General Mean

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Abstract

As I study elliptic integrals, I naturally encountered the Arithmetic-Geometric mean sequence. The question naturally arises about the utility of this type of sequence with other types of mean. This then leads to a study of the Generalized Mean, which is the subject of this note. Here I present some basic notes, products, and connections between various General Means.

Standard Means

The arithmetic mean is the average.

$$AM = \frac{a_1 + a_2 + ... + a_n}{n}$$

The geometric mean is

$$GeoM = (a_1 a_2 a_3 ... a_n)^{1/n}$$

The harmonic mean is

$$\frac{1}{HM} = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + ... + \frac{1}{a_n}\right)$$

The root mean squared is

$$RMS = \left(\frac{a_1^2 + a_2^2 + ... + a_n^2}{n}\right)^{1/2}$$

General Mean

In AMS55, (Abramowitz and Stegun), we find formula 3.1.14 for the generalized means

$$GM(t) = \left(\frac{1}{n} \sum_{k=1}^{n} a_k^t \right)^{1/t}$$

Specialization Down to Two Arguments

Each of the formulas can be specialized to two arguments.

$$\begin{aligned} AM(a,b) &= \frac{a+b}{2} \\ GeoM(a,b) &= \sqrt{ab} \\ HM(a,b) &= \left(\frac{(1/a)+(1/b)}{2} \right)^{-1} = \frac{2ab}{a+b} \\ RMS(a,b) &= \left(\frac{a^2+b^2}{2} \right)^{1/2} \\ GM(a,b,n) &= \left(\frac{a^n+b^n}{2} \right)^{1/n} \end{aligned}$$

From “An Atlas of Functions” by Keith Oldham, Jan Myland and Jerome Spanier, we find on page 647

$$\begin{aligned} GM(a,b,-1) &= HM(a,b) \\ GM(a,b,0) &= GeoM(a,b) \\ GM(a,b,1) &= AM(a,b) \\ GM(a,b,2) &= RMS(a,b) \end{aligned}$$

The proof that the geometric mean is GM(a,b,0) is given later in this paper.

Properties of the Generic Mean

Define

$$GM(a,b,n) = \left(\frac{a^n+b^n}{2} \right)^{1/n}$$

Sample C code to generate this function is

```
#include <math.h>
double GeneralMean(double a, double b, double n)
{
     if (fabs(n) < 1.0e-10) return sqrt(a*b);
     else return( pow( 0.5*(pow(a,n) + pow(b,n))  ,(1.0/n))  );
}
```

We see from the definition, that $GM(a,b,n)$ is symmetrical between a and b. We also see that $GM(a,b)$ is homogenous, and we can normalize or apply factors easily.

$$\begin{aligned} GM(a,b,n) &= GM(b,a,n) \\ \lambda G(a,b,n) &= G(\lambda a, \lambda b, n) \\ GM(a,b,n) &= a * GM(1,b/a,n) \\ GM(1,1/a,n) &= GM(1,a,n)/a \\ GM(1,1,n) &= 1 \end{aligned}$$

Without loss of generality, I can limit studies to $GM(1,x,n)$, which allows simple plotting, and easier analysis.

If we assume $a < b$, we have

$$a < GM(a,b,-1) < GM(a,b,0) < GM(a,b,1) < GM(a,b,2) < b$$

from the properties of the means listed above. This suggests that if $m < n$, then $GM(a,b,m) < GM(a,b,n)$.

Nothing in the original formula limits us to integer value for n.

Figure 1 is a plot of GM(1,x,n) for $n = -2.0, -1.0, -0.5, 0.0, 0.5, 1.0, 2.0$

We see all curves are well behaved, with positive slopes. We can see the positive slope criteria from differential equation for slope, where all factors are positive.

$$\begin{aligned} GM(1,x,n) &= \left(\frac{1^n + x^n}{2}\right)^{1/n} \\ \frac{\partial GM(1,x,n)}{\partial x} &= \left(\frac{1 + x^n}{2}\right)^{-1+1/n} x^{n-1} \end{aligned}$$

Zooming in around $n = 0$ to check the assertion of the Geometric Mean being the $n \to 0$ case, in Figure 2 we show GM for $n = 0.2, 0.1, 0.0, -0.1, -0.2$ cases. Proof that $GM(a,b,0) = \sqrt{ab}$ is at the end of the next section.

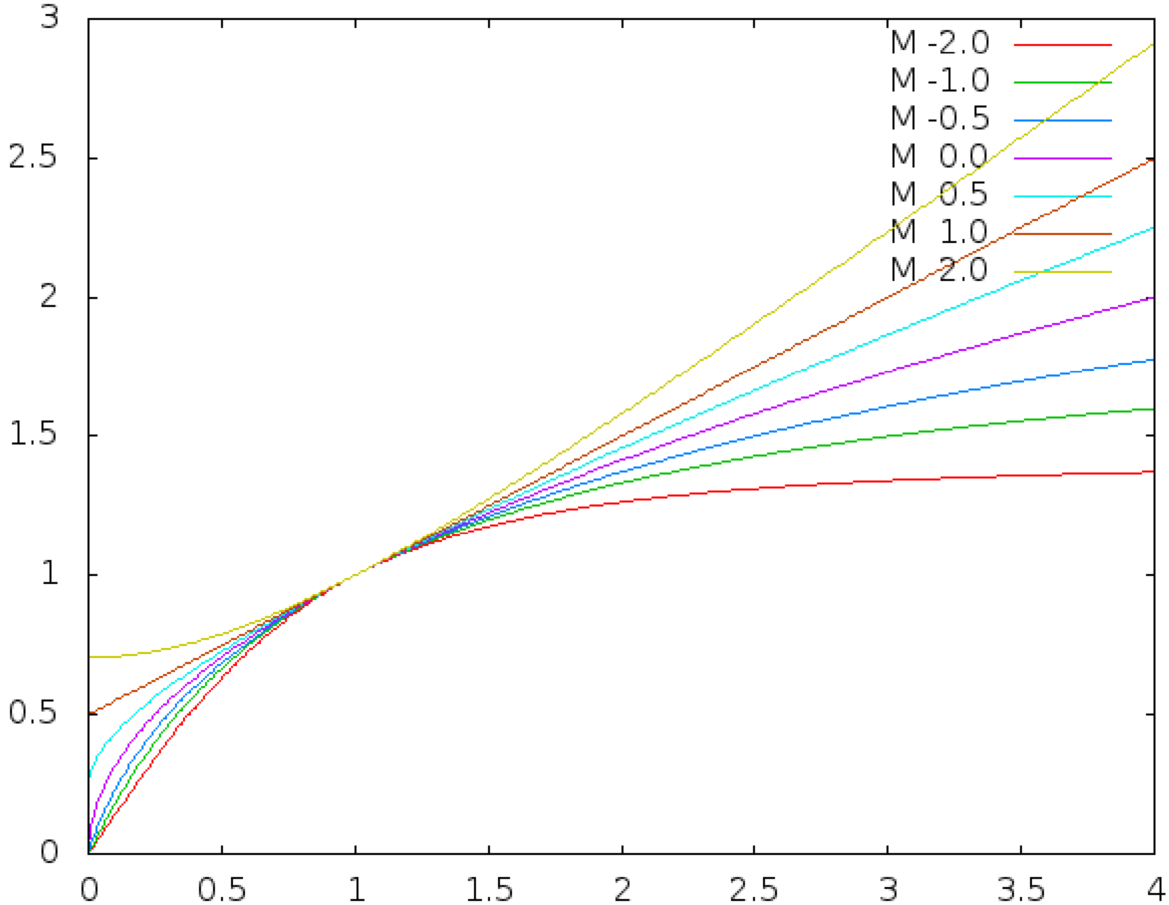

Figure 1: $y = GM(1, x, n)$ for $n = -2.0, -1.0, -0.5, 0.0, 0.5, 1.0, 2.0$

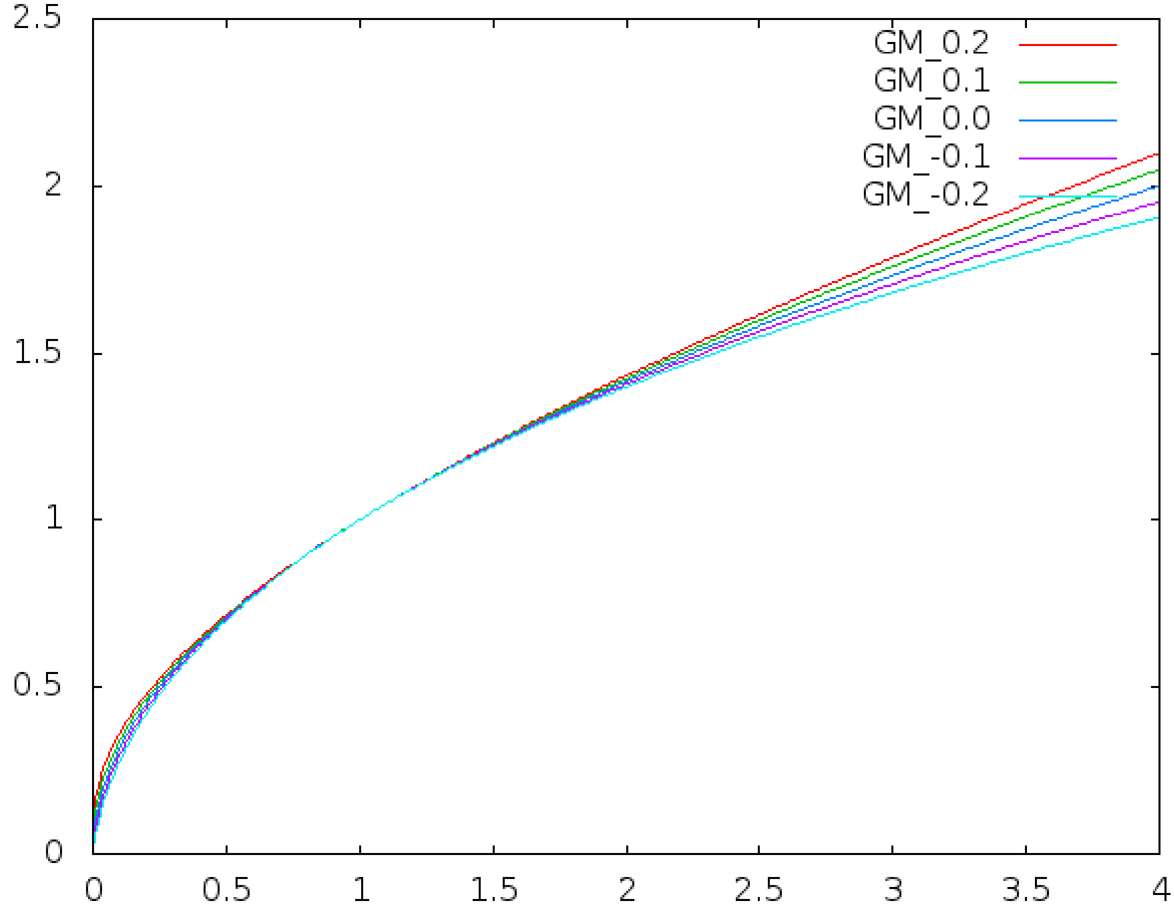

Figure 2: $y = GM(1, x, n)$ for $n = -0.2, -0.1, 0.0, 0.1, 0.2$

Product of Positive and Negative Exponent GMs

The product of GM(a,b,n)*GM(a,b,-n) is ab.

$$\begin{aligned} GM(a,b,n)*GM(a,b,-n) &= \left(\frac{a^n+b^n}{2}\right)^{1/n}\left(\frac{a^{-n}+b^{-n}}{2}\right)^{-1/n} \\ &= \left(\frac{a^n+b^n}{2}\right)^{1/n}\left(\frac{2}{a^{-n}+b^{-n}}\right)^{1/n} \\ &= \left(\frac{a^n+b^n}{a^{-n}+b^{-n}}\right)^{1/n} \\ &= \left(\frac{a^nb^n}{a^nb^n}\frac{a^n+b^n}{a^{-n}+b^{-n}}\right)^{1/n} \\ &= \left(a^nb^n\frac{a^n+b^n}{b^n+a^n}\right)^{1/n} \\ &= (a^nb^n)^{1/n} \\ &= ab \end{aligned}$$

We thus have the connection formula for negative n.

$$GM(a,b,-n) = \frac{ab}{GM(a,b,n)}$$

Returning to the question of $GM(a,b,0)=\sqrt{ab}$, we see

$$GM(a,b,n)*GM(a,b,-n) = ab$$

implies that as $n \to 0$, $GM(a,b,n) \to GM(a,b,-n)$, and

$$\begin{aligned} GM(a,b,0)^2 &= ab \\ GM(a,b,0) &= \sqrt{ab} \end{aligned}$$

Specific Implementations for $n = -2.0, -1.0, -0.5, 0.0, 0.5, 1.0, 2.0$

For reference, we build here specific implementation for the cases of $n = -2.0, -1.0, -0.5, 0.0, 0.5, 1.0, 2.0$.

$$
\begin{aligned}
GM(a,b,2) &= \sqrt{\frac{a^2+b^2}{2}} \quad \text{Root Mean Square} \\
GM(a,b,1) &= \frac{a+b}{2} \quad \text{Arithmetic Average} \\
GM(a,b,0.5) &= \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \\
&= \frac{a+b+2\sqrt{ab}}{4} \\
&= \frac{\left(\frac{a+b}{2}\right)+\sqrt{ab}}{2} \quad \text{Average of Arithmetic and Geometric Means} \\
GM(a,b,0) &= \sqrt{ab} \quad \text{Geometric Mean} \\
GM(a,b,-0.5) &= \left(\frac{\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}}{2}\right)^{-2} = \left(\frac{\sqrt{b}+\sqrt{a}}{2\sqrt{ab}}\right)^{-2} \\
&= \left(\frac{2\sqrt{ab}}{\sqrt{b}+\sqrt{a}}\right)^2 = \frac{ab}{GM(a,b,0.5)} = \frac{4ab}{a+b+2\sqrt{ab}} \\
GM(a,b,-1) &= \left(\frac{\frac{1}{a}+\frac{1}{b}}{2}\right)^{-1} = \frac{2ab}{a+b} = \frac{ab}{GM(a,b,1)} \quad \text{Harmonic Mean} \\
GM(a,b,-2) &= \left(\frac{a^{-2}+b^{-2}}{2}\right)^{-1/2} = \sqrt{\frac{2a^2b^2}{a^2+b^2}} = \frac{ab}{GM(a,b,2)}
\end{aligned}
$$

Special Relationship for GM(a,b,05) and GM(a,b,-0.5)

I noticed above that GM(a,b,0.5) is the arithmetic average of the geometric and arithmetic averages. I suspect a similar relationship occurs for GM(a,b,-0.5). The relationship proves to be the harmonic average of the harmonic average and geometric average.

$$
\begin{aligned}
HM(Geo(a,b), HM(a,b)) &= \frac{2\sqrt{ab}\left(\frac{2ab}{a+b}\right)}{\sqrt{ab}+\left(\frac{2ab}{a+b}\right)} \\
&= \frac{4(ab)^{3/2}}{\sqrt{ab}(a+b)+2ab} \\
&= \frac{4ab}{a+b+2\sqrt{ab}} = GM(a,b,-0.5)
\end{aligned}
$$

These suggest a half-angle formula involving n.

$$
\begin{aligned}
GM(a,b,0.5) &= GM(GM(a,b,0), GM(a,b,1), 1) \quad \text{checks} \\
GM(a,b,-0.5) &= GM(GM(a,b,0), GM(a,b,-1), -1) \quad \text{checks} \\
GM(a,b,1) &= GM(GM(a,b,0), GM(a,b,2), 2) \quad \text{checks} \\
GM(a,b,-1) &= GM(GM(a,b,0), GM(a,b,-2), -2) \quad \text{checks}
\end{aligned}
$$

We now prove a half-angle, or double-angle like formula for the general mean.

$$
\text{Prove} \quad GM(GM(a,b,0), GM(a,b,2n), 2n) \;=\; GM(a,b,n)
$$

$$
\begin{aligned}
\left[\frac{\left(\sqrt{ab}\right)^{2n}+\left[\left(\frac{a^{2n}+b^{2n}}{2}\right)^{1/(2n)}\right]^{2n}}{2}\right]^{1/(2n)} &= \left[\frac{\left(\sqrt{ab}\right)^{2n}+\left(\frac{a^{2n}+b^{2n}}{2}\right)}{2}\right]^{1/(2n)} \\
&= \left[\frac{a^n b^n+\left(\frac{a^{2n}+b^{2n}}{2}\right)}{2}\right]^{1/(2n)} \\
&= \left[\frac{2a^n b^n + a^{2n} + b^{2n}}{4}\right]^{1/(2n)} \\
&= \left(\frac{a^n+b^n}{2}\right)^{1/n} = GM(a,b,n)
\end{aligned}
$$

Large Values for n

Figure 3 show curves with large values of n. By the time we hit $n = 10$, we have pretty much ramped this curve.

Figure 4 show curves with large negative values of n. Same comment about ramping applies here as well.

AGM

The AGM process is a series of calculations, iterating upon the arithmetic and geometric means, rapidly converging on a single number.

Start with two numbers, a_0 and g_0. Form the sequence

$$\begin{aligned} a_{n+1} &= \frac{a_n + g_n}{2} \\ g_{n+1} &= \sqrt{a_n * g_n} \end{aligned}$$

When the absolute value of the difference between the pairs is small enough for your needs, you're done. As shown, a and g are symmetrical. In practice, we will order $a > g$, so that we can estimate the error (and calculate elliptic integrals of the first kind) using

$$c_n = \sqrt{a_n^2 - g_n^2}$$

In integral form, the AGM is

$$\mathrm{AGM(a, b)} = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2(\theta) + b^2 \sin^2(\theta)}}$$

This function has the geometric interpretation of the average of the inverse radius for an ellipse over 90 degrees.

AHM

The harmonic mean of a and b is

$$\mathrm{HM}(a, b) = \frac{2}{\frac{1}{a} + \frac{1}{b}} = \frac{2ab}{a + b}$$

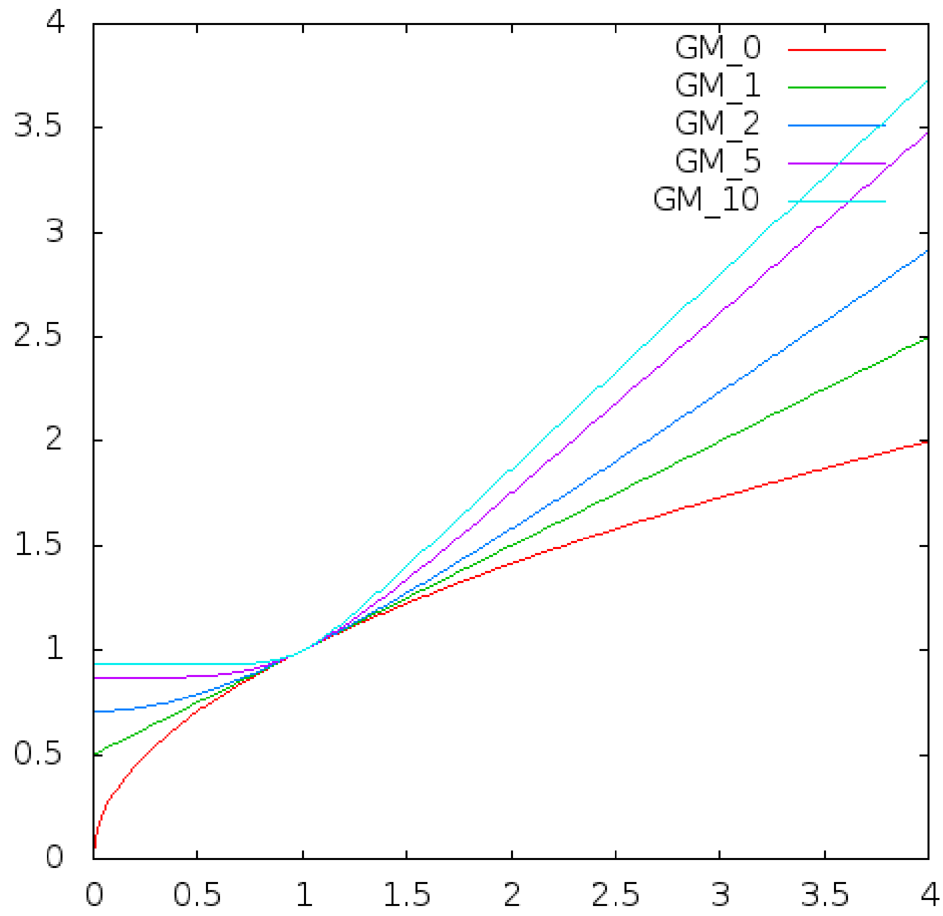

Figure 3: $y = GM(1, x, n)$ for $n = 0, 1, 2, 5, 10$

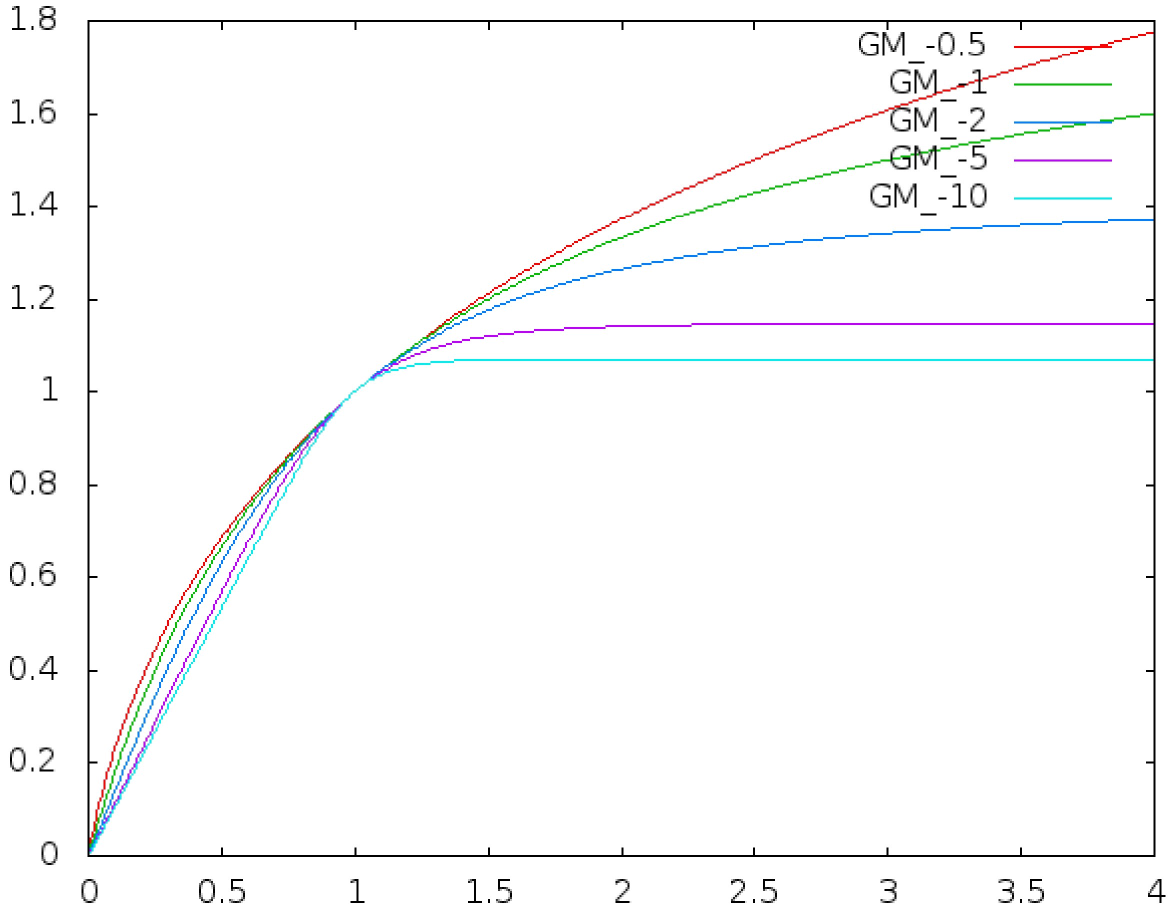

Figure 4: $y = GM(1, x, n)$ for $n = -0.5, -1, -2, -5, -10$

To form the AHM, we do the sequence starting with a_0 and h_0

$$\begin{aligned} a_{n+1} &= \frac{a_n + h_n}{2} \\ h_{n+1} &= \frac{2a_n h_n}{a_n + h_n} \end{aligned}$$

AHM(1,x) is seen numerically to be the square root of x. It turns out that doing an AGM process with pairs with opposite polarities of n seems to always go to harmonic mean.

For example, the sequence with GM(a,b,3) and GM(a,b,-3) goes to $\sqrt{ab}$. The sequence with GM(a,b,0.5) and GM(a,b,-0.5) goes to $\sqrt{ab}$.

GHM

The geometric mean of a and b is

$$\mathrm{GM}(a, b) = \sqrt{ab}$$

The harmonic mean of a and b is

$$\mathrm{HM}(a, b) = \frac{2}{\frac{1}{a} + \frac{1}{b}} = \frac{2ab}{a + b}$$

To form the GHM, we do the sequence starting with g_0 and h_0

$$\begin{aligned} g_{n+1} &= \sqrt{g_n h_n} \\ h_{n+1} &= \frac{2g_n h_n}{g_n + h_n} \end{aligned}$$

GHM(1,x) converges nicely. I expect this to be related to elliptic integrals of the second kind.

Plots of AGM, AHM and GHM

Figure 5 shows the function AGM(1,x), AHM(1,x), and GHM(1,x).

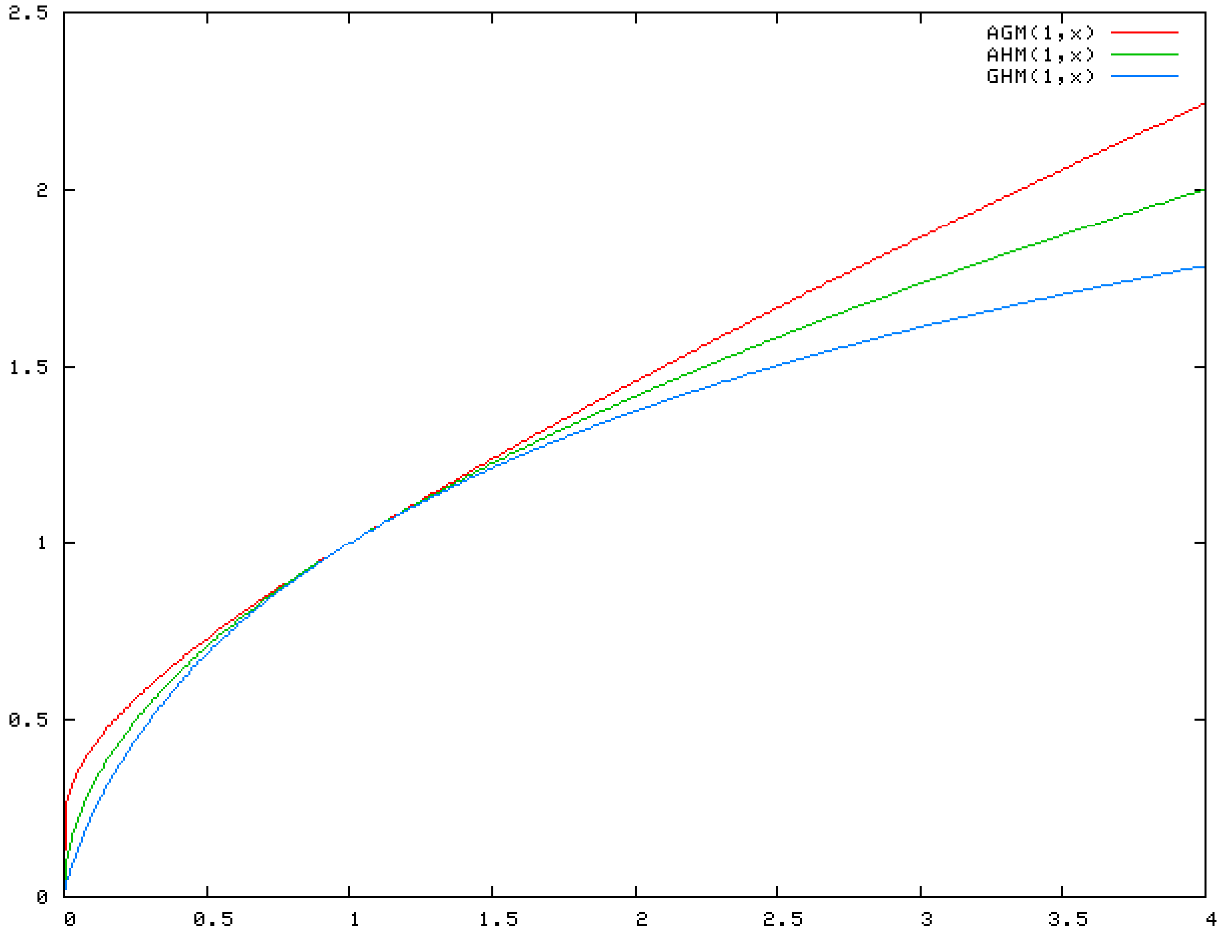

Figure 5: $y = \mathrm{AGM}(1, \mathrm{x})$, $y = \mathrm{AHM}(1, x) = \sqrt{x}$, $y = \mathrm{GHM}(1, x)$

C Code to Product Plots

Sample code in C to produce this plot is at `http://www.kurtnalty.com/Plot_AHM.c`.

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>

double AGM(double A, double B)
{
    double long a[30], g[30], c[30];
    int i;
    double long eps = 1.0e-8;

    for (i=0;i<30;i++) {    // clear arrays
        a[i] = 0.0;
        g[i] = 0.0;
        c[i] = 0.0;
    }

 // some initial values

    a[0] = A;  // I assume a[0] > g[0]
    g[0] = B;

// assumption check and correction, if necessary

    if (a[0] < g[0]) {
        c[0] = a[0];  // swap using currently unused c[0]
        a[0] = g[0];
        g[0] = c[0];
    }

    c[0] = sqrt(a[0]*a[0] - g[0]*g[0]);
 //   printf("%Lf  %Lf  %Lf \n",a[0], g[0], c[0]);

    for (i=1;i<30;i++) {    // open loop, do 30 iterations max
```

```
        a[i] = 0.5*(a[i-1] + g[i-1]);
        g[i] = sqrtl(a[i-1]*g[i-1]);
        c[i] = sqrtl(fabsl(a[i]*a[i] - g[i]*g[i]));
//        printf("%Lf  %Lf  %Lf \n",a[i], g[i], c[i]);
        if (fabsl(c[i]) < eps) break;
    }

    if (i < 30) return a[i];
    else {
        printf("x");
        return a[29];
    }
}


double AHM(double A, double B)
{
    double long a[30], g[30], c[30];
    int i;
    double long eps = 1.0e-8;

    for (i=0;i<30;i++) {    // clear arrays
        a[i] = 0.0;
        g[i] = 0.0;
        c[i] = 0.0;
    }

 // some initial values

    a[0] = A;  // I assume a[0] > g[0]
    g[0] = B;

// assumption check and correction, if necessary

    if (a[0] < g[0]) {
        c[0] = a[0];  // swap using currently unused c[0]
        a[0] = g[0];
        g[0] = c[0];
```

```
    }

    c[0] = sqrt(a[0]*a[0] - g[0]*g[0]);
 //   printf("%Lf  %Lf  %Lf \n",a[0], g[0], c[0]);

    for (i=1;i<30;i++) {    // open loop, do 30 iterations max
        a[i] = 0.5*(a[i-1] + g[i-1]);
//        g[i] = sqrtl(a[i-1]*g[i-1]);
        g[i] = 2.0*(a[i-1]*g[i-1])/(a[i-1] + g[i-1]);
        c[i] = sqrtl(fabsl(a[i]*a[i] - g[i]*g[i]));
//        printf("%Lf  %Lf  %Lf \n",a[i], g[i], c[i]);
        if (fabsl(c[i]) < eps) break;
    }

    if (i < 30) return a[i];
    else {
        printf("y");
        return a[29];
    }
}


double GHM(double A, double B)
{
    double long a[30], g[30], c[30];
    int i;
    double long eps = 1.0e-8;

    for (i=0;i<30;i++) {    // clear arrays
        a[i] = 0.0;
        g[i] = 0.0;
        c[i] = 0.0;
    }

 // some initial values

    a[0] = A;  // I assume a[0] > g[0]
    g[0] = B;
```

```
// assumption check and correction, if necessary

    if (a[0] < g[0]) {
        c[0] = a[0];  // swap using currently unused c[0]
        a[0] = g[0];
        g[0] = c[0];
    }

    c[0] = sqrt(a[0]*a[0] - g[0]*g[0]);
 //   printf("%Lf  %Lf  %Lf \n",a[0], g[0], c[0]);

    for (i=1;i<30;i++) {    // open loop, do 30 iterations max
        a[i] = sqrtl(a[i-1]*g[i-1]);
        g[i] = 2.0*(a[i-1]*g[i-1])/(a[i-1] + g[i-1]);
        c[i] = sqrtl(fabsl(a[i]*a[i] - g[i]*g[i]));
//        printf("%Lf  %Lf  %Lf \n",a[i], g[i], c[i]);
        if (fabsl(c[i]) < eps) break;
    }

    if (i < 30) return a[i];
    else {
        printf("z");
        return a[29];
    }
}

int main(void)
{
        double A, B, D, E, F;
        double x, dx;
        FILE* Output;

        Output = fopen("Plot_AHM.txt","w");

x = 0.0;
        dx = 0.01;
        for (x = 0.0; x<4.0; x+= dx) {
```

```
            A = 1.0;
            B = x;
            D = AGM(A,B);
            E = AHM(A,B);         // seems to be square root
            F = GHM(A,B);
            fprintf(Output,"%lf   %lf  %lf  %lf\n", x, D, E, F);
        }

        fclose(Output);

        return 0;
}
```