} \nabla^2 \phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) &= -\frac{\rho}{\epsilon} \\ \nabla^2 \phi + \frac{\partial}{\partial t}\left( \Phi - \mu\epsilon \frac{\partial \phi}{\partial t} \right) &= -\frac{\rho}{\epsilon} \end{aligned}$$

$$\nabla^2 \phi - \mu\epsilon \frac{\partial^2 \phi}{\partial t^2} = -\left( \frac{\partial \Phi}{\partial t} + \frac{\rho}{\epsilon} \right) \tag{9}$$

There is nothing new here. This is just a repackaging of a standard equation in a format suggestive of the wave equation, interpreting the Lorentz terms as an additional, dynamic source.

We do the same exercise for the magnetic potential. This is Griffiths' equation (7.50).

$$
\begin{aligned}
\left( \nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) - \nabla \left( \nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial \phi}{\partial t} \right) &= -\mu \mathbf{J} \\
\left( \nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) - \nabla \Phi &= -\mu \mathbf{J}
\end{aligned}
$$

$$
\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = \nabla \Phi - \mu \mathbf{J} \tag{10}
$$

Again, no new physics here, simply a change of notation. This definitely tidies the ugly equations, and provides some insight on the nature of the source terms.

We've done wave equations for $\mathbf{E}$ and $\mathbf{B}$. How about the same for $\Phi$?

$$
\begin{aligned}
\Phi &= \nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial \phi}{\partial t} \\
\nabla^2 \Phi &= \nabla \cdot (\nabla(\nabla \cdot \mathbf{A})) + \mu\epsilon \frac{\partial}{\partial t} \nabla^2 \phi \\
&= \nabla \cdot (\nabla \times (\nabla \times \mathbf{A}) + \nabla^2 \mathbf{A}) + \mu\epsilon \frac{\partial}{\partial t} \nabla^2 \phi \\
&= \nabla \cdot (\nabla^2 \mathbf{A}) + \mu\epsilon \frac{\partial}{\partial t} \nabla^2 \phi
\end{aligned}
$$

The divergence of a curl being zero was used to eliminate the first term. Continuing, we substitute for $\nabla^2 \phi$

$$
\begin{aligned}
\nabla^2 \phi &= \mu\epsilon \frac{\partial^2 \phi}{\partial t^2} - \left( \frac{\partial \Phi}{\partial t} + \frac{\rho}{\epsilon} \right) \\
\nabla^2 \Phi &= \nabla \cdot (\nabla^2 \mathbf{A}) + \mu\epsilon \frac{\partial}{\partial t} \nabla^2 \phi \\
&= \nabla \cdot (\nabla^2 \mathbf{A}) + \mu\epsilon \frac{\partial}{\partial t} \left( \mu\epsilon \frac{\partial^2 \phi}{\partial t^2} - \left( \frac{\partial \Phi}{\partial t} + \frac{\rho}{\epsilon} \right) \right) \\
&= \nabla \cdot (\nabla^2 \mathbf{A}) + (\mu\epsilon)^2 \frac{\partial^3 \phi}{\partial t^3} - \mu\epsilon \frac{\partial^2 \Phi}{\partial t^2} - \mu \frac{\partial \rho}{\partial t} \\
\nabla^2 \Phi + \mu\epsilon \frac{\partial^2 \Phi}{\partial t^2} &= \nabla \cdot (\nabla^2 \mathbf{A}) + (\mu\epsilon)^2 \frac{\partial^3 \phi}{\partial t^3} - \mu \frac{\partial \rho}{\partial t}
\end{aligned}
$$

Using the continuity of charge equation, we can rewrite this as

$$\begin{aligned}
\mu(\nabla \cdot \mathbf{J}) &= -\mu\frac{\partial \rho}{\partial t} \\
\nabla^2\Phi + \mu\epsilon\frac{\partial^2\Phi}{\partial t^2} &= \nabla \cdot (\nabla^2\mathbf{A}) + (\mu\epsilon)^2\frac{\partial^3\phi}{\partial t^3} - \mu\frac{\partial \rho}{\partial t} \\
\nabla^2\Phi + \mu\epsilon\frac{\partial^2\Phi}{\partial t^2} &= \nabla \cdot (\nabla^2\mathbf{A}) + (\mu\epsilon)^2\frac{\partial^3\phi}{\partial t^3} + \mu(\nabla \cdot \mathbf{J})
\end{aligned}$$

We see a significant difference in the propagation of the Lorentz field. Instead of diffusing, like **E** and **B**, it is coalescing. This is significant, and our historical forcing the Lorentz field to zero, or whatever was convenient for calculations, led us to miss this behavior.

Summarizing,

$$\begin{aligned}
\mathbf{B} &= \nabla \times \mathbf{A} \\
\mathbf{E} &= -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \\
\Phi &= \nabla \cdot \mathbf{A} + \mu\epsilon\frac{\partial \phi}{\partial t} \\
\nabla^2\phi - \mu\epsilon\frac{\partial^2\phi}{\partial t^2} &= -\left(\frac{\partial \Phi}{\partial t} + \frac{\rho}{\epsilon}\right) \\
\nabla^2\mathbf{A} - \mu\epsilon\frac{\partial^2\mathbf{A}}{\partial t^2} &= \nabla\Phi - \mu\mathbf{J} \\
\nabla^2\Phi + \mu\epsilon\frac{\partial^2\Phi}{\partial t^2} &= \nabla \cdot (\nabla^2\mathbf{A}) + (\mu\epsilon)^2\frac{\partial^3\phi}{\partial t^3} + \mu(\nabla \cdot \mathbf{J})
\end{aligned}$$

# References


[1] David J. Griffiths, *Introduction to Electrodynamics* 1981 Prentice Hall, ISBN 0-13-481374-X