

A NEW SERIES for the RECTIFICATION of
the ELLIPSIS; together with some
OBSERVATIONS on the EVOLUTION of the
FORMULA $(a^2 + b^2 - 2ab \cos \phi)^n$

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October 20, 1796

Abstract

This is a slightly modified transcription of Transactions of the Royal Society of Edinburgh, volume 4 (1796) pp 177-190. I (Kurt Nalty) have updated symbols for π , e and i , as well as changing differentials to Liebnitz 'd' form rather than the Newton dotted form. The digitized scan from which this was transcribed is courtesy of Google.

Cover Letter to Mr. John Playfair

DEAR SIR,

Having, as you know, bestowed a good deal of time and attention to the study of that part of physical astronomy which relates to the mutual disturbances of the planets, I have, naturally, been led to consider the various methods of resolving the formula $(a^2 + b^2 - 2ab \cos \phi)^n$ into infinite series of the form $A + B \cos(\phi) + C \cos(2\phi) + \&c$. In the course of these investigations, a series for the rectification of the ellipsis occurred to me, remarkable for its simplicity, as well as its rapid convergency. As I believe it to be new, I send it to you, enclosed, together with some remarks on the evolution of the formula just mentioned, which, if you think proper, you may submit to the consideration of the Royal Society.

JAMES IVORY.

[Read Nov. 8, 1796]

MAIN PAPER

LET ε denote the eccentricity of an ellipse, of which the semi-transverse axis is unity, and π the length of the semicircle, radius being unity: Then, if we put

$$e = \frac{1 - \sqrt{1 - \varepsilon^2}}{1 + \sqrt{1 - \varepsilon^2}}$$

half the periphery of the ellipsis will be

$$= \frac{\pi}{1 + e} \left(1 + \frac{1^2}{2^2} e^2 + \frac{1^2 \cdot 1^2}{2^2 \cdot 4^2} e^4 + \frac{1^2 \cdot 1^2 \cdot 3^2}{2^2 \cdot 4^2 \cdot 6^2} e^6 + \frac{1^2 \cdot 1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} e^8 + \&c. \right),$$

the coefficients being the squares of the coefficients of the radical $\sqrt{1 - \varepsilon^2}$.

THE common series, is

$$\pi \times \left(1 - \frac{1}{2} \cdot \frac{1}{2} \varepsilon^2 - \frac{1 \cdot 1}{2 \cdot 4} \cdot \frac{1 \cdot 3}{2 \cdot 4} \varepsilon^4 - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \varepsilon^6 - \&c. \right).$$

THE first of these series converges faster than the other on two accounts: first, because the coefficients decrease more rapidly; and, next, because e is very small in comparison of ε , even when ε is great; Thus, if ε be $\frac{4}{5}$, e will be $\frac{1}{4}$, and $e^2 = \frac{1}{16}$.

IN order to point out the way in which the preceding series was discovered, let us suppose $(a^2 + b^2 - 2ab \cos \phi)^n = A + B \cos(\phi) + C \cos(2\phi) + \&c$; and to determine the coefficients, $A, B, C, \&c$. Let us, with M. DE LA GRANGE, consider the quantity $(a^2 + b^2 - 2ab \cos \phi)$ as the product of two imaginary expressions $(a - be^{i\phi})$, and $(a - be^{-i\phi})$, where e denotes the number whose hyperbolic logarithm is unity. Then, by expanding the powers $(a - be^{i\phi})^n$, and $(a - be^{-i\phi})^n$ into the series $a^n \left(1 - \alpha \cdot \frac{b}{a} e^{i\phi} + \beta \cdot \frac{b^2}{a^2} e^{i2\phi} - \gamma \cdot \frac{b^3}{a^3} e^{i3\phi} + \&c. \right)$ and $a^n \left(1 - \alpha \cdot \frac{b}{a} e^{-i\phi} + \beta \cdot \frac{b^2}{a^2} e^{-i2\phi} - \gamma \cdot \frac{b^3}{a^3} e^{-i3\phi} + \&c. \right)$, we have $\alpha = n$, $\beta = \frac{n \cdot (n-1)}{1 \cdot 2}$, $\gamma = \frac{n \cdot (n-1) \cdot (n-2)}{1 \cdot 2 \cdot 3} \&c$.

THEN multiplying these two series together, and putting $2 \cos(m\phi)$ for its imaginary value $e^{im\phi} + e^{-im\phi}$, we shall find, on equating the terms,

$$A = a^{2n} \times \left(1 + \alpha^2 \cdot \frac{b^2}{a^2} + \beta^2 \cdot \frac{b^4}{a^4} + \gamma^2 \cdot \frac{b^6}{a^6} + \&c. \right),$$

$$B = -2a^{2n} \times \left(\alpha \cdot \frac{b}{a} + \alpha\beta \cdot \frac{b^3}{a^3} + \beta\gamma \cdot \frac{b^5}{a^5} + \&c. \right),$$

and so on.

OF the several series for A, B, C &c. the first deserves particular attention, on account of the simplicity of the law of its terms. It deserves the more attention, too, that the whole fluent $\int_0^\pi (a^2 + b^2 - 2ab \cos(\phi))^n d\phi$, generated while ϕ from 0 becomes π , half the circumference of the circle, is $A\pi$: all the other terms of the fluent then vanishing.

SUPPOSE now, in an ellipsis, the semi-transverse = 1, the eccentricity = ε , and ϕ an arch of the circumscribing circle, reckoned from the extremity of the transverse: then the fluxion of the correspondent arch of the ellipsis, cut off by the same ordinate, will be = $d\phi \sqrt{1 - \varepsilon^2 \cos^2 \phi}$.

IN this expression, I write $\frac{1}{2} + \frac{1}{2} \cos(2\phi)$, for $\cos^2(\phi)$: and put the result, $d\phi \sqrt{1 - \frac{\varepsilon^2}{2} - \frac{\varepsilon^2}{2} \cos(2\phi)} = d\phi \sqrt{a^2 + b^2 - 2ab \cos(2\phi)}$, a and b being indeterminate quantities.

TO determine a and b , we have $a^2 + b^2 = 1 - \frac{\varepsilon^2}{2}$, and $2ab = \frac{\varepsilon^2}{2}$: whence $a + b = 1$, and $a - b = \sqrt{1 - \varepsilon^2}$ so that $a = \frac{1 + \sqrt{1 - \varepsilon^2}}{2}$ and $b = \frac{1 - \sqrt{1 - \varepsilon^2}}{2}$.

I THUS obtain $d\phi \sqrt{1 - \varepsilon^2 \cos^2(\phi)} = d\phi \sqrt{a^2 + b^2 - 2ab \cos(2\phi)}$: and, taking the whole fluent, while ϕ from 0 becomes π , it is manifest, from what has been premised, that the semiperiphery of the ellipsis is =

$$\pi \times \alpha \times \left(1 + \frac{1}{2^2} \cdot \frac{b^2}{a^2} + \frac{1^2 \cdot 1^2}{2^2 \cdot 4^2} \cdot \frac{b^4}{a^4} + \frac{1^2 \cdot 1^2 \cdot 3^2}{2^2 \cdot 4^2 \cdot 6^2} \cdot \frac{b^6}{a^6} + \&c. \right),$$

or putting $\frac{b}{a} = e = \frac{1 - \sqrt{1 - \varepsilon^2}}{1 + \sqrt{1 - \varepsilon^2}}$, and $\alpha = \frac{a}{a+b} = \frac{1}{1+b/a} = \frac{1}{1+e}$, the semiperimeter of the ellipsis

$$= \frac{\pi}{1+e} \times \left(1 + \frac{1^2}{2^2} e^2 + \frac{1^2 \cdot 1^2}{2^2 \cdot 4^2} e^4 + \frac{1^2 \cdot 1^2 \cdot 3^2}{2^2 \cdot 4^2 \cdot 6^2} e^6 + \&c. \right)$$

IN this series, as was before observed, e is a small fraction even when ε is very considerable, and the coefficients are more simple in the law of the progression, and converge faster, (especially in the first terms), than in the common series.

IF we suppose the ellipsis to be infinitely flattened, in which case $\varepsilon = 1$ and $e = 1$, and the semiperiphery = 2, this series gives

$$2 = \frac{\pi}{2} \times \left(1 + \frac{1^2}{2^2} + \frac{1^2 \cdot 1^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 1^2 \cdot 3^2}{2^2 \cdot 4^2 \cdot 6^2} + \&c. \right)$$

and so,

$$\frac{4}{\pi} = 1 + \frac{1^2}{2^2} + \frac{1^2 \cdot 1^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 1^2 \cdot 3^2}{2^2 \cdot 4^2 \cdot 6^2} + \frac{1^2 \cdot 1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} + \&c.$$

BUT, we may remark, that as we have here obtained the sum of the squares of the coefficients of the binomial when the exponent is $\frac{1}{2}$; so, from the same source, we may determine the sum of the squares of the coefficients corresponding to any other exponent, at least by a fluent.

FOR taking the whole fluent when $\phi = \pi$, we have

$$\int_0^\pi (a^2 + b^2 - 2ab \cos(\phi))^n d\phi = a^{2n} \pi \left(1 + \alpha^2 \cdot \frac{b^2}{a^2} + \beta^2 \cdot \frac{b^4}{a^4} + \gamma^2 \cdot \frac{b^6}{a^6} + \&c. \right)$$

and so when $a = 1$, and $b = 1$,

$$\int_0^\pi \frac{(a^2 + b^2 - 2ab \cos(\phi))^n}{\pi} d\phi = 1 + \alpha^2 + \beta^2 + \gamma^2 + \&c.$$

NOW, when $a = 1$, and $b = 1$,

$$\int_0^\pi (a^2 + b^2 - 2ab \cos(\phi))^n d\phi = 2^{2n} \times \int_0^\pi \left(\sin \left(\frac{\phi}{2} \right) \right)^{2n} d\phi,$$

because $2 \left(\sin \left(\frac{\phi}{2} \right) \right)^2 = 1 - \cos(\phi)$: we thus obtain

$$\frac{2^{2n} \times \int_0^\pi \left(\sin \frac{\phi}{2} \right)^{2n} d\phi}{\pi} = 1 + \alpha^2 + \beta^2 + \gamma^2 + \&c.$$

the whole fluent to be taken when $\phi = \pi$, or $\frac{\phi}{2} = \frac{\pi}{2}$.

IF we put $x = \sin \frac{\phi}{2}$, we shall have

$$\frac{2^{2n} \times \int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}} dx}{\pi/2} = 1 + \alpha^2 + \beta^2 + \gamma^2 + \&c,$$

the whole fluent to be taken when $x = 1$; and in this formula n is any number, fractional or integral, positive or negative; and $\alpha, \beta, \gamma, \&c.$ the coefficients of the binomial raised to a power of which the exponent is n .

WHEN n is a whole positive number,

$$\int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \cdot \frac{\pi}{2}, \quad \text{in the case when } x = 1 :$$

And so, $2^{2n} \times \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} = 1 + \alpha^2 + \beta^2 + \gamma^2 + \&c.$

Now, $2^{2n} \times \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}$ is no other than the coefficient of the middle term of a binomial, raised to the power expressed by $2n$: Hence we a very curious property of these numbers: viz. *that the sum of the squares of the coefficients of a binomial, the exponent being n , is equal to the the coefficient of the middle term of a binomial of which the exponent is $2n$.*

ANOTHER remark, which I have to offer on this subject, may be considered not only as curious, in an analytical point of view, but as, in some measure, accomplishing an object that has much engaged the attention of mathematicians.

IN the computation of the planetary disturbances, it becomes necessary to evolve the fraction $(a^2 + b^2 - 2ab \cos \phi)^{-3/2}$ into a series of this form, $A + B \cos(\phi) + C \cos(2\phi) + \&c.$ The quantities a and b represent the distances of the disturbing planets from the sun; and when these bear so great a proportion to one another, (as in the case of Jupiter and Saturn, or Venus and the Earth), that the fraction $\frac{b}{a}$ is large, it becomes extremely difficult to compute the coefficients $A, B, \&c.$ by series, on account of the great number of terms that must be taken in. This matter not a little perplexed the first geometers who considered this subject, and they were obliged to approximate to the quantities sought by the method of quadratures, and by other artifices.

TWO things are to be attended to with regard to the quantities A, B, C &c. The first is, That it is not necessary to compute all of them separately by series, or by other methods: They form a recurring series; and the two first being so computed, all the rest may be derived from them. The second thing is, That the quantities A and B having been computed for any exponent n , the corresponding quantities are thence derived, by easy formulæ, for the exponents $n+1, n+2; n-1, n-2$; and in general for the exponent $n+m, m$ being any integer number, positive or negative.

FROM these remarks, it follows, that the whole difficulty lies in the computation of the first quantities A and B; and that we are not confined to a given exponent n , but may choose any one in the series, $n+1, n+2, \&c.; n-1, n-2 \&c.$; that will render the computation most easy and expeditious.

THUS, in order to compute the quantities A and B, for the exponent $-\frac{3}{2}$, M. DE LA GRANGE makes choice of exponent $+\frac{1}{2}$, which, in the whole series of exponents $+\frac{3}{2}, +\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \&c.$ is the most favourable for computation, on account of the convergency of the coefficients of the series for A and B.

IN considering these subject, however, I have fallen upon a method of computing the quantities A and B for the exponent $-\frac{1}{2}$ by series that converges so fast, that, even taking the most unfavourable case that occurs in the theory of the planets, two or three terms give the values required with a sufficient degree of exactness. This is what I am now to communicate.

WE are then to consider the expression

$$(a^2 + b^2 - 2ab \cos \phi)^{-1/2} = \frac{1}{\sqrt{a^2 + b^2 - 2ab \cos \phi}} :$$

for the sake of simplicity in calculation, I write $\frac{b}{a} = c$, throwing out a altogether; and I suppose

$$\frac{1}{\sqrt{1 + c^2 - c \cos \phi}} = A + B \cos(\phi) + C \cos(2\phi) + \&c.$$

LET ψ be an angle, so related to ϕ , that $\sin(\psi - \phi) = c \sin(\psi)$: It is obvious, from this formula, that $\psi = \phi$ when $\sin(\psi) = 0$, that is, when ψ is equal to 0, or to π , 2π , &c.

WE have then, $\cos(\psi - \phi) = \sqrt{1 - c^2 \sin^2(\psi)}$: and taking the fluxions,

$$d\psi - d\phi = \frac{c \cos(\psi) \times d\psi}{\cos(\psi - \phi)} = \frac{c \cos(\psi) \times d\psi}{\sqrt{1 - c^2 \sin^2 \psi}} :$$

$$\text{whence } d\phi = d\psi \times \frac{\sqrt{1 - c^2 \sin^2(\psi)} - c \cos(\psi)}{\sqrt{1 - c^2 \sin^2 \psi}}$$

BUT

$$\begin{aligned} \left(\sqrt{1 - c^2 \sin^2 \psi} - c \cos \psi \right)^2 &= 1 - c^2 \sin^2 \psi + c^2 \cos^2 \psi - 2c \cos \psi \sqrt{1 - c^2 \sin^2 \psi} \\ &= 1 + c^2 - 2c^2 \sin^2 \psi - 2c \cos \psi \sqrt{1 - c^2 \sin^2 \psi} , \\ &\quad (\text{because } c^2 \cos^2 \psi = c^2 - c^2 \sin^2 \psi) \\ &= 1 + c^2 - 2c \times \left(c \sin \psi \times \sin \psi + \cos \psi \sqrt{1 - c^2 \sin^2 \psi} \right) . \end{aligned}$$

Now, if we write for $c \sin \psi$ its equal, $\sin(\psi - \phi)$, and for $\sqrt{1 - c^2 \sin^2 \psi}$ its equal, $\cos(\psi - \phi)$, we shall have

$$\begin{aligned} c \sin \psi \times \sin \psi + \cos \psi \times \sqrt{1 - c^2 \sin^2 \psi} &= \sin(\psi - \phi) \times \sin \psi + \cos \psi \times \cos(\psi - \phi) \\ &= \cos \phi : \end{aligned}$$

which being substituted, there comes out

$$\left(\sqrt{1 - c^2 \sin^2 \psi} - c \cos \psi\right)^2 = 1 + c^2 - 2c \cos \phi.$$

OUR fluxional formula thus becomes

$$d\phi = \frac{\sqrt{1 + c^2 - 2c \cos \phi}}{\sqrt{1 - c^2 \sin^2 \psi}} d\psi \quad : \text{whence}$$

$$\frac{d\phi}{\sqrt{1 + c^2 - 2c \cos \phi}} = \frac{d\psi}{\sqrt{1 - c^2 \sin^2 \psi}}.$$

I NEXT transform the quantity $\sqrt{1 - c^2 \sin^2 \psi}$ as in the investigation for the elliptic series, and putting

$$c' = \frac{1 - \sqrt{1 - c^2}}{1 + \sqrt{1 - c^2}}, \quad \text{I find}$$

$$\sqrt{1 - c^2 \sin^2 \psi} = \frac{\sqrt{1 + c'^2 + 2c' \cos(2\psi)}}{1 + c'}, \quad \text{and so}$$

$$\frac{d\phi}{\sqrt{1 + c^2 - 2c \cos \psi}} = \frac{(1 + c')d\psi}{\sqrt{1 + c'^2 + 2c' \cos(2\psi)}}$$

Now, taking the fluents when $\phi = \pi$, and $\psi = \pi$, we shall have

$$\int_0^\pi \frac{d\phi}{\sqrt{1 + c^2 - 2c \cos \phi}} = A \times \pi \quad :$$

And according to the method of M. DE LA GRANGE,

$$\int_0^\pi \frac{d\psi}{\sqrt{1 + c'^2 + 2c' \cos(2\psi)}} = \pi \times \left(1 + \frac{1^2}{2^2} c'^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} c'^4 + \&c.\right) \quad :$$

Hence

$$A = (1 + c') \times \left(1 + \frac{1^2}{2^2} c'^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} c'^4 + \&c.\right).$$

And in this value of A , c' will be a small fraction, even though c be large; and the series will therefore converge very fast.

BUT, taking the value of A directly in a series, we have

$$A = 1 + \frac{1^2}{2^2}c^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}c^4 + \&c.$$

And so,

$$1 + \frac{1^2}{2^2}c^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}c^4 + \&c. = (1 + c') \times \left(1 + \frac{1^2}{2^2}c'^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}c'^4 + \&c. \right)$$

Now, if two series being exactly alike, it is evident that we may transform the one, as we have transformed the other, and that, if we put

$$c'' = \frac{1 - \sqrt{1 - c'^2}}{1 + \sqrt{1 - c'^2}}, \text{ we shall have}$$

$$1 + \frac{1^2}{2^2}c'^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}c'^4 + \&c. = (1 + c'') \times \left(1 + \frac{1^2}{2^2}c''^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}c''^4 + \&c. \right) : \text{ whence}$$

$$A = (1 + c')(1 + c'') \left(1 + \frac{1^2}{2^2}c''^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}c''^4 + \&c. \right)$$

IT is manifest we may proceed in this manner as far as we please, and that, if we put

$$c''' = \frac{1 - \sqrt{1 - c''^2}}{1 + \sqrt{1 - c''^2}} ;$$

$$c'''' = \frac{1 - \sqrt{1 - c'''^2}}{1 + \sqrt{1 - c'''^2}}$$

and so on, we shall have the value of A in an infinite product,

$$A = (1 + c') \times (1 + c'') \times (1 + c''') \times (1 + c''') \times \&c,$$

the quantities c' , c'' , c''' , &c. converging very rapidly.

NOTHING more seems to be wished for, with regard to the computation of the quantity A : since we can, by methods sufficiently simple, exhibit the

value of it in series that shall converge as fast as we please. By a similar mode of reasoning, I find the series

$$1 - \frac{1^2}{2^2}\gamma^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}\gamma^4 - \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}\gamma^6 + \&c.$$

(which occurs in determining the time of a body's descent in the arch of a circle),

$$= (1 - c) \times \left(1 + \frac{1^2}{2^2}c^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}c^4 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}c^6 + \&c. \right)$$

where

$$c = \frac{\sqrt{1 + \gamma^2} - 1}{\sqrt{1 + \gamma^2} + 1} :$$

so that the summation of this series also is accomplished by the method above.

I HAVE now only to explain the method of computing B . For this purpose I resume

$$\frac{1}{\sqrt{1 + c^2 - 2c \cos \phi}} = A + B \cos(\phi) + C \cos(2\phi) + \&c.$$

Multiply by $2 \cos(\phi)$, and there results

$$\frac{2 \cos(\phi)}{\sqrt{1 + c^2 - 2c \cos \phi}} = B + (2A + C) \cos(\phi) + \&c.$$

whence it is manifest that the whole fluent

$$\int_0^\pi \frac{2 \cos \phi}{\sqrt{1 + c^2 - 2c \cos \phi}} d\phi = \pi B$$

when $\phi = \pi$, is equal to $B \times \pi$.

FROM the preceding investigation we have

$$\frac{d\phi}{\sqrt{1 + c^2 - 2c \cos \phi}} = \frac{d\psi}{\sqrt{1 - c^2 \sin^2 \psi}}, \quad \text{and}$$

$$\cos \phi = c \sin^2(\psi) + \cos(\psi) \sqrt{1 - c^2 \sin^2(\psi)}$$

whence

$$\frac{2 \cos \phi}{\sqrt{1 + c^2 - 2c \cos \phi}} d\phi = \frac{2c \sin^2 \psi}{\sqrt{1 - c^2 \sin^2 \psi}} d\psi + 2 \cos(\psi) d\psi.$$

Again,

$$2 \sin^2 \psi = 1 - \cos(2\psi), \quad \text{and}$$

$$\frac{1}{\sqrt{1 - c^2 \sin^2 \psi}} = \frac{1 + c'}{\sqrt{1 + c'^2 + 2c' \cos(2\psi)}}$$

with c' being

$$c' = \frac{1 - \sqrt{1 - c^2}}{1 + \sqrt{1 - c^2}}$$

These substitutions being made, we get

$$\begin{aligned} \frac{2 \cos \phi}{\sqrt{1 + c^2 - 2c \cos \phi}} d\phi &= c \times \frac{(1 + c')}{\sqrt{1 + c'^2 + 2c' \cos(2\psi)}} d\psi \\ &\quad - c \times \frac{(1 + c') \cos(2\psi)}{\sqrt{1 + c'^2 + 2c' \cos(2\psi)}} d\psi \\ &\quad + 2 \cos(\psi) d\psi \end{aligned}$$

SUPPOSE now,

$$\frac{1}{\sqrt{1 + c^2 + 2c \cos(2\psi)}} = A' - B' \cos(2\psi) + c' \cos(4\psi) - \&c.$$

it is evident, from what goes before, that, taking the fluents of the above fluxions, when ϕ and $\psi = \pi$, we shall have

$$B \times \pi = c \times (1 + c') \times \left(A' + \frac{B'}{2} \right) \times \pi, \quad \text{and so}$$

$$B = c \times (1 + c') \times \left(A' + \frac{B'}{2} \right)$$

THE values of A' and B' , in series according to the method of M. DE LA GRANGE, are

$$A' = 1 + \frac{1^2}{2^2}c'^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}c'^4 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}c'^6 + \&c.$$

$$\frac{1}{2}B' = \left(\frac{1}{2}c' + \frac{1}{2} \cdot \frac{1 \cdot 3}{2 \cdot 4}c'^3 + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}c'^5 + \&c. \right)$$

which converges very fast, on account of the smallness of c' in respect of c .

IF, however, it is required to find the value of B by series still more converging, we may easily do so : For it is manifest that B and B' are similar functions of c and c' : and that if we make

$$c'' = \frac{1 - \sqrt{1 - c'^2}}{1 + \sqrt{1 - c'^2}}$$

$$c''' = \frac{1 - \sqrt{1 - c''^2}}{1 + \sqrt{1 - c''^2}}$$

and so on, and put A'' , A''' , &c; B'' , B''' , &c. to denote the corresponding values of A' and B' , we shall then have

$$B = c \cdot (1 + c') \left(A' + \frac{B'}{2} \right)$$

$$B' = c' \cdot (1 + c'') \left(A'' + \frac{B''}{2} \right)$$

$$B'' = c'' \cdot (1 + c''') \left(A''' + \frac{B'''}{2} \right) \&c. :$$

Now, remarking that

$$A' = (1 + c'')A'';$$

$$A'' = (1 + c''')A''' \&c.$$

we have the following values of B :

$$B = c \times \left(1 + \frac{c'}{2}\right) \cdot (1 + c') \cdot (1 + c'') A'' + \frac{c}{2} \cdot \frac{c'}{2} (1 + c')(1 + c'') B''.$$

$$B = c \times \left(1 + \frac{c'}{2} + \frac{c'}{2} \cdot \frac{c''}{2}\right) (1 + c') (1 + c'') (1 + c''') A''' \\ + \frac{c}{2} \cdot \frac{c'}{2} \cdot \frac{c''}{2} (1 + c')(1 + c'')(1 + c''') B'''.$$

And we may proceed in this manner to find the value of B in series that shall converge as fast as we please.

As the quantities c' , c'' , c''' , &c. diminish very fast, the series A' , A'' , A''' will approach rapidly to unity, and B' , B'' , B''' will decrease rapidly to nothing : Hence we have ultimately,

$$B = c \times \left(1 + \frac{c'}{2} + \frac{c'}{2} \cdot \frac{c''}{2} + \frac{c'}{2} \cdot \frac{c''}{2} \cdot \frac{c'''}{2} + \&c.\right) \times (1 + c')(1 + c'')(1 + c''')\&c.$$

or, since

$$A = (1 + c')(1 + c'')(1 + c''')\&c. : \\ B = c \times \left(1 + \frac{c'}{2} + \frac{c'}{2} \cdot \frac{c''}{2} + \frac{c'}{2} \cdot \frac{c''}{2} \cdot \frac{c'''}{2} + \&c.\right) \times A.$$

WE shall best see the degree of convergency of the quantities c , c' , c'' , &c. if we take the infinite series by which they are derived one from another. Now, if

$$y = \frac{1 - \sqrt{1 - x^2}}{1 + \sqrt{1 - x^2}}, \text{ then also} \\ y = \frac{x^2}{4} + \frac{x^4}{8} + \frac{5x^6}{64} + \frac{7x^8}{128} + \&c. :$$

whence it is obvious, that in the series of quantities c , c' , c'' , &c. the fourth part of the square of any term is nearly equal to the following term, and the rapidity with which the series decreases is therefore very great.

THE method, then, that results from the preceding investigation for computing A and B , is shortly this :

$$\text{Put } c' = \frac{1 - \sqrt{1 - c^2}}{1 + \sqrt{1 - c^2}} : \text{ and compute}$$

$$1 + \frac{1^2}{2^2}c'^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}c'^4 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}c'^6 + \&c. = M,$$

$$\text{and } \frac{1}{2}c' + \frac{1}{2} \cdot \frac{1 \cdot 3}{2 \cdot 4}c'^3 + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}c'^5 + \&c. = N.$$

$$\begin{aligned} \text{Then } A &= (1 + c') \times M, \text{ and} \\ B &= c \times (1 + c') \times (M + N). \end{aligned}$$

THE series M and N will converge so fast, even in the most unfavourable case that occurs in the theory of the planets, that the first three terms will give the sums sufficiently exact; and it will therefore not be necessary to have recourse to the more converging series A'' and B'' .

SUCH is the method that I had first imagined, for facilitating these sorts of computations. I have since found, however, that by means of the common tables of sines and tangents, the quantities A and B may be computed in a still easier way from the expressions,

$$\begin{aligned} A &= (1 + c')(1 + c'')(1 + c''')\&c. \\ B &= c \times \left(1 + \frac{c'}{2} + \frac{c'}{2} \cdot \frac{c''}{2} + \frac{c'}{2} \cdot \frac{c''}{2} \cdot \frac{c'''}{2} + \&c \right) \times A. \end{aligned}$$

FOR if $c = \sin(m)$, then $\sqrt{1 - c^2} = \cos(m)$, and $c' = \frac{1 - \cos(m)}{1 + \cos(m)} = \tan^2\left(\frac{m}{2}\right)$: consequently $1 + c' = \sec^2\left(\frac{m}{2}\right)$. In like manner, if $c' = \sin(m')$, $c'' = \sin(m'')$, &c. we shall have $\sin(m') = \tan^2\left(\frac{m'}{2}\right)$; $\sin m'' = \tan^2\left(\frac{m''}{2}\right)$, and so on : And $1 + c'' = \sec^2\left(\frac{m'}{2}\right)$; $1 + c''' = \sec^2\left(\frac{m''}{2}\right)$, and so on. Thus :

$$A = \sec^2\left(\frac{m}{2}\right) \times \sec^2\left(\frac{m'}{2}\right) \times \sec^2\left(\frac{m''}{2}\right) \times \&c.$$

TO find the logarithm of A, we have then only to add together the logarithm secants of the angles $\frac{m}{2}$, $\frac{m'}{2}$, $\frac{m''}{2}$, &c. to diminish the sum by as many times the radius as there are secants, and to take twice the remainder. As the angles $\frac{m}{2}$, $\frac{m'}{2}$, $\frac{m''}{2}$, &c. decrease very fast, it will seldom be necessary to compute more than two or three of them.

THE series $\left(1 + \frac{c'}{2} + \frac{c'}{2} \cdot \frac{c''}{2} + \frac{c'}{2} \cdot \frac{c''}{2} \cdot \frac{c'''}{2} + \&c.\right)$ is also readily computed from the tables; because the logarithms of c' , c'' , c''' , &c. being the sines of the angles m' , m'' , &c. are all found in the tables.

As an example, let $c = 0.72333$: which is the fraction that arises from dividing the mean distance of Venus from the sun, by the mean distance of the Earth; and this is the most unfavourable case that occurs in the theory of the planets : Then to compute A, I find, in the table of natural sines, that 0.72333 corresponds to $46^\circ 19' 48\frac{1}{3}''$: we have therefore

$m = 46^\circ 19' 48\frac{1}{3}''$ $\text{L. tan } \frac{m}{2} = \text{l. tan } 23^\circ 9' 54\frac{1}{6}'' = 9.6313206$ <hr style="width: 50%; margin-left: auto; margin-right: 0;"/> $\text{L. sin } m' = 9.2626412$ $m' = 10^\circ 32' 57''$ $\text{L. tan } \frac{m'}{2} = \text{l. tan } 5^\circ 16' 28\frac{1}{2}'' = 8.9652949$ <hr style="width: 50%; margin-left: auto; margin-right: 0;"/> $\text{L. sin } m'' = 7.9395898$ $m'' = 0^\circ 29' 18''$ $\text{L. tan } \frac{m''}{2} = \text{l. tan } 0^\circ 14' 39'' = 7.6295664$ <hr style="width: 50%; margin-left: auto; margin-right: 0;"/> $\text{L. sin } m''' = 5.2591328$		$\text{L. sec } \frac{m}{2} = 10.0365070$ $\text{L. sec } \frac{m'}{2} = 10.0018429$ $\text{L. sec } \frac{m''}{2} = 10.0000039$ <hr style="width: 50%; margin-left: auto; margin-right: 0;"/> 0.0383538 <hr style="width: 50%; margin-left: auto; margin-right: 0;"/> 2 $\text{L. A} = 0.0767076$
<p>As m''' will only be a few seconds, it may be neglected. Hence $A = 1.19318$</p>		

To compute B , let $S = 1 + \frac{c'}{2} + \frac{c'}{2} \cdot \frac{c''}{2} + \&c.$

	$1 = 1.000000$
$\text{L. } c' = \text{l. sin } m' = 9.2626412$	$\frac{c'}{2} = 0.091540$
$\text{L. } c'' = \text{l. sin } m'' = 7.9305898$	$\frac{c'}{2} \cdot \frac{c''}{2} = 0.000390$
$1.c'.c'' = 7.1932310$	$S = 1.091830$

$$B = c \times S \times A.$$

$$\begin{aligned}
L.c &= 1.8593365 \\
L.S &= 0.0381948 \\
L.A &= 0.0767076 \\
\hline
L.B &= 1.9742389, \text{ and } B = 0.942408
\end{aligned}$$

Calculations Repeated Using Modern Logarithms

The logarithms used at the end of the paper have an offset of 10, to eliminate negative signs. This section repeats his calculation using (now) standard base 10 logarithms.

As an example, let $c = 0.72333$: which is the fraction that arises from dividing the mean distance of Venus from the sun, by the mean distance of the Earth; and this is the most unfavourable case that occurs in the theory of the planets : Then to compute A, I find, in the table of natural sines, that 0.72333 corresponds to $46^\circ 19' 48\frac{1}{3}''$: we have therefore

$ \begin{aligned} & m = 46^\circ 19' 48\frac{1}{3}'' \\ \log\left(\tan \frac{m}{2}\right) &= \log\left(\tan 23^\circ 9' 54\frac{1}{6}''\right) = -0.36867934 \\ & \qquad \qquad \qquad \times 2 \\ & \qquad \qquad \qquad \hline & \log(\sin m') = -0.73735868 \\ & m' = 10^\circ 32' 57'' \\ \log\left(\tan \frac{m'}{2}\right) &= \log\left(\tan 5^\circ 16' 28\frac{1}{2}''\right) = -1.034702856 \\ & \qquad \qquad \qquad \times 2 \\ & \qquad \qquad \qquad \hline & \log(\sin m'') = -2.06940571 \\ & m'' = 0^\circ 29' 18'' \\ \log\left(\tan \frac{m''}{2}\right) &= \log(\tan 0^\circ 14' 39'') = -2.3704278 \\ & \qquad \qquad \qquad \times 2 \\ & \qquad \qquad \qquad \hline & \log(\sin m''') = -4.7408556 \end{aligned} $	$ \begin{aligned} \log\left(\sec \frac{m}{2}\right) &= 0.03650709 \\ \log\left(\sec \frac{m'}{2}\right) &= 0.00184291 \\ \log\left(\sec \frac{m''}{2}\right) &= 0.0000039 \\ & \qquad \qquad \qquad \hline & 0.0383538 \\ & \qquad \qquad \qquad \times 2 \\ & \qquad \qquad \qquad \hline \log A &= 0.0767076 \end{aligned} $
<p>As m''' will only be a few seconds, it may be neglected. Hence A = 1.19318</p>	

To compute B , let $S = 1 + \frac{c'}{2} + \frac{c'}{2} \cdot \frac{c''}{2} + \&c.$

	$1 = 1.000000$
$\log(c') = \log(\sin m') = -0.73735868$	$\frac{c'}{2} = 0.091540$
$\log(c'') = \log(\sin m'') = -2.06940571$	$\frac{c'}{2} \cdot \frac{c''}{2} = 0.000390$
$\log(c'.c'') = -2.806764392$	$S = 1.091830$

$$B = c \times S \times A.$$

$\log(c)$	$=$	-0.14066352
$\log(S)$	$=$	0.0381948
$\log(A)$	$=$	0.0767076

$$\log(B) = -0.02576112, \text{ and } B = 0.942408$$

C Code Verification

AS an example of the superiority of Mr. Ivory's formula, I include a small C program calculating the length along the ellipse by both the conventional formula, as well as Mr. Ivory's.

```
/*
```

```
    This program verifies the two series expression for the length
    of the ellipse of James Ivory's 1796 paper.
```

```
*/
```

```
#include <stdio.h>
#include <stdlib.h>
```



```

#include <math.h>

int main(void)
{

    int i;
    double k, m, f1, f2, sum1, sum2, a, e, e2;

    printf("Please enter a : "); // minor radius of unit ellipse

    scanf("%lf",&a);

    k = sqrt(1.0 - a*a); // eccentricity of the ellipse
    m = k*k;

    // Ivory's transform
    e = (1.0 - sqrt(1.0 - k*k))/(1.0 + sqrt(1.0 - k*k));
    e2 = e*e;

    // s1 = 2.0*Legendre_Elliptic_Integral_Second_Kind(M_PI_2, 'k', k);
    // printf("Semi-perimeter for a = %lf via MyMathLib is %lf \n \n",a, s1);

    sum1 = 0.0;
    sum2 = 0.0;
    printf("sum1 = %lf \t sum2 = %lf \n", sum1, sum2);

    sum1 = 1.0*M_PI; // classic formula, as found in AMS 55, p591
    sum2 = ((M_PI)/(1.0+e)); // Ivory's formula, as found in 1796
    printf("sum1 = %lf \t sum2 = %lf \n", sum1, sum2);

    f1 = 1.0;
    f2 = 1.0;

    for (i=0;i<20;i++) {
        f1 *= ((2.0*i + 1.0)/(2.0*i + 2.0))*((2.0*i + 1.0)/(2.0*i + 2.0))*m;
        sum1 -= M_PI*f1/(2.0*i + 1.0);

        f2 *= ((2.0*i + 1.0)/(2.0*i + 2.0))*((2.0*i + 1.0)/(2.0*i + 2.0))*e2;
    }
}

```

```

        sum2 += ((M_PI)/(1.0+e))*f2/((2.0*i + 1.0)*(2.0*i + 1.0));

        printf("sum1 = %lf \t sum2 = %lf \n", sum1, sum2);
    }
    return 0;
}

/*

Please enter a : 0.5
sum1 = 0.000000    sum2 = 0.000000
sum1 = 3.141593    sum2 = 2.356194
sum1 = 2.552544    sum2 = 2.421644
sum1 = 2.469709    sum2 = 2.422099
sum1 = 2.443823    sum2 = 2.422111
sum1 = 2.433206    sum2 = 2.422112
sum1 = 2.428189    sum2 = 2.422112
sum1 = 2.425602    sum2 = 2.422112
sum1 = 2.424187    sum2 = 2.422112
sum1 = 2.423378    sum2 = 2.422112
sum1 = 2.422901    sum2 = 2.422112
sum1 = 2.422612    sum2 = 2.422112
sum1 = 2.422433    sum2 = 2.422112
sum1 = 2.422321    sum2 = 2.422112
sum1 = 2.422249    sum2 = 2.422112
sum1 = 2.422203    sum2 = 2.422112
sum1 = 2.422173    sum2 = 2.422112
sum1 = 2.422153    sum2 = 2.422112
sum1 = 2.422140    sum2 = 2.422112
sum1 = 2.422131    sum2 = 2.422112
sum1 = 2.422125    sum2 = 2.422112
sum1 = 2.422121    sum2 = 2.422112

*/

```