Foch and Ivanenko's Luminal Electron

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Abstract

David Delphenich `http://www.neo-classical-physics.info` has translated the 1929 paper *On a possible geometric interpretation of relativistic quantum theory* by V. Fock and D. Ivanenko. This paper provides a geometric interpretation of the Dirac equation, and points out that the expected velocity of the electron is always c. This note is an extract of their work.

Dirac Matrices are Directional Elements

Fock and Ivanenko identify the Dirac matrices γ_ν as directional elements. (This is the approach of Minkowski geometric algebra.) A generic fourspace vector $\vec{u}$ is given by

$$\vec{u} \;=\; \sum_{\nu=1}^{4} \vec{\gamma}_\nu u_\nu$$

Quoting the authors, "If we regard the components u_ν as numbers, then we have a geometric operator before us here whose eigenvalue, up to sign, coincides with the absolute value of the vector."

Let's examine this a bit.

In text format, one set of the Dirac matrices with signature (+,-,-,-) is

```
   Unity                   xyzt
[  1  0  0  0]          [  0  0  I  0]
[  0  1  0  0]          [  0  0  0  I]
[  0  0  1  0]          [  I  0  0  0]
[  0  0  0  1]          [  0  I  0  0]
```

```
        x              y              z              t
[  0  0  0  1] [  0  0  0 -I] [  0  0  1  0] [  1  0  0  0]
[  0  0  1  0] [  0  0  I  0] [  0  0  0 -1] [  0  1  0  0]
[  0 -1  0  0] [  0  I  0  0] [ -1  0  0  0] [  0  0 -1  0]
[ -1  0  0  0] [ -I  0  0  0] [  0  1  0  0] [  0  0  0 -1]

       xy             xz             yz             xt             yt             zt
[ -I  0  0  0] [  0  1  0  0] [  0 -I  0  0] [  0  0  0 -1] [  0  0  0  I] [  0  0 -1  0]
[  0  I  0  0] [ -1  0  0  0] [ -I  0  0  0] [  0  0 -1  0] [  0  0 -I  0] [  0  0  0  1]
[  0  0 -I  0] [  0  0  0  1] [  0  0  0 -I] [  0 -1  0  0] [  0  I  0  0] [ -1  0  0  0]
[  0  0  0  I] [  0  0 -1  0] [  0  0 -I  0] [ -1  0  0  0] [ -I  0  0  0] [  0  1  0  0]

       xyz            xyt            xzt            yzt
[  0  0 -I  0] [ -I  0  0  0] [  0  1  0  0] [  0 -I  0  0]
[  0  0  0 -I] [  0  I  0  0] [ -1  0  0  0] [ -I  0  0  0]
[  I  0  0  0] [  0  0  I  0] [  0  0  0 -1] [  0  0  0  I]
[  0  I  0  0] [  0  0  0 -I] [  0  0  1  0] [  0  0  I  0]
```

For a generic Minkowski multivector,

$$\begin{aligned}
&Ae_q + \\
&Be_x + Ce_y + De_z + Ee_t + \\
&Fe_{xy} + Ge_{xz} + He_{yz} + Je_{xt} + Ke_{yt} + Le_{zt} + \\
&Me_{xyz} + Ne_{xyt} + Pe_{xzt} + Re_{yzt} + \\
&Se_{xyzt}
\end{aligned}$$

the associated matrix representation is

```
[ +A+E-I*F-I*N & +G+P-I*H-I*R & +D-L-I*M+I*S & +B-J-I*C+I*K ]
[ -G-P-I*H-I*R & +A+E+I*F+I*N & +B-J+I*C-I*K & -D+L-I*M+I*S ]
[ -D-L+I*M+I*S & -B-J+I*C+I*K & +A-E-I*F+I*N & +G-P-I*H+I*R ]
[ -B-J-I*C-I*K & +D+L+I*M+I*S & -G+P-I*H+I*R & +A-E+I*F-I*N ]
```

For the specific case of a four-vector, we have the multivector

$$Be_x + Ce_y + De_z + Ee_t$$

and the associated matrix

$$\begin{pmatrix}
+E & 0 & +D & +B - I*C \\
0 & +E & +B + I*C & -D \\
-D & -B + I*C & -E & 0 \\
-B - I*C & +D & 0 & -E
\end{pmatrix}$$

I point out that B, C, D, and E are simply symbols with no pre-assigned meaning. They could be coordinates, or velocities, or whatever item is of interest. However, the spatial assignment aligns with the spacetime axis.

Independent of interpretation, this matrix will have a set of eigenvalues and eigenvectors. The eigen equation is

$$\begin{pmatrix} +E & 0 & +D & +B - I*C \\ 0 & +E & +B + I*C & -D \\ -D & -B + I*C & -E & 0 \\ -B - I*C & +D & 0 & -E \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \lambda \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

or

$$\begin{pmatrix} +E - \lambda & 0 & +D & +B - I*C \\ 0 & +E - \lambda & +B + I*C & -D \\ -D & -B + I*C & -E - \lambda & 0 \\ -B - I*C & +D & 0 & -E - \lambda \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

Taking the determinant of the left matrix to be zero yields the characteristic polynomial.

$$\begin{vmatrix} +E - \lambda & 0 & +D & +B - I*C \\ 0 & +E - \lambda & +B + I*C & -D \\ -D & -B + I*C & -E - \lambda & 0 \\ -B - I*C & +D & 0 & -E - \lambda \end{vmatrix} = 0$$

or

$$(\lambda^2 + B^2 + C^2 + D^2 - E^2)^2 = 0$$

We have a degenerate case, where the unique eigenvalues are

$$\lambda = \pm\sqrt{-B^2 - C^2 - D^2 + E^2}$$

Returning to Fock and Ivanenko's paper, they then examine the tangent fourspace differential distance.

$$d\vec{s} = \sum_{\nu=1}^{4} \vec{\gamma}_\nu dx_\nu$$

Letting $B = dx, C = dy, D = dz$, and $E = dt$, they see the eigenvectors for this differential distance is $\pm d\tau = \pm\sqrt{(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2}$. Dividing by proper time, they now have the unit tangent in spacetime

$$\frac{d\vec{s}}{d\tau} = \sum_{\nu=1}^{4} \vec{\gamma}_\nu \frac{dx_\nu}{d\tau} = \sum_{\nu=1}^{4} \vec{\gamma}_\nu v_\nu$$

Quoting the authors, "If we regard the v_ν as numbers here, then the eigenvalues of this geometric operator are equal to ± c." In my case, due to the measurement of time in meters, I have eigenvectors equal to ±1. Using the quantum mechanical concept that the eigenvalues are the only allowed states, we see that velocity described by the expression above can only be $\pm c$. This is an important insight, as it means a particle described by the above must be subject to curvature, with average speed different from instantaneous luminal speed.

Going from linear geometry to quantum geometry using the canonical momentum, they have the components of proper velocity as

$$v_\nu = \frac{1}{m}\left(p_\nu + \frac{q}{c}A_\nu\right) = \frac{1}{m}\left(-i\hbar\frac{\partial}{\partial x_\nu} + \frac{q}{c}A_\nu\right)$$

The proper velocity vector is then

$$\frac{d\vec{s}}{d\tau} = \frac{1}{m}\sum_{\nu=1}^{4}\vec{\gamma}_\nu\left(p_\nu + \frac{q}{c}A_\nu\right) = \frac{1}{m}\sum_{\nu=1}^{4}\vec{\gamma}_\nu\left(-i\hbar\frac{\partial}{\partial x_\nu} + \frac{q}{c}A_\nu\right)$$

Their next step, they set the above expression equal to the eigenvalue c, then use as an operator on the wavefunction to recover the Dirac equation.

$$\frac{1}{m}\sum_{\nu=1}^{4}\vec{\gamma}_\nu\left(-i\hbar\frac{\partial}{\partial x_\nu} + \frac{q}{c}A_\nu\right) = c$$
$$\frac{1}{m}\sum_{\nu=1}^{4}\vec{\gamma}_\nu\left(-i\hbar\frac{\partial}{\partial x_\nu} + \frac{q}{c}A_\nu\right)\psi = c\psi$$

In this fashion, they obtain the Dirac equation from eigenvalues of the unit tangent in spacetime.

References

[1] "Über eine mögliche geometrische Deutung der relativistischen Quantentheorie," Zeit. f. Phys. 54 (1929), 798-802.