Frenet Formula Examples Using Geometric Algebra

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Wedge Versus Geometric Product

In geometric algebra, we commonly find two products. The wedge product is an anti-symmetric product of basis vectors, commonly used in integration. The geometric product, or Clifford product, is a combination of symmetric and anti-symmetric products similar to an extension of complex numbers to multi-dimensional spaces.

Three Dimensional Wedge Product

In three dimensional Euclidean geometric algebra, we have one scalar e_q, three vector basis e_x, e_y, and e_z three bivector basis e_{xy}, e_{xz}, and e_{yz}, and one trivector e_{xyz}. The scalar factor commutes with all other terms. However, the vector terms anti-commute among themselves, implying that squared wedge factors become zero. For example, $e_x \wedge e_y = -e_y \wedge e_x = e_{xy}$ and $e_x \wedge e_y \wedge e_z = e_{xyz} = -e_x \wedge e_z \wedge e_y$, yet $e_x \wedge e_x = 0$.

The choice of default orderings for the bivectors and trivector varies among different authors, or within an author at different points in time. In this work, in three-space, we are using e_{zx}, instead of e_{xz}.

In text format, suppressing the letter e in the basis names, we have a wedge multiplication table

```
q      x      y      z      xy     zx     yz     xyz
x      0      xy    -zx     0      0      xyz    0
y     -xy     0      yz     0      xyz    0      0
z      zx    -yz     0      xyz    0      0      0
xy     0      0      xyz    0      0      0      0
zx     0      xyz    0      0      0      0      0
yz     xyz    0      0      0      0      0      0
xyz     0     0      0      0      0      0      0
```

In component form, suitable for programming languages such as C, we can express the product $c = a \wedge b$ as

```
c.q   =  a.q*b.q;
c.x   =  a.q*b.x   + a.x*b.q;
c.y   =  a.q*b.y   + a.y*b.q;
c.z   =  a.q*b.z   + a.z*b.q;
c.xy  =  a.q*b.xy  + a.x*b.y  - a.y*b.x  + a.xy*b.q;
c.zx  =  a.q*b.zx  - a.x*b.z  + a.z*b.x  + a.zx*b.q;
c.yz  =  a.q*b.yz  + a.y*b.z  - a.z*b.y  + a.yz*b.q;
c.xyz =  a.q*b.xyz + a.x*b.yz + a.y*b.zx + a.xy*b.z + a.z*b.xy + a.zx*b.y + a.yz*b.x + a.xyz*b.q;
```

In this work, we will be primarily using the wedge product.

Three Dimensional Geometric Product

The geometric product combines dot and wedge products in a linear combination reminiscent of complex numbers. In three dimensional Euclidean geometric algebra, we still have one scalar e_q, three vector basis e_x, e_y, and e_z three bivector basis e_{xy}, e_{xz}, and e_{yz}, and one trivector e_{xyz}. Unlike the wedge product, the basis vectors square to one, and anti-commute among themselves. For example, $e_x e_x = 1$, but $e_x e_y = -e_y e_x = e_{xy}$. The basis bivectors square to negative one, and anti-commute. The trivector e_{xyz} squares to negative one, and commutes with all algebra elements, exactly mimicing $i = \sqrt{-1}$.

The choice of default orderings for the bivectors and trivector varies among different authors, or within an author at different points in time. The choice of default orderings for the bivectors and multivectors, in this case, is made by multiplying the vectors by the trivector to get the bivectors. Because the trivector mimics i, we can form complex numbers from (e_q, e_{xyz}), (e_x, e_{yz}), (e_y, e_{zx}), and (e_z, e_{xy}) components. In three-space, we are using e_{zx}, instead of e_{xz}.

In text format, suppressing the letter e in the basis names, we have a multiplication table

```
q     x     y     z     xy    zx    yz    xyz
x     q     xy    -zx   y     -z    xyz   yz
y     -xy   q     yz    -x    xyz   z     zx
z     zx    -yz   q     xyz   x     -y    xy
xy    -y    x     xyz   -q    yz    -zx   -z
zx    z     xyz   -x    -yz   -q    xy    -y
yz    xyz   -z    y     zx    -xy   -q    -x
xyz   yz    zx    xy    -z    -y    -x    -q
```

In component form, suitable for programming languages such as C, we can express the product $c = ab$ as

```
c.q   =  + a.q*b.q   + a.x*b.x   + a.y*b.y   + a.z*b.z   - a.xy*b.xy  - a.zx*b.zx  - a.yz*b.yz  - a.xyz*b.xyz
c.x   =  + a.q*b.x   + a.x*b.q   - a.y*b.xy  + a.z*b.zx  + a.xy*b.y   - a.zx*b.z   - a.yz*b.xyz - a.xyz*b.yz
c.y   =  + a.q*b.y   + a.x*b.xy  + a.y*b.q   - a.z*b.yz  - a.xy*b.x   - a.zx*b.xyz + a.yz*b.z   - a.xyz*b.zx
c.z   =  + a.q*b.z   - a.x*b.zx  + a.y*b.yz  + a.z*b.q   - a.xy*b.xyz + a.zx*b.x   - a.yz*b.y   - a.xyz*b.xy
c.xy  =  + a.q*b.xy  + a.x*b.y   - a.y*b.x   + a.z*b.xyz + a.xy*b.q   + a.zx*b.yz  - a.yz*b.zx  + a.xyz*b.z
c.zx  =  + a.q*b.zx  - a.x*b.z   + a.y*b.xyz + a.z*b.x   - a.xy*b.yz  + a.zx*b.q   + a.yz*b.xy  + a.xyz*b.y
c.yz  =  + a.q*b.yz  + a.x*b.xyz + a.y*b.z   - a.z*b.y   + a.xy*b.zx  - a.zx*b.xy  + a.yz*b.q   + a.xyz*b.x
c.xyz =  + a.q*b.xyz + a.x*b.yz  + a.y*b.zx  + a.z*b.xy  + a.xy*b.z   + a.zx*b.y   + a.yz*b.x   + a.xyz*b.q
```

Frenet-Serret Formulas in 3D

In conventional three dimensional geometry, we can describe a curve by means of the Frenet-Serret formulas. We parameterize the curve by path-length s, and at each point on the curve define the scalar curvature κ, scalar torsion τ, and a local orthogonal frame consisting of the unit tangent $\vec{u}$, normal $\vec{n}$ and binormal $\vec{b}$. The curvature measures the deviation of the curve from a straight line, and the torsion measure the deviation of the curve from a plane. Two curves which have the same histories of $\kappa(s)$ and $\tau(s)$ are congruent, regardless of origin or attitude.

I prefer to use a local left hand Frenet frame of orthogonal vectors: tangent $\vec{u}$, normal $\vec{n}$, and binormal $\vec{b} = \vec{n} \times \vec{u}$.

The left hand coordinates have a simpler sign convention, especially in higher dimensions, than a right hand coordinate frame. Notice that ds, u and higher terms are independent of the choice of an origin; these are locally defined expressions.

Using a left-handed local frame with unit tangent $\vec{u}$, unit normal $\vec{n}$, and unit binormal $\vec{b}$, the Frenet formulas in three space (not geometric algebra) are

$$
\begin{aligned}
(ds)^2 &= d\vec{r} \cdot d\vec{r} = dx^2 + dy^2 + dz^2 \\
\frac{d\vec{r}}{ds} &= \vec{u} \\
\frac{d\vec{u}}{ds} &= \kappa \vec{n} \\
\frac{d\vec{n}}{ds} &= \tau \vec{b} - \kappa \vec{u} \\
\frac{d\vec{b}}{ds} &= -\tau \vec{n}
\end{aligned}
$$

Numerically generating curves from this set of differential equations is easy, using Runge-Kutta or better numerical integrators.

Commonly, we generate curves using time integration of velocity, acceleration and jerk. Our connection to length based parameterization is simply $ds = v\ dt$.

Using time based derivatives, we can express curvature as a vector as

$$\vec{\kappa} = \frac{\vec{a} \times \vec{v}}{v^3} = \kappa \vec{n}$$

We can express scalar torsion as

$$\tau = \frac{(\vec{j} \cdot (\vec{a} \times \vec{v}))}{(\vec{a} \times \vec{v}) \cdot (\vec{a} \times \vec{v})}$$

where v is the scalar velocity, $\vec{v}$ is the vector velocity, $\vec{a}$ is the vector acceleration, and $\vec{j}$ is the vector jerk.

Curvature and Torsion In Geometric Algebra

We can do the same process in geometric algebra, with slightly different definitions. In the formulas below, I will use a Cartesian orthgonal three space with basis vector e_x, e_y and e_z.

As in the standard approach, we define a differential pathlength along the curve.

$$ds^2 = dx^2 + dy^2 + dz^2$$

Tangent

We define a unit tangent

$$\vec{u} = \frac{dx}{ds}e_x + \frac{dy}{ds}e_y + \frac{dz}{ds}e_z$$

We can define a vector differential for ds using this unit tangent.

$$d\vec{s} = \vec{u}ds$$

Because the unit tangent has a constant length of one, the derivative of length squared is zero, and the unit tangent and its derivative are orthogonal.

$$\begin{aligned} \vec{u} \cdot \vec{u} &= 1 \\ \frac{d}{ds}(\vec{u} \cdot \vec{u}) &= 2\vec{u} \cdot \frac{d\vec{u}}{ds} = 0 \\ \vec{u} &\perp \frac{d\vec{u}}{ds} \end{aligned}$$

In geometric algebra, the product of two vectors is the sum of the dot product and wedge product.

$$\vec{a}\vec{b} = \vec{a} \cdot \vec{b} + \vec{a} \wedge \vec{b}$$

For the special case of orthogonal vectors, such as $\vec{u}$ and $d\vec{u}/ds$, the geometric product and wedge product coincide. We will preferentially use the wedge product in these notes.

Curvature

Taking the wedge product of $\vec{u}$ and $d\vec{u}/ds$, we find the curvature bivector

$$\vec{u} \wedge \frac{d\vec{u}}{ds} = \vec{u}\frac{d\vec{u}}{ds} = \kappa(\vec{u} \wedge \vec{n}) = \kappa\vec{u}\vec{n}$$

or, simply, the bivector curvature is

$$\bar{\bar{\kappa}} = \vec{u} \wedge \frac{d\vec{u}}{ds}$$

.

For a circle, the radius of curvature is the inverse of the curvature, $\rho = 1/\kappa$. The incremental distance along a circular arc is related to angle and radius, $ds = \rho d\theta$. We can invert this equation, to find

$$\begin{aligned} ds &= \rho d\theta \\ d\theta &= \frac{1}{\rho} ds = \kappa ds \end{aligned}$$

When we generalize to geometric algebra, we find that angles become bivectors in the plane of rotation. I will use double overbars to indicate bivectors, here.

$$d\bar{\bar{\theta}} = \bar{\bar{\kappa}} ds = \kappa \vec{u}\vec{n} ds$$

It is important to point out that κ and ρ are locally defined in terms of differential elements, and are not tied to any particular coordinate zero location.

We now look at the interesting unit bivector element expression inspired by distance along arc length,

$$\begin{aligned} d\bar{\bar{s}} &= \rho \ d\bar{\bar{\theta}} \\ &= \rho \ \kappa \ \vec{u} \ \vec{n} \ ds \\ &= \vec{u} \ \vec{n} \ ds \end{aligned}$$

We see that this unit bivector associated with curvature is similar to the unit tangent for linear terms.

Torsion

The triple wedge product

$$\vec{u} \wedge \frac{d\vec{u}}{ds} \wedge \frac{d^2\vec{u}}{ds^2} = \kappa^2 \ \tau \ \vec{u} \ \vec{n} \ \vec{b}$$

defines geometric torsion. In three-space, torsion is a single pseudoscalar.

Frenet Frame Recovery

In standard three dimensional space, the three unit vectors $\vec{u}$, $\vec{n}$ and $\vec{b}$ define a local coordinate system at each space along the curve. We have $\vec{u}$ from the unit tangent, and can recover $\vec{n}$ and $\vec{b}$ by use of the geometric product. Start with

$$\vec{u} \wedge \frac{d\vec{u}}{ds} = \kappa \vec{u}\vec{n}$$

Since $\vec{u}$ and $d\vec{u}/ds$ are orthogonal, the geometric and wedge products coincide.

$$\vec{u} \wedge \frac{d\vec{u}}{ds} = \vec{u}\frac{d\vec{u}}{ds} = \kappa \vec{u}\vec{n}$$

Geometrically premultiply by the unit tangent, and find

$$\vec{u}\left(\vec{u}\frac{d\vec{u}}{ds}\right) = \frac{d\vec{u}}{ds} = \kappa \vec{n}$$

and the result

$$\vec{n} = \frac{1}{\kappa}\frac{d\vec{u}}{ds}$$

which matches the standard result, and does not require wedge or geometric products.

From the torsion trivector, we have

$$\vec{u} \wedge \frac{d\vec{u}}{ds} \wedge \frac{d^2\vec{u}}{ds^2} = \kappa^2 \ \tau \ \vec{u} \ \vec{n} \ \vec{b}$$

Geometrically premultiply by the unit tangent, followed by the normal, and find

$$\kappa^2 \ \tau \ \vec{b} = \vec{n} \ \vec{u} \left(\vec{u} \wedge \frac{d\vec{u}}{ds} \wedge \frac{d^2\vec{u}}{ds^2}\right)$$

Examples

We now do a few examples.

Straight Line

A line has a tangent, but no curvature or torsion. Parameterize the line by $x = s$, $y = 0$, and $z = 0$. We find our unit tangent components.

$$\begin{aligned} u_x &= \frac{dx}{ds} = 1 \\ u_y &= 0 \\ u_z &= 0 \end{aligned}$$

We find that the second derivative and higher derivatives with respect to s are zero. Consequently,

$$\begin{aligned} \frac{d\vec{u}}{ds} &= 0 \\ \bar{\bar{\kappa}} &= \vec{u} \wedge \frac{d\vec{u}}{ds} = 0 \end{aligned}$$

Circle in XY Plane

We can describe a circle in the XY plane by

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= 0 \\ ds &= r\ d\theta \end{aligned}$$

We have the unit tangent components

$$\begin{aligned} dx &= -r\sin\theta\ d\theta \\ dy &= r\cos\theta\ d\theta \\ dz &= 0 \end{aligned}$$

$$\begin{aligned} u_x &= \frac{dx}{ds} = -\sin\theta \\ u_y &= \frac{dy}{ds} = \cos\theta \\ u_z &= 0 \end{aligned}$$

We calculate our unit tangent derivatives with respect to s

$$\begin{aligned} \frac{du_x}{ds} &= \frac{-\cos\theta}{r} \\ \frac{du_y}{ds} &= \frac{-\sin\theta}{r} \\ \frac{du_z}{ds} &= 0 \end{aligned}$$

We do our wedge product, remembering that squared basis zero out, and that $a_{xy} = -a_{yx}$,

$$\begin{aligned}
\bar{\bar{\kappa}} &= \vec{u} \wedge \frac{d\vec{u}}{ds} \\
&= (-\sin\theta \ a_x + \cos\theta \ a_y) \wedge \left(-\frac{\cos\theta}{r} \ a_x - \frac{\sin\theta}{r} \ a_y \right) \\
&= \frac{\sin^2\theta}{r} a_{xy} - \frac{\cos^2\theta}{r} a_{yx} = \frac{1}{r} \ a_{xy}
\end{aligned}$$

We see that the magnitude of the curvature is the inverse of the radius, as expected for a circle.

Doing the second unit tangent derivatives with respect to s, we have

$$\begin{aligned}
\frac{d^2 u_x}{ds^2} &= \frac{\sin\theta}{r^2} \\
\frac{d^2 u_y}{ds^2} &= \frac{-\cos\theta}{r^2} \\
\frac{d^2 u_z}{ds^2} &= 0
\end{aligned}$$

The torsion related wedge product is zero due to repeated basis vector wedge products.

$$\begin{aligned}
\vec{u} \wedge \frac{d\vec{u}}{ds} \wedge \frac{d^2\vec{u}}{ds^2} &= \left(\vec{u} \wedge \frac{d\vec{u}}{ds} \right) \wedge \frac{d^2\vec{u}}{ds^2} = \kappa^2 \tau \vec{u}\vec{n}\vec{b} \\
&= \frac{1}{r} \ a_{xy} \wedge \frac{d^2\vec{u}}{ds^2} \\
&= \frac{1}{r} \ a_{xy} \wedge \left(\frac{\sin\theta}{r^2} \ a_x - \frac{\cos\theta}{r^2} a_y \right) = 0
\end{aligned}$$

Consequently, I am happy setting $\tau = 0$. For our Frenet frame, we have the unit tangent and normal defined.

$$\begin{aligned}
\vec{u} &= -\sin\theta \ a_x + \cos\theta \ a_y \\
\vec{n} &= -\cos\theta \ a_x - \sin\theta \ a_y
\end{aligned}$$

Spiral

The spiral has constant curvature and torsion. We can parameterize a spiral by

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= \alpha\theta \end{aligned}$$

Our derivatives are

$$\begin{aligned} dx &= -r\sin\theta \; d\theta \\ dy &= r\cos\theta \; d\theta \\ dz &= \alpha \; d\theta \\ ds &= \sqrt{r^2+\alpha^2} \; d\theta \end{aligned}$$

Our unit tangent components are

$$\begin{aligned} u_x &= \frac{-r\sin\theta}{\sqrt{r^2+\alpha^2}} \\ u_y &= \frac{r\cos\theta}{\sqrt{r^2+\alpha^2}} \\ u_z &= \frac{\alpha}{\sqrt{r^2+\alpha^2}} \end{aligned}$$

Our derivatives of the unit tangent components with respect to s are

$$\begin{aligned} \frac{du_x}{ds} &= \frac{-r\cos\theta}{r^2+\alpha^2} \\ \frac{du_y}{ds} &= \frac{-r\sin\theta}{r^2+\alpha^2} \\ \frac{du_z}{ds} &= 0 \end{aligned}$$

Our normal vector is

$$\vec{n} = -\cos\theta \; a_x - \sin\theta \; a_y$$

Our curvature related wedge is

$$
\begin{aligned}
\bar{\bar{\kappa}} &= \vec{u} \wedge \frac{d\vec{u}}{ds} \\
&= \left(\frac{-r\sin\theta}{\sqrt{r^2+\alpha^2}} a_x + \frac{r\cos\theta}{\sqrt{r^2+\alpha^2}} a_y + \frac{\alpha}{\sqrt{r^2+\alpha^2}} a_z \right) \wedge \left(\frac{-r\cos\theta}{r^2+\alpha^2} a_x + \frac{-r\sin\theta}{r^2+\alpha^2} a_y \right) \\
&= \frac{r}{(r^2+\alpha^2)^{3/2}} \left(r a_{xy} - \alpha\cos\theta \; a_{zx} + \alpha\sin\theta \; a_{yz} \right)
\end{aligned}
$$

This has magnitude

$$
\kappa = \frac{r}{r^2+\alpha^2}
$$

consistent with the magnitude of $d\vec{u}/ds$. Now I want to look at the bivector components of curvature. In the xy plane, we have a constant curvature component which is a bit less than the total curvature.

$$
\kappa_{xy} = \frac{r^2}{(r^2+\alpha^2)^{3/2}}
$$

The other two components of curvature are complementary harmonic content.

$$
\begin{aligned}
\kappa_{yz} &= +\frac{\alpha r\sin\theta}{(r^2+\alpha^2)^{3/2}} \\
\kappa_{zx} &= -\frac{\alpha r\cos\theta}{(r^2+\alpha^2)^{3/2}}
\end{aligned}
$$

The total curvature is constant. These variable components reflect the changing direction of the bivector curvature as we advance along the spiral.

Our second derivative components of the tangent with respect to s are

$$
\begin{aligned}
\frac{d^2u_x}{ds^2} &= \frac{+r\sin\theta}{(r^2+\alpha^2)^{3/2}} \\
\frac{d^2u_y}{ds^2} &= \frac{-r\cos\theta}{(r^2+\alpha^2)^{3/2}} \\
\frac{d^2u_z}{ds^2} &= 0
\end{aligned}
$$

Our torsion related wedge product is

$$\begin{aligned}
\kappa^2 \tau \vec{u}\vec{n}\vec{b} &= \vec{u} \wedge \frac{d\vec{u}}{ds} \wedge \frac{d^2\vec{u}}{ds^2} \\
&= \left(\vec{u} \wedge \frac{d\vec{u}}{ds} \right) \wedge \frac{d^2\vec{u}}{ds^2} \\
&= \frac{r}{(r^2+\alpha^2)^{3/2}} \left(r a_{xy} - \alpha \cos\theta \; a_{zx} + \alpha \sin\theta \; a_{yz} \right) \wedge \frac{r}{(r^2+\alpha^2)^{3/2}} \left(\sin\theta a_x - \cos\theta a_y \right) \\
&= \frac{r^2}{(r^2+\alpha^2)^3} \left(\alpha \cos^2\theta \; a_{xyz} + \alpha \sin^2\theta \; a_{xyz} \right) \\
&= \frac{r^2 \alpha}{(r^2+\alpha^2)^3} a_{xyz}
\end{aligned}$$

In three-space, the trivector $\vec{u}\vec{n}\vec{b} = a_{xyz}$. We know $\kappa = r/(r^2+\alpha^2)$. From this, we find

$$\tau = \frac{\alpha}{r^2+\alpha^2}$$

Fun Relationships for Spirals

We have

$$\kappa = \frac{r}{r^2+\alpha^2}$$
$$\tau = \frac{\alpha}{r^2+\alpha^2}$$
$$\kappa^2 + \tau^2 = \frac{1}{r^2+\alpha^2}$$
$$\left(\kappa^2+\tau^2\right)\left(r^2+\alpha^2\right) = 1$$

We can interpret this as showing a relationship between compound curvature and compound radii.

Four Dimensional Curves of Constant Curvatures

Curves of constant curvature κ, torsion τ and lift γ trace curves on the surface of a hypersphere with fixed radius.

$$R^2 = \frac{\tau^2+\gamma^2}{\kappa^2\gamma^2}$$

The trajectory parameterized by pathlength (self-history) has two frequency components and two orthogonal radii.

$$\omega_1 = \sqrt{\frac{(\kappa^2+\tau^2+\gamma^2)+\sqrt{(\kappa^2+\tau^2+\gamma^2)^2-4\kappa^2\gamma^2}}{2}}$$
$$\omega_2 = \sqrt{\frac{(\kappa^2+\tau^2+\gamma^2)-\sqrt{(\kappa^2+\tau^2+\gamma^2)^2-4\kappa^2\gamma^2}}{2}}$$

$$r_1 = \frac{\sqrt{(\kappa^2-\omega_2^2)^2+\kappa^2\tau^2}}{\omega_1(\omega_1^2-\omega_2^2)} = \frac{1}{\omega_1}\sqrt{\frac{\kappa^2-\omega_2^2}{\omega_1^2-\omega_2^2}}$$
$$r_2 = \frac{\sqrt{(\kappa^2-\omega_1^2)^2+\kappa^2\tau^2}}{\omega_2(\omega_1^2-\omega_2^2)} = \frac{1}{\omega_2}\sqrt{\frac{\omega_1^2-\kappa^2}{\omega_1^2-\omega_2^2}}$$

We have the relationship

$$(r_1\omega_1)^2+(r_2\omega_2)^2 = 1$$

Given three curvatures, we can calculate frequencies, then radii, and we can then align our coordinates system for easy calculation of our basis vectors.

$$x = r_1\cos(\omega_1 s)$$
$$y = r_1\sin(\omega_1 s)$$
$$z = r_2\cos(\omega_2 s)$$
$$t = r_2\sin(\omega_2 s)$$

$$\tilde{u} = \frac{d\tilde{r}}{ds}$$
$$u_x = -r_1\omega_1\sin(\omega_1 s)$$
$$u_y = r_1\omega_1\cos(\omega_1 s)$$
$$u_z = -r_2\omega_2\sin(\omega_2 s)$$
$$u_t = r_2\omega_2\cos(\omega_2 s)$$

$$
\begin{aligned}
\tilde{n} &= \frac{1}{\kappa}\frac{d\tilde{u}}{ds} \\
n_x &= \left(-r_1\omega_1^2/\kappa\right)\cos(\omega_1 s) \\
n_y &= \left(-r_1\omega_1^2/\kappa\right)\sin(\omega_1 s) \\
n_z &= \left(-r_2\omega_2^2/\kappa\right)\cos(\omega_2 s) \\
n_t &= \left(-r_2\omega_2^2/\kappa\right)\sin(\omega_2 s)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{b} &= \frac{1}{\tau}\left(\frac{d\tilde{n}}{ds} + \kappa\tilde{u}\right) \\
b_x &= \left[\left(r_1\omega_1^3/\kappa - \kappa\omega_1 r_1\right)/\tau\right]\sin(\omega_1 s) \\
b_y &= \left[\left(-r_1\omega_1^3/\kappa + \kappa\omega_1 r_1\right)/\tau\right]\cos(\omega_1 s) \\
b_z &= \left[\left(r_2\omega_2^3/\kappa - \kappa\omega_2 r_2\right)/\tau\right]\sin(\omega_2 s) \\
b_t &= \left[\left(-r_2\omega_2^3/\kappa + \kappa\omega_2 r_2\right)/\tau\right]\cos(\omega_2 s)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{w} &= \frac{1}{\gamma}\left(\frac{d\tilde{b}}{ds} + \tau\tilde{n}\right) \\
w_x &= \left[\left(\left[\left(r_1\omega_1^4/\kappa - \kappa\omega_1^2 r_1\right)/\tau\right] - \tau\omega_1^2 r_1/\kappa\right)/\gamma\right]\cos(\omega_1 s) \\
w_y &= \left[\left(\left[\left(r_1\omega_1^4/\kappa - \kappa\omega_1^2 r_1\right)/\tau\right] - \tau\omega_1^2 r_1/\kappa\right)/\gamma\right]\sin(\omega_1 s) \\
w_z &= \left[\left(\left[\left(r_2\omega_2^4/\kappa - \kappa\omega_2^2 r_2\right)/\tau\right] - \tau\omega_2^2 r_2/\kappa\right)/\gamma\right]\cos(\omega_2 s) \\
w_t &= \left[\left(\left[\left(r_2\omega_2^4/\kappa - \kappa\omega_2^2 r_2\right)/\tau\right] - \tau\omega_2^2 r_2/\kappa\right)/\gamma\right]\sin(\omega_2 s)
\end{aligned}
$$

Our four dimensional multivector has one scalar component, four vector components, six bivector components, four trivector components, and one pseudo-scalar component. For this example, I will use $\kappa = 3$, $\tau = 4$, $\gamma = 5$, as for this Pythagorean set of values, the trajectories integrate into a filament, as the two frequencies are harmonically locked. For more details, see my note "ThreeCurvature.pdf" at `http://www.kurtnalty.com/ThreeCurvatures.pdf`.

For these curvature values, $\kappa = 3$, $\tau = 4$, $\gamma = 5$, we have frequencies $\omega_1 = 6.7082$, $\omega_2 = 2.23607$ with a 3:1 ratio, and $r_1 = 0.0471405$ and $r_2 = 0.424264$ with a 1:9 ratio. Figure 1 shows a projection of the trajectory from four down to two dimensions illustrating the filamentary solution.

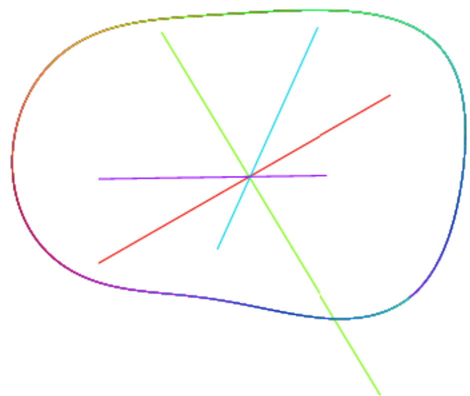

Figure 1: 3000 Points of 345 Solution

As pointed out earlier, the tangent components are

$$\begin{aligned}
\tilde{u} &= \frac{d\tilde{r}}{ds} \\
u_x &= -r_1\omega_1 \sin(\omega_1 s) \\
u_y &= r_1\omega_1 \cos(\omega_1 s) \\
u_z &= -r_2\omega_2 \sin(\omega_2 s) \\
u_t &= r_2\omega_2 \cos(\omega_2 s)
\end{aligned}$$

The pathlength derivative of $\tilde{u}$ is

$$\begin{aligned}
\frac{du_x}{ds} &= -r_1\omega_1^2 \cos(\omega_1 s) \\
\frac{du_y}{ds} &= -r_1\omega_1^2 \sin(\omega_1 s) \\
\frac{du_z}{ds} &= -r_2\omega_2^2 \cos(\omega_2 s) \\
\frac{du_t}{ds} &= -r_2\omega_2^2 \sin(\omega_2 s)
\end{aligned}$$

The curvature bivector has six components

$$\begin{aligned}
\kappa_{xy} &= \omega_1^3 r_1^2 \\
\kappa_{xz} &= \omega_1\omega_2 r_1 r_2 \left(+\omega_2 \sin(\omega_1 s)\cos(\omega_2 s) - \omega_1 \cos(\omega_1 s)\sin(\omega_2 s)\right) \\
\kappa_{yz} &= \omega_1\omega_2 r_1 r_2 \left(-\omega_2 \cos(\omega_1 s)\cos(\omega_2 s) - \omega_1 \sin(\omega_1 s)\sin(\omega_2 s)\right) \\
\kappa_{xt} &= \omega_1\omega_2 r_1 r_2 \left(+\omega_2 \sin(\omega_1 s)\sin(\omega_2 s) + \omega_1 \cos(\omega_1 s)\cos(\omega_2 s)\right) \\
\kappa_{yt} &= \omega_1\omega_2 r_1 r_2 \left(-\omega_2 \cos(\omega_1 s)\sin(\omega_2 s) + \omega_1 \sin(\omega_1 s)\cos(\omega_2 s)\right) \\
\kappa_{zt} &= \omega_2^3 r_2^2
\end{aligned}$$

The primary planes of curvature, namely XY and ZT, have a constant component of curvature, just as was the case for the spiral. The four other planes have changing components with pathlength, and the overall curvature is equal to κ.

Figure 2 shows the XT and YZ curvature components, while Figure 3 shows the XZ and YT components. Notice the various reverse symmetries here.

We now want to calculate the torsional related wedge product.

$$\vec{u} \wedge \frac{d\vec{u}}{ds} \wedge \frac{d^2\vec{u}}{ds^2} = \kappa^2 \tau \vec{u}\vec{n}\vec{b}$$

Our second derivative with respect to path length is

$$\begin{aligned}
\frac{d^2 u_x}{ds^2} &= r_1 \omega_1^3 \sin(\omega_1 s) \\
\frac{d^2 u_y}{ds^2} &= -r_1 \omega_1^3 \cos(\omega_1 s) \\
\frac{d^2 u_z}{ds^2} &= r_2 \omega_2^3 \sin(\omega_2 s) \\
\frac{d^2 u_t}{ds^2} &= -r_2 \omega_2^3 \cos(\omega_2 s)
\end{aligned}$$

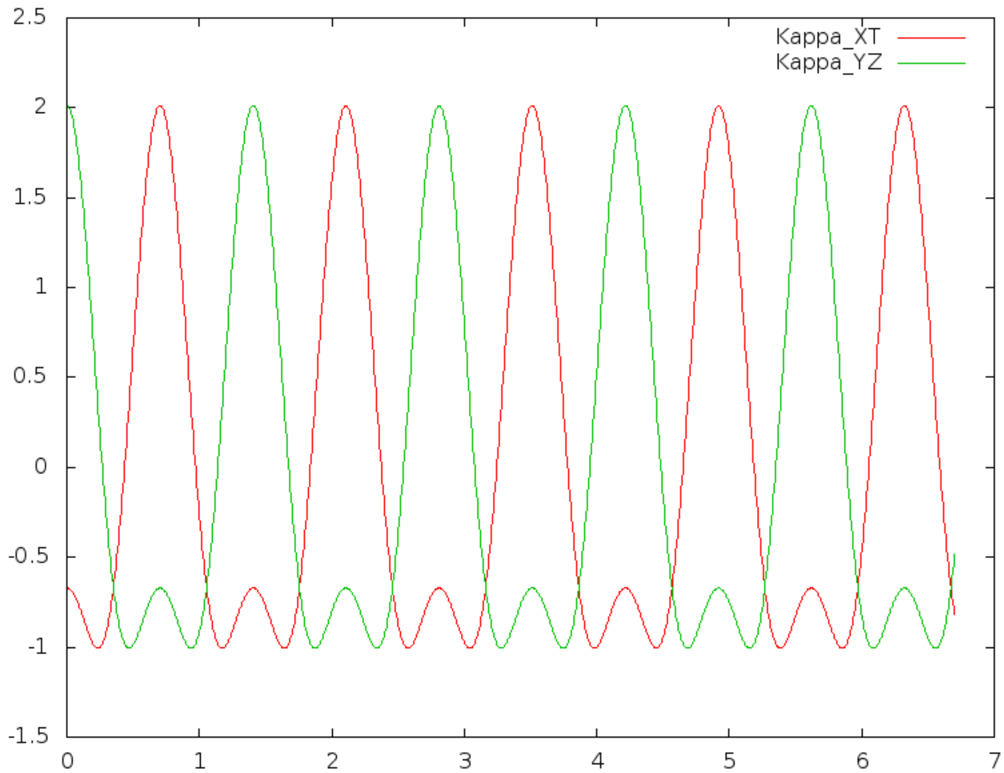

Figure 2: XT and YZ Curvature Components Versus Pathlength

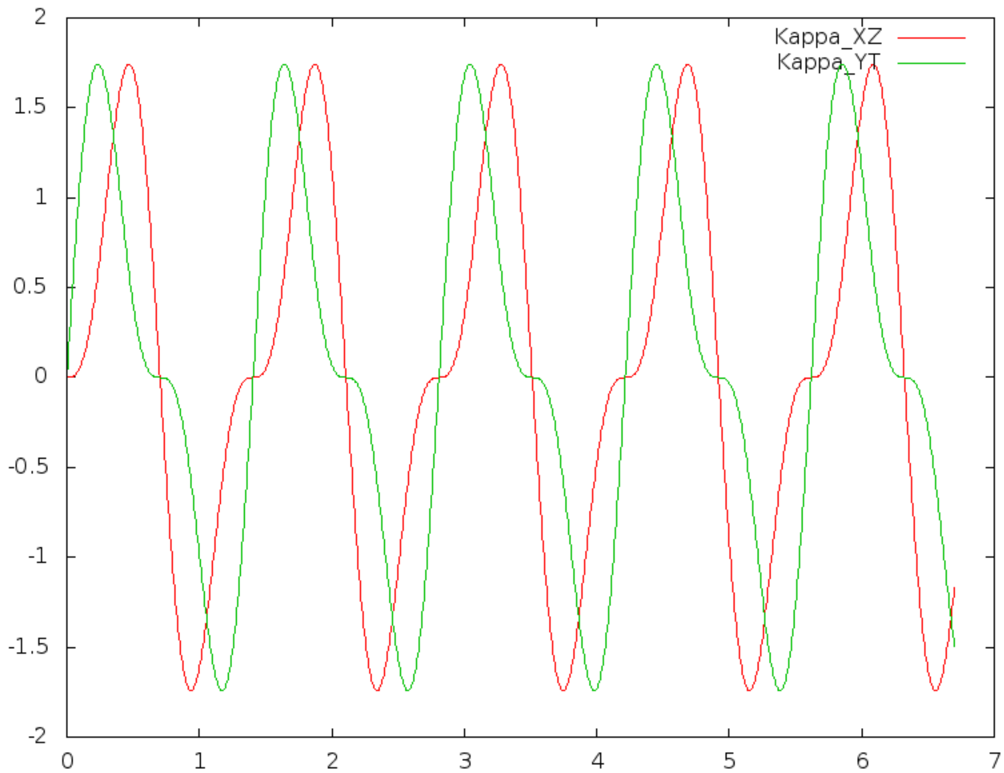

Figure 3: XZ and YT Curvature Components Versus Pathlength

The curvature bivector $\vec{u} \wedge \vec{u}'$ has six components

$$
\begin{aligned}
\kappa_{xy} &= \omega_1^3 r_1^2 \\
\kappa_{xz} &= \omega_1\omega_2 r_1 r_2 \left(+\omega_2 \sin(\omega_1 s)\cos(\omega_2 s) - \omega_1 \cos(\omega_1 s)\sin(\omega_2 s)\right) \\
\kappa_{yz} &= \omega_1\omega_2 r_1 r_2 \left(-\omega_2 \cos(\omega_1 s)\cos(\omega_2 s) - \omega_1 \sin(\omega_1 s)\sin(\omega_2 s)\right) \\
\kappa_{xt} &= \omega_1\omega_2 r_1 r_2 \left(+\omega_2 \sin(\omega_1 s)\sin(\omega_2 s) + \omega_1 \cos(\omega_1 s)\cos(\omega_2 s)\right) \\
\kappa_{yt} &= \omega_1\omega_2 r_1 r_2 \left(-\omega_2 \cos(\omega_1 s)\sin(\omega_2 s) + \omega_1 \sin(\omega_1 s)\cos(\omega_2 s)\right) \\
\kappa_{zt} &= \omega_2^3 r_2^2
\end{aligned}
$$

Wedging these four components against the six components of the curvature bivector, yields 24 terms, only 12 of which are non-zero. The torsion trivector will suffer a bit of abuse of notation here. The magnitude of the torsion trivector is really $\kappa^2\tau$. I will place a factor of κ^2 in front of the τ components to reflect this fact.

$$
\begin{aligned}
\kappa^2\tau_{xyz} &= \kappa_{xy}\frac{d^2u_z}{ds^2} - \kappa_{xz}\frac{d^2u_y}{ds^2} + \kappa_{yz}\frac{d^2u_x}{ds^2} \\
\kappa^2\tau_{xyt} &= \kappa_{xy}\frac{d^2u_t}{ds^2} - \kappa_{xt}\frac{d^2u_y}{ds^2} + \kappa_{yt}\frac{d^2u_x}{ds^2} \\
\kappa^2\tau_{xzt} &= \kappa_{xz}\frac{d^2u_t}{ds^2} - \kappa_{xt}\frac{d^2u_z}{ds^2} + \kappa_{zt}\frac{d^2u_x}{ds^2} \\
\kappa^2\tau_{yzt} &= \kappa_{yz}\frac{d^2u_t}{ds^2} - \kappa_{yt}\frac{d^2u_z}{ds^2} + \kappa_{yt}\frac{d^2u_x}{ds^2}
\end{aligned}
$$

From the expression above, we would expect a lot of harmonic content. Interestingly enough, most sines and cosines get absorbed, leaving a simple harmonic term for each component above. Doing the calculation for $\kappa^2\tau_{xyz}$,

$$
\begin{aligned}
\kappa^2\tau_{xyz} &= +\omega_1^3\omega_2^3 r_1^2 r_2 \sin(\omega_2 s) \\
&\quad +\omega_1^4\omega_2 r_1^2 r_2 \cos(\omega_1 s)\left(\omega_2 \sin(\omega_1 s)\cos(\omega_2 s) - \omega_1 \cos(\omega_1 s)\sin(\omega_2 s)\right) \\
&\quad +\omega_1^4\omega_2 r_1^2 r_2 \sin(\omega_1 s)\left(-\omega_2 \cos(\omega_1 s)\cos(\omega_2 s) - \omega_1 \sin(\omega_1 s)\sin(\omega_2 s)\right) \\
&= +\omega_1^3\omega_2^3 r_1^2 r_2 \sin(\omega_2 s) \\
&\quad +\omega_1^4\omega_2 r_1^2 r_2 \cos(\omega_1 s)\left(-\omega_1 \cos(\omega_1 s)\sin(\omega_2 s)\right) \\
&\quad +\omega_1^4\omega_2 r_1^2 r_2 \sin(\omega_1 s)\left(-\omega_1 \sin(\omega_1 s)\sin(\omega_2 s)\right) \\
&= +\omega_1^3\omega_2^3 r_1^2 r_2 \sin(\omega_2 s) \\
&\quad +\omega_1^4\omega_2 r_1^2 r_2 \left(-\omega_1 \sin(\omega_2 s)\right) \\
&= +\omega_1^3\omega_2^3 r_1^2 r_2 \sin(\omega_2 s) \\
&\quad -\omega_1^5\omega_2 r_1^2 r_2 \sin(\omega_2 s) \\
&= +(\omega_1^3\omega_2^3 - \omega_1^5\omega_2) r_1^2 r_2 \sin(\omega_2 s)
\end{aligned}
$$

In a similar fashion,

$$
\begin{aligned}
\kappa^2\tau_{xyt} &= -\omega_1^3\omega_2^3 r_1^2 r_2 \cos(\omega_2 s) \\
&\quad +\omega_1^4\omega_2 r_1^2 r_2 \cos(\omega_1 s)\left(\omega_2 \sin(\omega_1 s)\sin(\omega_2 s) + \omega_1 \cos(\omega_1 s)\cos(\omega_2 s)\right) \\
&\quad +\omega_1^4\omega_2 r_1^2 r_2 \sin(\omega_1 s)\left(-\omega_2 \cos(\omega_1 s)\sin(\omega_2 s) + \omega_1 \sin(\omega_1 s)\cos(\omega_2 s)\right) \\
&= -\omega_1^3\omega_2^3 r_1^2 r_2 \cos(\omega_2 s) \\
&\quad +\omega_1^4\omega_2 r_1^2 r_2 \cos(\omega_1 s)\left(\omega_1 \cos(\omega_1 s)\cos(\omega_2 s)\right) \\
&\quad +\omega_1^4\omega_2 r_1^2 r_2 \sin(\omega_1 s)\left(\omega_1 \sin(\omega_1 s)\cos(\omega_2 s)\right) \\
&= -\omega_1^3\omega_2^3 r_1^2 r_2 \cos(\omega_2 s) \\
&\quad +\omega_1^5\omega_2 r_1^2 r_2 \cos(\omega_2 s) \\
&= -(\omega_1^3\omega_2^3 - \omega_1^5\omega_2) r_1^2 r_2 \cos(\omega_2 s)
\end{aligned}
$$

Now the next component

$$\begin{aligned}
\kappa^2\tau_{xzt} &= -\omega_1\omega_2^4 r_1 r_2^2 \cos(\omega_2 s)\,(+\omega_2 \sin(\omega_1 s)\cos(\omega_2 s) - \omega_1 \cos(\omega_1 s)\sin(\omega_2 s)) \\
&\quad -\omega_1\omega_2^4 r_1 r_2^2 \sin(\omega_2 s)\,(+\omega_2 \sin(\omega_1 s)\sin(\omega_2 s) + \omega_1 \cos(\omega_1 s)\cos(\omega_2 s)) \\
&\quad +\omega_1^3\omega_2^3 r_1 r_2^2 \sin(\omega_1 s) \\
&= -\omega_1\omega_2^5 r_1 r_2^2 \cos(\omega_2 s)\,(+\sin(\omega_1 s)\cos(\omega_2 s)) \\
&\quad -\omega_1\omega_2^5 r_1 r_2^2 \sin(\omega_2 s)\,(+\sin(\omega_1 s)\sin(\omega_2 s)) \\
&\quad +\omega_1^3\omega_2^3 r_1 r_2^2 \sin(\omega_1 s) \\
&= -\omega_1\omega_2^5 r_1 r_2^2 \sin(\omega_1 s) \\
&\quad +\omega_1^3\omega_2^3 r_1 r_2^2 \sin(\omega_1 s) \\
&= (\omega_1^3\omega_2^3 - \omega_1\omega_2^5) r_1 r_2^2 \sin(\omega_1 s)
\end{aligned}$$

Now we finish up

$$\begin{aligned}
\kappa^2\tau_{yzt} &= +\omega_1\omega_2^4 r_1 r_2^2 \cos(\omega_2 s)\,(+\omega_2 \cos(\omega_1 s)\cos(\omega_2 s) + \omega_1 \sin(\omega_1 s)\sin(\omega_2 s)) \\
&\quad +\omega_1\omega_2^4 r_1 r_2^2 \sin(\omega_2 s)\,(+\omega_2 \cos(\omega_1 s)\sin(\omega_2 s) - \omega_1 \sin(\omega_1 s)\cos(\omega_2 s)) \\
&\quad -\omega_1^3\omega_2^3 r_1 r_2^2 \cos(\omega_1 s) \\
&= +\omega_1\omega_2^4 r_1 r_2^2 \cos(\omega_2 s)\,(+\omega_2 \cos(\omega_1 s)\cos(\omega_2 s)) \\
&\quad +\omega_1\omega_2^4 r_1 r_2^2 \sin(\omega_2 s)\,(+\omega_2 \cos(\omega_1 s)\sin(\omega_2 s)) \\
&\quad -\omega_1^3\omega_2^3 r_1 r_2^2 \cos(\omega_1 s) \\
&= +\omega_1\omega_2^5 r_1 r_2^2 \cos(\omega_1 s) - \omega_1^3\omega_2^3 r_1 r_2^2 \cos(\omega_1 s) \\
&= (\omega_1\omega_2^5 r_1 r_2^2 - \omega_1^3\omega_2^3 r_1 r_2^2) \cos(\omega_1 s) \\
&= (-\omega_1^3\omega_2^3 + \omega_1\omega_2^5) r_1 r_2^2 \cos(\omega_1 s)
\end{aligned}$$

Summarizing, the components for $\vec{u} \wedge d\vec{u}/ds$ are

$$\begin{aligned}
\kappa^2\tau_{xyz} &= (+\omega_1^3\omega_2^3 - \omega_1^5\omega_2) r_1^2 r_2 \sin(\omega_2 s) \\
\kappa^2\tau_{xyt} &= (-\omega_1^3\omega_2^3 + \omega_1^5\omega_2) r_1^2 r_2 \cos(\omega_2 s) \\
\kappa^2\tau_{xzt} &= (+\omega_1^3\omega_2^3 - \omega_1\omega_2^5) r_1 r_2^2 \sin(\omega_1 s) \\
\kappa^2\tau_{yzt} &= (-\omega_1^3\omega_2^3 + \omega_1\omega_2^5) r_1 r_2^2 \cos(\omega_1 s)
\end{aligned}$$

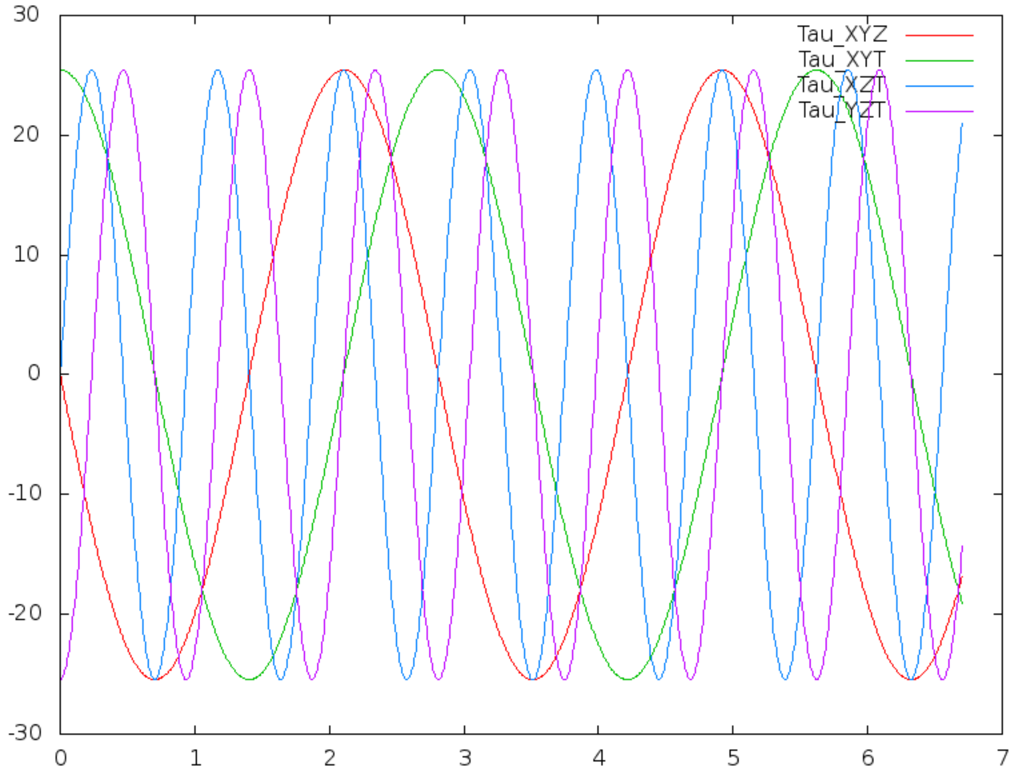

Figure 4: Torsion Components Versus Pathlength

Figure 4 shows the four torsional components versus pathlength. Notice the two frequencies, and quadrature within each frequency. The magnitude of the torsion is constant, at our expected value.

Our last calculation is lift. In four dimensions, lift has only one component, in the a_{xyzt} direction. Our wedge formula for lift is

$$\vec{u} \wedge \frac{d\vec{u}}{ds} \wedge \frac{d^2\vec{u}}{ds^2} \wedge \frac{d^3\vec{u}}{ds^3} = \kappa^3 \tau^2 \gamma \; a_{xyzt}$$

Our third derivative of $\vec{u}$ is

$$\begin{aligned}
\frac{d^3u_x}{ds^3} &= r_1\omega_1^4\cos(\omega_1 s)\\
\frac{d^3u_y}{ds^3} &= r_1\omega_1^4\sin(\omega_1 s)\\
\frac{d^3u_z}{ds^3} &= r_2\omega_2^4\cos(\omega_2 s)\\
\frac{d^3u_t}{ds^3} &= r_2\omega_2^4\sin(\omega_2 s)
\end{aligned}$$

The components for $\vec{u}\wedge d\vec{u}/ds$ are

$$\begin{aligned}
\kappa^2\tau_{xyz} &= (+\omega_1^3\omega_2^3-\omega_1^5\omega_2)r_1^2r_2\sin(\omega_2 s)\\
\kappa^2\tau_{xyt} &= (-\omega_1^3\omega_2^3+\omega_1^5\omega_2)r_1^2r_2\cos(\omega_2 s)\\
\kappa^2\tau_{xzt} &= (+\omega_1^3\omega_2^3-\omega_1\omega_2^5)r_1r_2^2\sin(\omega_1 s)\\
\kappa^2\tau_{yzt} &= (-\omega_1^3\omega_2^3+\omega_1\omega_2^5)r_1r_2^2\cos(\omega_1 s)
\end{aligned}$$

When we do our wedge product, we will have only one component, with four terms.

$$(\kappa^2\tau_{xyz}a_{xyz}+\kappa^2\tau_{xyt}a_{xyt}+\kappa^2\tau_{xzt}a_{xzt}+\kappa^2\tau_{yzt}a_yzt)\wedge(a_x+\frac{d^3u_y}{ds^3}a_y+\frac{d^3u_z}{ds^3}a_z+\frac{d^3u_t}{ds^3}a_t)=$$

$$\left(\kappa^2\tau_{xyz}\frac{d^3u_t}{ds^3}-\kappa^2\tau_{xyt}\frac{d^3u_z}{ds^3}+\kappa^2\tau_{xzt}\frac{d^3u_y}{ds^3}-\kappa^2\tau_{yzt}\frac{d^3u_x}{ds^3}\right)a_{xyzt}$$

Substituting in our expressions, we find

$$\kappa^3\tau^2\gamma=(+\omega_1^3\omega_2^3-\omega_1^5\omega_2)r_1^2r_2^2\omega_2^4+(+\omega_1^3\omega_2^3-\omega_1\omega_2^5)r_1^2r_2^2\omega_1^4$$

$$=r_1^2r_2^2\omega_1^3\omega_2^3\left(\omega_1^2-\omega_2^2\right)^2$$

For our example values of $\kappa=3$, $\tau=4$ and $\gamma=5$, both sides of the above come to 2160, as expected.

A simple C program illustrating these examples is posted at `http://www.kurtnalty.com/GA_4D_Demo.c`