Garret Sobczyk's 2x2 Matrix Derivation

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Abstract

Using matrices to represent geometric algebras is known, but not necessarily the best practice. While I have used small computer programs to scan through candidate matrices to find representations, I have not been able to derive such matrices from first principles. Garret Sobczyk at `www.GarretStar.com`, however, has. I repeat his derivation here, for 2x2 matrix representation of the 2D geometric algebra, including more intermediate steps for easier understanding.

2D Euclidean Geometric Algebra Basis

Start with a two dimensional, Euclidean geometric algebra. We have two vector directions, e_1 and e_2 which correspond to our standard x and y directions. Our geometric multivector elements are scalars, vectors e_1 and e_2, and a single bivector $e_1 \wedge e_2 = e_{12}$.

In this algebra, scalar multiplication is commutative and associative, vectors square to scalar one, and the product of two vectors resulting in a bivector is anti-commutative, associative and squares to negative one. The order sensitive multiplication table for this algebra is shown below.

	1	e_1	e_2	e_1e_2
1	1	e_1	e_2	e_1e_2
e_1	e_1	1	e_1e_2	e_2
e_2	e_2	$-e_1e_2$	1	$-e_1$
e_1e_2	e_1e_2	$-e_2$	e_1	-1

Given this notation, the generic 2D multivector can be written as

$$g = \alpha + xe_1 + ye_2 + \beta e_1e_2$$

Sobczyk now introduces a linear combination of two of these basis, where he rotates the scalar and e_2 elements 45 degrees, and applies a small scaling factor.

$$\begin{aligned} u_+ &= \frac{1+e_2}{2} \\ u_- &= \frac{1-e_2}{2} \end{aligned}$$

Adding these two basis recovers the 1, subtracting these two basis recovers e_2. Each of the basis square to themselves, making these combinations idempotents like $0*0=0$ and $1*1=1$. More fun, the product of these two basis is zero, making these combinations mutual annihilators. These two properties of projection and annihilation greatly simplify power series formulas, as found in exponentiation.

$$\begin{aligned} u_+ + u_- &= \frac{1+e_2}{2} + \frac{1-e_2}{2} = 1 \\ u_+ - u_- &= \frac{1+e_2}{2} - \frac{1-e_2}{2} = e_2 \qquad \text{(notice below } \mathrm{e}_2 * \mathrm{e}_2 = 1\text{)} \\ u_+ * u_+ &= \frac{1+e_2}{2} * \frac{1+e_2}{2} = \frac{1+2e_2+e_2*e_2}{4} = \frac{1+2e_2+1}{4} = u_+ \\ u_- * u_- &= \frac{1-e_2}{2} * \frac{1-e_2}{2} = \frac{1-2e_2+e_2*e_2}{4} = \frac{1-2e_2+1}{4} = u_- \\ u_+ * u_- &= \frac{1+e_2}{2} * \frac{1-e_2}{2} = \frac{1*1-e_2*e_2}{4} = \frac{1*1-1}{4} = 0 \end{aligned}$$

Having seen how the u_+ and u_- basis work among themselves, let us write out the preproduct and postproduct expressions among the u_+, u_-, e_1 and e_1e_2 terms.

$$\begin{aligned} u_+e_1 &= \frac{1+e_2}{2}e_1 = \frac{e_1+e_2e_1}{2} = \frac{e_1-e_1e_2}{2} = e_1\frac{1-e_2}{2} = e_1u_- \\ e_1u_+ &= e_1\frac{1+e_2}{2} = \frac{e_1+e_1e_2}{2} = \frac{e_1-e_2e_1}{2} = \frac{1-e_2}{2}e_1 = u_-e_1 \end{aligned}$$

Now for e_2,

$$\begin{aligned}
u_+e_2 &= \frac{1+e_2}{2}e_2 = \frac{e_2+e_2*e_2}{2} = \frac{e_2+1}{2} = u_+ \\
e_2u_+ &= e_2\frac{1+e_2}{2} = \frac{e_2+e_2*e_2}{2} = \frac{e_2+1}{2} = u_+ \\
u_-e_2 &= \frac{1-e_2}{2}e_2 = \frac{e_2-e_2*e_2}{2} = \frac{e_2-1}{2} = -u_- \\
e_2u_- &= e_2\frac{1-e_2}{2} = \frac{e_2-e_2*e_2}{2} = \frac{e_2-1}{2} = -u_-
\end{aligned}$$

Finally for the bivector e_1e_2,

$$\begin{aligned}
u_+e_1e_2 &= \frac{1+e_2}{2}e_1e_2 = \frac{e_1e_2+e_2e_1e_2}{2} = \frac{e_1e_2-e_1}{2} = -e_1u_- \\
e_1e_2u_+ &= e_1e_2\frac{1+e_2}{2} = \frac{e_1e_2+e_1e_2e_2}{2} = \frac{e_1e_2+e_1}{2} = e_1u_+ \\
u_-e_1e_2 &= \frac{1-e_2}{2}e_1e_2 = \frac{e_1e_2-e_2e_1e_2}{2} = \frac{e_1e_2+e_1}{2} = e_1u_+ \\
e_1e_2u_- &= e_1e_2\frac{1-e_2}{2} = \frac{e_1e_2-e_1e_2e_2}{2} = \frac{e_1e_2-e_1}{2} = -e_1u_-
\end{aligned}$$

So, a little summary sheet. . .

$u_+ = (1+e_2)/2$	$u_- = (1-e_2)/2$	$u_+ + u_- = 1$	$u_+ - u_- = e_2$
$u_+u_+ = u_+$	$u_-u_- = u_-$	$u_+u_- = 0$	$u_-u_+ = 0$
$u_+e_1 = e_1u_-$	$e_1u_+ = u_-e_1$	$u_-e_1 = e_1u_+$	$e_1u_- = u_+e_1$
$u_+e_2 = u_+$	$e_2u_+ = u_+$	$u_-e_2 = -u_-$	$e_2u_- = -u_-$
$u_+e_1e_2 = -e_1u_-$	$e_1e_2u_+ = e_1u_+$	$u_-e_1e_2 = e_1u_+$	$e_1e_2u_- = -e_1u_-$

Sobczyk now presents two matrix expressions using geometric algebra elements.

$$\begin{pmatrix} 1 \\ e_1 \end{pmatrix} u_+ \begin{pmatrix} 1 & e_1 \end{pmatrix} = \begin{pmatrix} u+ & u_+e_1 \\ e_1u_+ & e_1u_+e_1 \end{pmatrix} = \begin{pmatrix} u+ & e_1u_- \\ e_1u_+ & u_- \end{pmatrix}$$

Sobcyzk calls the right hand matrix the spectral basis for 2D. The next matrix expression is

$$\begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix} = u_+ + e_1u_+e_1 = u_+ + e_1e_1u_- = u_+ + u_- = 1$$

Now the fun begins. Start with the true statement for the generic multivector $g = 1 * g * 1$.

$$g = 1g1 = \left[\begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix}\right] g \left[\begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix}\right]$$

Being an associative algebra, we can refactor our multiplications, keeping the order intact.

$$g = 1g1 = \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ \left[\begin{pmatrix} 1 \\ e_1 \end{pmatrix} g \begin{pmatrix} 1 & e_1 \end{pmatrix}\right] u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix}$$

$$g = 1g1 = \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ \begin{pmatrix} g & ge_1 \\ e_1 g & e_1 g e_1 \end{pmatrix} u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix}$$

We now want to examine the middle three product terms. Begin by looking at each term in the center matrix.

$$\begin{aligned}
g &= \alpha + xe_1 + ye_2 + \beta e_1 e_2 \\
e_1 g &= \alpha e_1 + e_1 x e_1 + e_1 y e_2 + \beta e_1 e_1 e_2 = x + \alpha e_1 + \beta e_2 + y e_1 e_2 \\
g e_1 &= \alpha e_1 + x e_1 e_1 + y e_2 e_1 + \beta e_1 e_2 e_1 = x + \alpha e_1 - \beta e_2 - y e_1 e_2 \\
e_1 g e_1 &= e_1 x + e_1 \alpha e_1 - e_1 \beta e_2 - e_1 y e_1 e_2 = \alpha + x e_1 - y e_2 - \beta e_1 e_2
\end{aligned}$$

We now postmultiply by u_+, and separate product terms to isolate leading u_- and u_+ terms. Start with g.

$$\begin{aligned}
g u_+ &= (\alpha + xe_1 + ye_2 + \beta e_1 e_2) u_+ \\
&= \alpha u_+ + x e_1 u_+ + y e_2 u_+ + \beta e_1 e_2 u_+ \\
&= u_+ \alpha + u_- e_1 x + y u_+ + \beta e_1 u_+ \\
&= u_+ \alpha + u_- x e_1 + y u_+ + u_- e_1 \beta \\
&= u_+ (\alpha + y) + u_- (x + \beta) e_1
\end{aligned}$$

Now, premultiply by u_+, project and annihilate.

$$\begin{aligned}
u_+ (g u_+) &= u_+ (u_+ (\alpha + y) + u_- (x + \beta) e_1) \\
&= u_+ u_+ (\alpha + y) + u_+ u_- (x + \beta) e_1 \\
&= u_+ (\alpha + y) + 0 * (x + \beta) e_1 \\
&= u_+ (\alpha + y)
\end{aligned}$$

In a similar fashion, we process $e_1 g$

$$\begin{aligned}(e_1 g)u_+ &= (x + \alpha e_1 + \beta e_2 + y e_1 e_2)u_+ \\ &= x u_+ + \alpha e_1 u_+ + \beta e_2 u_+ + y e_1 e_2 u_+ \\ &= u_+ x + \alpha u_- e_1 + \beta u_+ + y e_1 u_+ \\ &= u_+ x + u_+ \beta + u_- \alpha e_1 + y u_- e_1 \\ &= u_+(x + \beta) + u_-(\alpha + y)e_1\end{aligned}$$

Now, premultiply by u_+, project and annihilate.

$$\begin{aligned}u_+(e_1 g u_+) &= u_+(u_+(x + \beta) + u_-(\alpha + y)e_1) \\ &= u_+(x + \beta)\end{aligned}$$

We now rapidly do the remaining terms.

$$\begin{aligned}(g e_1)u_+ &= (x + \alpha e_1 - \beta e_2 - y e_1 e_2)u_+ \\ &= x u_+ + \alpha e_1 u_+ - \beta e_2 u_+ - y e_1 e_2 u_+ \\ &= u_+ x + u_- \alpha e_1 - u_+ \beta - u_- y e_1 \\ &= u_+(x - \beta) + u_-(\alpha - y)e_1\end{aligned}$$

Now, premultiply by u_+, project and annihilate.

$$\begin{aligned}u_+(g e_1)u_+ &= u_+(u_+(x - \beta) + u_-(\alpha - y)e_1) \\ &= u_+(x - \beta)\end{aligned}$$

Finish with

$$\begin{aligned}(e_1 g e_1)u_+ &= (\alpha + x e_1 - y e_2 - \beta e_1 e_2)u_+ \\ &= \alpha u_+ + x e_1 u_+ - y e_2 u_+ - \beta e_1 e_2 u_+ \\ &= u_+ \alpha + x u_- e_1 - y u_+ - \beta e_1 u_+ \\ &= u_+ \alpha + u_- x e_1 - u_+ y - \beta u_- e_1 \\ &= u_+(\alpha - y) + u_-(x - \beta)e_1\end{aligned}$$

Now, premultiply by u_+, project and annihilate.

$$\begin{aligned}u_+(e_1 g e_1)u_+ &= u_+(u_+(\alpha - y) + u_-(x - \beta)e_1) \\ &= u_+(\alpha - y)\end{aligned}$$

We thus have the nice result that

$$u_+ \begin{pmatrix} g & ge_1 \\ e_1g & e_1ge_1 \end{pmatrix} u_+ = \begin{pmatrix} u_+(\alpha+y) & u_+(x-\beta) \\ u_+(x+\beta) & u_+(\alpha-y) \end{pmatrix}$$

Pulling out the common factor u_+, we have

$$u_+ \begin{pmatrix} g & ge_1 \\ e_1g & e_1ge_1 \end{pmatrix} u_+ = u_+ \begin{pmatrix} \alpha+y & x-\beta \\ x+\beta & \alpha-y \end{pmatrix}$$

This real matrix is the 2D Euclidean geometric algebra, with respect to the spectral basis.

I point out that being an array of scalars (real numbers, not geometrical objects), this matrix commutes with the geometric number u_+.

$$u_+ \begin{pmatrix} \alpha+y & x-\beta \\ x+\beta & \alpha-y \end{pmatrix} = \begin{pmatrix} \alpha+y & x-\beta \\ x+\beta & \alpha-y \end{pmatrix} u_+ = u_+ \begin{pmatrix} \alpha+y & x-\beta \\ x+\beta & \alpha-y \end{pmatrix} u_+$$

Sobczyk uses the notation $[g]$ for this real matrix.

$$[g] = \begin{pmatrix} \alpha+y & x-\beta \\ x+\beta & \alpha-y \end{pmatrix}$$

So, a little recap is in order before we proceed.

$$g = 1g1 = \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ \left[\begin{pmatrix} 1 \\ e_1 \end{pmatrix} g \begin{pmatrix} 1 & e_1 \end{pmatrix} \right] u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix}$$

which becomes

$$g = \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+[g] \begin{pmatrix} 1 \\ e_1 \end{pmatrix}$$

Sobczyk now does a preproduct and postproduct on g.

$$\begin{pmatrix} 1 \\ e_1 \end{pmatrix} g \begin{pmatrix} 1 & e_1 \end{pmatrix} = \begin{pmatrix} 1 \\ e_1 \end{pmatrix} \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+[g] \begin{pmatrix} 1 \\ e_1 \end{pmatrix} \begin{pmatrix} 1 & e_1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ e_1 \end{pmatrix} g \begin{pmatrix} 1 & e_1 \end{pmatrix} = \begin{pmatrix} 1 & e_1 \\ e_1 & 1 \end{pmatrix} u_+[g] \begin{pmatrix} 1 & e_1 \\ e_1 & 1 \end{pmatrix}$$

We now sandwich multiply by u_+, and find

$$u_+\begin{pmatrix}1\\e_1\end{pmatrix}g\begin{pmatrix}1 & e_1\end{pmatrix}u_+ = u_+\begin{pmatrix}1 & e_1\\e_1 & 1\end{pmatrix}u_+[g]\begin{pmatrix}1 & e_1\\e_1 & 1\end{pmatrix}u_+$$

On the right hand side of the equation, I bring the left u_+ into the matrix.

$$u_+\begin{pmatrix}1\\e_1\end{pmatrix}g\begin{pmatrix}1 & e_1\end{pmatrix}u_+ = \begin{pmatrix}u_+ & u_+e_1\\u_+e_1 & u_+\end{pmatrix}u_+[g]\begin{pmatrix}1 & e_1\\e_1 & 1\end{pmatrix}u_+$$

Now I drag the u_+ across the e_1 terms, creating u_- elements.

$$u_+\begin{pmatrix}1\\e_1\end{pmatrix}g\begin{pmatrix}1 & e_1\end{pmatrix}u_+ = \begin{pmatrix}u_+ & e_1u_-\\e_1u_- & u_+\end{pmatrix}u_+[g]\begin{pmatrix}1 & e_1\\e_1 & 1\end{pmatrix}u_+$$

We now bring the u_+ term left of the $[g]$ in the matrix from the right.

$$u_+\begin{pmatrix}1\\e_1\end{pmatrix}g\begin{pmatrix}1 & e_1\end{pmatrix}u_+ = \begin{pmatrix}u_+u_+ & e_1u_-u_+\\e_1u_-u_+ & u_+u_+\end{pmatrix}[g]\begin{pmatrix}1 & e_1\\e_1 & 1\end{pmatrix}u_+$$

We project and annihilate.

$$u_+\begin{pmatrix}1\\e_1\end{pmatrix}g\begin{pmatrix}1 & e_1\end{pmatrix}u_+ = \begin{pmatrix}u_+ & 0\\0 & u_+\end{pmatrix}[g]\begin{pmatrix}1 & e_1\\e_1 & 1\end{pmatrix}u_+$$

Refactor out the u_+ term

$$u_+\begin{pmatrix}1\\e_1\end{pmatrix}g\begin{pmatrix}1 & e_1\end{pmatrix}u_+ = u_+\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}[g]\begin{pmatrix}1 & e_1\\e_1 & 1\end{pmatrix}u_+$$

And now absorb the unit matrix.

$$u_+\begin{pmatrix}1\\e_1\end{pmatrix}g\begin{pmatrix}1 & e_1\end{pmatrix}u_+ = u_+[g]\begin{pmatrix}1 & e_1\\e_1 & 1\end{pmatrix}u_+$$

Since the $[g]$ matrix consists of real numbers, the multivector u_+ and scalar array $[g]$ commute, unlike the multivector u_+ and the multivector g. So, we move u_+ to the right hand side of $[g]$, and repeat the previous steps.

$$u_+\begin{pmatrix}1\\e_1\end{pmatrix}g\begin{pmatrix}1 & e_1\end{pmatrix}u_+ = [g]u_+\begin{pmatrix}1 & e_1\\e_1 & 1\end{pmatrix}u_+$$

Bring the left u_+ into the matrix.

$$u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix} g \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ = [g] \begin{pmatrix} u_+ & u_+ e_1 \\ u_+ e_1 & u_+ \end{pmatrix} u_+$$

Drag u_+ across e_1.

$$u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix} g \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ = [g] \begin{pmatrix} u_+ & e_1 u_- \\ e_1 u_- & u_+ \end{pmatrix} u_+$$

Bring in the right u_+.

$$u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix} g \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ = [g] \begin{pmatrix} u_+ u_+ & e_1 u_- u_+ \\ e_1 u_- u_+ & u_+ u_+ \end{pmatrix}$$

Project and annihilate.

$$u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix} g \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ = [g] \begin{pmatrix} u_+ & 0 \\ 0 & u_+ \end{pmatrix}$$

Factor and absorb.

$$u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix} g \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ = [g] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u_+ = [g] u_+ = u_+ [g]$$

We have the very pretty result

$$u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix} g \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ = [g] u_+ = u_+ [g]$$

To get a stand alone formula for $[g]$, Sobczyk develops a formula for $u_-[g]$, then adds the terms $u_+[g] + u_-[g] = (u_+ + u_-)[g] = 1[g] = [g]$.

To develop the formula for $u_-[g]$, Sobczyk once again does a sandwich product with e_1 (conjugation with respect to e_1).

$$e_1 u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix} g \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ e_1 = e_1 u_+ [g] e_1$$

Doing the right hand side first, we drag e_1 across u_+.

$$e_1 u_+ [g] e_1 \quad = \quad u_- e_1 [g] e_1$$

Since $[g]$ is a scalar array, $[g]$ and e_1 commute.

$$e_1 u_+ [g] e_1 \quad = \quad u_- e_1 [g] e_1 = u_- [g] e_1 e_1$$

Now, since e_1 squares to one, we have

$$e_1 u_+ [g] e_1 \quad = \quad u_- e_1 [g] e_1 = u_- [g] e_1 e_1 = u_- [g]$$

We now work on the left side.

$$e_1 u_+ \begin{pmatrix} 1 \\ e_1 \end{pmatrix} g \begin{pmatrix} 1 & e_1 \end{pmatrix} u_+ e_1 = e_1 u_+ [g] e_1 = u_- [g]$$

Drag e_1 across the u_+ terms.

$$u_- e_1 \begin{pmatrix} 1 \\ e_1 \end{pmatrix} g \begin{pmatrix} 1 & e_1 \end{pmatrix} e_1 u_- = u_- [g]$$

Bring e_1 into the column and row matrices.

$$u_- \begin{pmatrix} e_1 \\ e_1 e_1 \end{pmatrix} g \begin{pmatrix} e_1 & e_1 e_1 \end{pmatrix} u_- = u_- [g]$$

Clean up the squares

$$u_- \begin{pmatrix} e_1 \\ 1 \end{pmatrix} g \begin{pmatrix} e_1 & 1 \end{pmatrix} u_- = u_- [g]$$

Do the order sensitive matrix products

$$u_- \begin{pmatrix} e_1 g e_1 & e_1 g \\ g e_1 & g \end{pmatrix} u_- = u_- [g]$$

We now have the generic result

$$[g] = u_+ \begin{pmatrix} g & g e_1 \\ e_1 g & e_1 g e_1 \end{pmatrix} u_+ + u_- \begin{pmatrix} e_1 g e_1 & e_1 g \\ g e_1 & g \end{pmatrix} u_-$$

Consistency Check

We previously evaluated

$$u_+ \begin{pmatrix} g & ge_1 \\ e_1 g & e_1 g e_1 \end{pmatrix} u_+ = u_+ \begin{pmatrix} \alpha + y & x - \beta \\ x + \beta & \alpha - y \end{pmatrix}$$

We want to verify

$$u_- \begin{pmatrix} e_1 g e_1 & e_1 g \\ g e_1 & g \end{pmatrix} u_- = u_- \begin{pmatrix} \alpha + y & x - \beta \\ x + \beta & \alpha - y \end{pmatrix}$$

Repeating from previously

$$\begin{aligned} g &= \alpha + x e_1 + y e_2 + \beta e_1 e_2 \\ e_1 g &= \alpha e_1 + e_1 x e_1 + e_1 y e_2 + \beta e_1 e_1 e_2 = x + \alpha e_1 + \beta e_2 + y e_1 e_2 \\ g e_1 &= \alpha e_1 + x e_1 e_1 + y e_2 e_1 + \beta e_1 e_2 e_1 = x + \alpha e_1 - \beta e_2 - y e_1 e_2 \\ e_1 g e_1 &= e_1 x + e_1 \alpha e_1 - e_1 \beta e_2 - e_1 y e_1 e_2 = \alpha + x e_1 - y e_2 - \beta e_1 e_2 \end{aligned}$$

We start with the $e_1 g e_1$ term.

$$\begin{aligned} e_1 g e_1 &= \alpha + x e_1 - y e_2 - \beta e_1 e_2 \\ (e_1 g e_1) u_- &= \alpha u_- + x e_1 u_- - y e_2 u_- - \beta e_1 e_2 u_- \\ &= u_- \alpha + u_+ e_1 x + y u_- + \beta e_1 u_- \\ &= u_- \alpha + u_+ e_1 x + y u_- + u_+ \beta e_1 \\ &= u_-(\alpha + y) + u_+(x e_1 + \beta e_1) \end{aligned}$$

Project and annihilate, and get the expected result.

$$u_-(e_1 g e_1) u_- = u_-(\alpha + y)$$

Continue with the $e_1 g$ term.

$$\begin{aligned} e_1 g &= x + \alpha e_1 + \beta e_2 + y e_1 e_2 \\ (e_1 g) u_- &= x u_- + \alpha e_1 u_- + \beta e_2 u_- + y e_1 e_2 u_- \\ (e_1 g) u_- &= x u_- + \alpha u_+ e_1 - \beta u_- - y e_1 u_- \\ (e_1 g) u_- &= x u_- + \alpha u_+ e_1 - \beta u_- - y u_+ e_1 \\ (e_1 g) u_- &= u_-(x - \beta) + u_+(e_1 \alpha - y e_1) \end{aligned}$$

Project and annihilate, and get the expected result.

$$u_-(e_1 g)u_- = u_-(x - \beta)$$

Continue with the ge_1 term.

$$\begin{aligned}
ge_1 &= x + \alpha e_1 - \beta e_2 - y e_1 e_2 \\
(ge_1)u_- &= x u_- + \alpha e_1 u_- - \beta e_2 u_- - y e_1 e_2 u_- \\
(ge_1)u_- &= u_- x + u_+ \alpha e_1 + u_- \beta + y e_1 u_- \\
(ge_1)u_- &= u_- x + u_+ \alpha e_1 + u_- \beta + u_+ y e_1 \\
(ge_1)u_- &= u_-(x + \beta) + u_+(\alpha e_1 + y e_1)
\end{aligned}$$

Project and annihilate, and get the expected result.

$$u_-(ge_1)u_- = u_-(x + \beta)$$

Finish with the g term.

$$\begin{aligned}
g &= \alpha + x e_1 + y e_2 + \beta e_1 e_2 \\
g u_- &= \alpha u_- + x e_1 u_- + y e_2 u_- + \beta e_1 e_2 u_- \\
g u_- &= u_- \alpha + u_+ x e_1 - u_- y - \beta e_1 u_- \\
g u_- &= u_- \alpha + u_+ x e_1 - u_- y - u_+ \beta e_1 \\
g u_- &= u_-(\alpha - y) + u_+(x e_1 - \beta e_1)
\end{aligned}$$

Project and annihilate, and get the expected result.

$$u_- g u_- = u_-(\alpha - y)$$

Conclusion

Garret Sobczyk has demonstrated why a matrix representation for 2D Euclidean geometric algebra is

$$[g] = \begin{pmatrix} \alpha + y & x - \beta \\ x + \beta & \alpha - y \end{pmatrix}$$