Extending Electromagnetism

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Abstract

These note explore the extension of electromagnetism by use of the geometric and wedge products with multivector potentials.

3D Euclidean Geometric Algebra Basis

Three dimensional Euclidean geometrical algebra has a scalar (1), three vectors (e_x, e_y and e_z), three bivectors (e_xe_y, e_ze_x, and e_ye_z), and one trivector ($e_xe_ye_z$) defining the geometry. Multivector multiplication is associative, but not necessarily commutative.

In 3D Euclidean space, by definition, the three vector elements individually square to +1.

$$\begin{aligned} e_x * e_x &= e_xe_x = 1 \\ e_y * e_y &= e_ye_y = 1 \\ e_z * e_z &= e_ze_z = 1 \end{aligned}$$

In contrast to the cross product, the product of different vector basis is an anti-commutating bivector.

$$\begin{aligned} e_x * e_y &= e_xe_y = -e_ye_x \\ e_y * e_z &= e_ye_z = -e_ze_y \\ e_z * e_x &= e_ze_x = -e_xe_z \end{aligned}$$

These bivectors square to -1, as illustrated by

$$
\begin{aligned}
(e_x e_y) * (e_x e_y) &= e_x e_y e_x e_y \\
&= -e_y e_x e_x e_y \\
&= -e_y 1 e_y = -e_y e_y \\
&= -1
\end{aligned}
$$

The trivector $e_x e_y e_z$ squares to negative one, and commutes will all multivector components. This trivector, mimicing the behavior of i, is commonly written as I, sometimes as i, sometimes as j in the literature. In our case, whenever I see an i in a parent equation prior geometric algebra, I will suspect this to translate into the trivector in the post geometric algebra format. When I want to emphasize the correlation to older equations, I will use the capital $I = e_x e_y e_z$.

With our bivectors, I have a preference to use $e_x e_y, e_y e_z, e_z e_x$ as the preferred order of products, which leads to component equations with obvious dot product and couple terms.

In multiplication table format, the order-sensitive multiplication among these elements, with prefactors on the left column and postfactors on top row, is

	1	e_x	e_y	e_z	$e_x e_y$	$e_z e_x$	$e_y e_z$	$e_x e_y e_z$
1	1	e_x	e_y	e_z	$e_x e_y$	$e_z e_x$	$e_y e_z$	$e_x e_y e_z$
e_x	e_x	1	$e_x e_y$	-$e_z e_x$	e_y	$-e_z$	$e_x e_y e_z$	$e_y e_z$
e_y	e_y	$-e_x e_y$	1	$e_y e_z$	$-e_x$	$e_x e_y e_z$	e_z	$e_z e_x$
e_z	e_z	$e_z e_x$	$-e_y e_z$	1	$e_x e_y e_z$	e_x	$-e_y$	$e_x e_y$
$e_x e_y$	$e_x e_y$	$-e_y$	e_x	$e_x e_y e_z$	-1	$e_y e_z$	$-e_z e_x$	$-e_z$
$e_z e_x$	$e_z e_x$	e_z	$e_x e_y e_z$	$-e_x$	$-e_y e_z$	-1	$e_x e_y$	$-e_y$
$e_y e_z$	$e_y e_z$	$e_x e_y e_z$	$-e_z$	e_y	$e_z e_x$	$-e_x e_y$	-1	$-e_x$
$e_x e_y e_z$	$e_x e_y e_z$	$e_y e_z$	$e_z e_x$	$e_x e_y$	$-e_z$	$-e_y$	$-e_x$	-1

Maxwell in 3D Euclidean GA

The extended Maxwell equations, including magnetic monopoles, are

$$\begin{aligned}\vec{\nabla}\cdot\vec{E} &= \frac{\rho_e}{\epsilon}\\ \vec{\nabla}\cdot\vec{B} &= \mu\rho_m\\ \vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t}-\mu\vec{j}_m\\ \vec{\nabla}\times\vec{B} &= \mu\epsilon\frac{\partial\vec{E}}{\partial t}+\mu\vec{j}_e\end{aligned}$$

Oliver Heaviside noted a duality in the Maxwell equations, where $c\vec{B}\rightarrow\vec{E}$ and $\vec{E}\rightarrow -c\vec{B}$ interchanges $\vec{B}$ and $\vec{E}$, leaving the form of the equations unchanged. The extended Maxwell Equations, re-written to emphasis the these dualities with $c = 1/\sqrt{\mu\epsilon}$, $z = \sqrt{\mu/\epsilon}$, $\mu = z/c$ and $\epsilon = 1/(zc)$

$$\begin{aligned}\vec{\nabla}\cdot\vec{E} &= z\left(c\rho_e\right)\\ \vec{\nabla}\cdot\left(c\vec{B}\right) &= z\rho_m\\ \vec{\nabla}\times\vec{E} &= -\frac{1}{c}\frac{\partial(c\vec{B})}{\partial t}-\mu\vec{j}_m\\ \vec{\nabla}\times\left(c\vec{B}\right) &= \frac{1}{c}\frac{\partial\vec{E}}{\partial t}+\mu\left(c\vec{j}_e\right)\end{aligned}$$

In this format, ct is a better fit than t, and $c\vec{B}$ is a better fit than $\vec{B}$.

Joseph Larmor noted that Heaviside's transformation matched a ninety degree rotation of a complex number, and suggested a continous rotation based upon a complex number format for the electromagnetic field. This transformation also maintained the form of Maxwell equations, and allowed us to eliminate either electric or magnetic charge by suitable choice of phase angle. This allows a complexified version of the extended Maxwell equations.

Larmor's format defined $\vec{F} = \vec{E} + ic\vec{B}$.

The generalized complexified Maxwell equations then become

$$\begin{aligned}\vec{\nabla}\cdot\vec{F} &= z\left(c\rho_e + i\rho_m\right)\\ \vec{\nabla}\times\vec{F} &= i\left(\frac{1}{c}\frac{\partial\vec{F}}{\partial t}+\mu\left(c\vec{j}_e + i\vec{j}_m\right)\right)\end{aligned}$$

We now convert the complexified Maxwell equations to multivector format. Replacing i by $I = e_x e_y e_z$, the dot product terms remains simply

$$\vec{\nabla} \cdot \vec{F} = z\left(c\rho_e + I\rho_m\right)$$

To convert the cross product, replace i by $I = e_x e_y e_z$, and use $\vec{\nabla} \times \vec{F} = -I\vec{\nabla} \wedge \vec{F}$ to obtain

$$\begin{aligned}
\vec{\nabla} \times \vec{F} &= I\left(\frac{1}{c}\frac{\partial \vec{F}}{\partial t} + \mu\left(c\vec{j}_e + I\vec{j}_m\right)\right) \\
-I\nabla \wedge F &= I\left(\frac{1}{c}\frac{\partial F}{\partial t} + \mu\left(c\vec{j}_e + I\vec{j}_m\right)\right) \\
\nabla \wedge F &= -\left(\frac{1}{c}\frac{\partial F}{\partial t} + \mu\left(c\vec{j}_e + I\vec{j}_m\right)\right) \\
\frac{1}{c}\frac{\partial F}{\partial t} + \nabla \wedge F &= -\mu\left(c\vec{j}_e + (e_x e_y e_z)\vec{j}_m\right)
\end{aligned}$$

This interesting equation states that current creates a field gradient in the time direction, and twists the field in the spatial directions.

Because the $\nabla \cdot F$ expression only contains scalar and pseudoscalar terms, and both terms commute with everything, we can express the geometric product gradient as $\nabla F = \nabla \cdot F + \nabla \wedge F$, and obtain

$$\begin{aligned}
\frac{1}{c}\frac{\partial F}{\partial t} + \nabla \wedge F &= -\mu\left(c\vec{j}_e + (e_x e_y e_z)\vec{j}_m\right) \\
\frac{1}{c}\frac{\partial F}{\partial t} + \nabla \wedge F + \nabla \cdot F &= \nabla \cdot F - \mu\left(c\vec{j}_e + (e_x e_y e_z)\vec{j}_m\right) \\
& z\left(c\rho_e + I\rho_m\right) - \mu\left(c\vec{j}_e + (e_x e_y e_z)\vec{j}_m\right) \\
\frac{1}{c}\frac{\partial F}{\partial t} + \nabla F &= z\left(c\rho_e + (e_x e_y e_z)\rho_m\right) - \mu\left(c\vec{j}_e + (e_x e_y e_z)\vec{j}_m\right)
\end{aligned}$$

Cleaning up the right hand side by consolidating scalar and pseudoscalar components, along with substituting $\mu = z/c$, we have the Maxwell equations in three dimensional, Euclidean multivector format as

$$\frac{1}{c}\frac{\partial F}{\partial t} + \nabla F = z\left(c\rho_e - \vec{j}_e\right) + z\left(\rho_m - \frac{\vec{j}_m}{c}\right)(e_x e_y e_z)$$

Field Multivector

The field multivector above is defined as $\vec{F} = \vec{E} + Ic\vec{B}$. Written in component form, we have

$$F = (0, E_x, E_y, E_z, cB_{xy}, cB_{zx}, cB_{yz}, 0)$$

where the bivector components of B correspond the value of the traditional Maxwell B of the missing index. For example, B_z in conventional Maxwell corresponds to B_{xy} in geometric Maxwell. Eric Langyel [4] uses the term antivector to describe these bivector terms. This correlation of vector terms to bivector terms only occurs in three dimensions due to the $(1, 3, 3, 1)$ structure of geometric algebra in three dimensions (meaning one scalar, three vector components, three bivector components and one trivector component), and does not apply in higher dimensions (such as spacetime, which has a $(1, 4, 6, 4, 1)$ structure of one scalar, four vector, six bivector, four trivector and one quadvector components).

The field, having zero components in the scalar and trivector terms, could be incomplete.

Alternatively, if the field is complete, the field may have nilpotent characteristics. Nilpotents square to zero ($N^2 = 0$), and are characterized by zero scalar and trivector components, with the additional characteristic that vector and bivector terms are equal magnitude, and orthogonal. Radiation, from a point charge in the far field, satisfies these characteristics, as seen in Griffiths [1], page 372.

Fourvector versus Multivector Derivatives

Examine the left hand operator of the Maxwell equation in multivector format.

$$\frac{1}{c}\frac{\partial F}{\partial t} + \nabla F = z\left(c\rho_e - \vec{j}_e\right) + z\left(\rho_m - \frac{\vec{j}_m}{c}\right)(e_x e_y e_z)$$

Most authors look at

$$\frac{1}{c}\frac{\partial}{\partial t} + \nabla$$

and see a fourvector associated with a Minkowski metric, and recast Maxwell equations in covariant, spacetime format. Instead, in this exercise, I will suggest we look at this operator as an incomplete multivector operator. In component form,

$$\frac{1}{c}\frac{\partial}{\partial t}+\nabla=\left(\frac{1}{c}\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z},0,0,0,0\right)$$

I suspect that the four zeroes in the right hand side of the operator indicate missing terms in electromagnetics. I suspect that the imaginary numbers found in quantum mechanics correspond to the trivector, and that we can expand our operator above to properly include quantum mechanics.

The form of greatest interest to me at the present time, is

$$\partial=\left(\frac{1}{c}\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z},k\frac{\partial^2}{\partial x\partial y},k\frac{\partial^2}{\partial z\partial x},k\frac{\partial^2}{\partial y\partial z},m\frac{\partial^3}{\partial x\partial y\partial z}\right)$$

When using the geometric derivative, I prefer all terms have the same physical units. For the derivative above, all units are inverse meters. The scale factor c is a natural fit for the time derivative. For the second order derivatives, a scale factor k is shown, without a predetermined value. I suspect k will prove to be the on the order of the classical electron radius, or perhaps the Planck distance, but as yet, no value is determined. Likewise, the trivector component has a scale factor m, which may be k^2, or some other constant. At this time, these constants are placeholders.

From Potential to Fields

The conventional electromagnetic potential consists of a scalar potential ϕ, and a vector potential $c\vec{A}$. I am scaling $\vec{A}$ by c to have consistent units of volts for the potential, and to match the earlier format where we find $c\vec{B}$ to be a better fit for the Maxwell equations. A natural translation to multivector format,with zeroes to indicate the unused fields, is

$$\mathbf{A}=(\phi,cA_x,cA_y,cA_z,0,0,0,0)$$

My working assumption here, is that the unused fields will be filled in by quantum mechanic related terms.

We begin by using our incomplete differential operator on our incomplete potential. We recover our expected classical fields, plus a bonus.

$$\mathbf{F} = \left(\frac{1}{c}\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z},0,0,0,0\right)(\phi,cA_x,cA_y,cA_z,0,0,0,0)$$

The resulting scalar portion is a measure of Lorentz gauge violation. This goes to zero in the far field, but is significant in the near field, as well in electrostatics and magnetostatics.

$$F_q = \frac{\partial \phi}{\partial (ct)} + \frac{\partial c A_x}{\partial x} + \frac{\partial c A_y}{\partial y} + \frac{\partial c A_z}{\partial z}$$

Next, we have the negative of the three components of the electric field.

$$\begin{aligned} F_x &= \left(\frac{\partial \phi}{\partial x} + \frac{\partial c A_x}{\partial (ct)} \right) = -E_x \\ F_y &= \left(\frac{\partial \phi}{\partial y} + \frac{\partial c A_y}{\partial (ct)} \right) = -E_y \\ F_z &= \left(\frac{\partial \phi}{\partial z} + \frac{\partial c A_z}{\partial (ct)} \right) = -E_z \end{aligned}$$

For the bivector components, we get the c scaled magnetic field, as expected.

$$\begin{aligned} F_{xy} &= \left(\frac{\partial c A_y}{\partial x} - \frac{\partial c A_x}{\partial y} \right) = c B_{xy} \\ F_{zx} &= \left(\frac{\partial c A_x}{\partial z} - \frac{\partial c A_z}{\partial x} \right) = c B_{zx} \\ F_{yz} &= \left(\frac{\partial c A_z}{\partial y} - \frac{\partial c A_y}{\partial z} \right) = c B_{yz} \end{aligned}$$

We end with a trivector of zero, (indicating no monopole contribution in the conventional model)

$$F_{xyz} = 0$$

Missing Terms in Electrodynamics

I find it highly unlikely that the truncated derivative and field descriptions above present a complete description of electromagnetics. Indeed, my working assumption in this paper is that the missing terms are our bridge to quantum mechanics. Rather than writing the full expected set of terms on paper at once, I will write out three more subsets of interactions, which will fit the typesetting environment a bit better.

The geometric product, being associative and distributive, allows use to express the full product as the sum of partial terms. For example, $(\mathbf{A}+\mathbf{B})(\mathbf{C}+\mathbf{D}) = \mathbf{AC}+\mathbf{AD}+\mathbf{BC}+\mathbf{BD}$. Compared to the full derivative acting on the full potential, our expression above, conventional electromagnetics, is just the **AC** term.

We now provide the **AD** term.

The AD Term

We now look at terms where the conventional differential operator acts on the extended potential terms.

$$\mathbf{G} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, 0, 0, 0, 0\right)(0, 0, 0, 0, cA_{xy}, cA_{zx}, cA_{yz}, cA_{xyz})$$

The resulting scalar portion is zero.

$$G_q = 0$$

Next, we have the vector portion.

$$G_x = \left(\frac{\partial cA_{zx}}{\partial z} - \frac{\partial cA_{xy}}{\partial y}\right)$$

$$G_y = \left(\frac{\partial cA_{xy}}{\partial x} - \frac{\partial cA_{yz}}{\partial z}\right)$$

$$G_z = \left(\frac{\partial cA_{yz}}{\partial y} - \frac{\partial cA_{zx}}{\partial x}\right)$$

Now we have the bivector portion.

$$G_{xy} = \left(\frac{\partial cA_{xy}}{\partial (ct)} + \frac{\partial cA_{xyz}}{\partial z}\right)$$

$$G_{zx} = \left(\frac{\partial cA_{zx}}{\partial (ct)} + \frac{\partial cA_{xyz}}{\partial y}\right)$$

$$G_{yz} = \left(\frac{\partial cA_{yz}}{\partial (ct)} + \frac{\partial cA_{xyz}}{\partial x}\right)$$

We end with a trivector.

$$G_{xyz} = \frac{\partial cA_{xyz}}{\partial (ct)} + \frac{\partial cA_{yz}}{\partial x} + \frac{\partial cA_{zx}}{\partial y} + \frac{\partial cA_{xy}}{\partial z}$$

The BC Term

We now look at terms where the extended differential operator second half acts on the conventional potential terms.

$$\mathbf{H} = \left(0,0,0,0,\frac{k\partial^2}{\partial x\partial y},\frac{k\partial^2}{\partial z\partial x},\frac{k\partial^2}{\partial y\partial z},\frac{m\partial^3}{\partial x\partial y\partial z},\right)(\phi, cA_x, cA_y, cA_z, 0,0,0,0)$$

The resulting scalar portion is zero.

$$H_q = 0$$

Next, we have the vector portion.

$$\begin{aligned} H_x &= \left(\frac{k\partial^2 cA_y}{\partial x\partial y} - \frac{k\partial^2 cA_z}{\partial z\partial x}\right) \\ H_y &= \left(\frac{k\partial^2 cA_z}{\partial y\partial z} - \frac{k\partial^2 cA_x}{\partial x\partial y}\right) \\ H_z &= \left(\frac{k\partial^2 cA_x}{\partial z\partial x} - \frac{k\partial^2 cA_y}{\partial y\partial z}\right) \end{aligned}$$

Now we have the bivector portion.

$$\begin{aligned} H_{xy} &= \left(\frac{k\partial^2 \phi}{\partial x\partial y} + \frac{m\partial^3 cA_z}{\partial x\partial y\partial z}\right) \\ H_{zx} &= \left(\frac{k\partial^2 \phi}{\partial z\partial x} + \frac{m\partial^3 cA_y}{\partial x\partial y\partial z}\right) \\ H_{yz} &= \left(\frac{k\partial^2 \phi}{\partial y\partial z} + \frac{m\partial^3 cA_x}{\partial x\partial y\partial z}\right) \end{aligned}$$

We end with a trivector.

$$H_{xyz} = \frac{k\partial^2 cA_x}{\partial y\partial z} + \frac{k\partial^2 cA_y}{\partial z\partial x} + \frac{k\partial^2 cA_z}{\partial x\partial y} + \frac{m\partial^3 \phi}{\partial x\partial y\partial z}$$

The BD Term

We now finish by looking at terms where the extended differential operator second half acts on the extended potential second half terms.

$$\mathbf{K} = \left(0,0,0,0,\frac{k\partial^2}{\partial x\partial y},\frac{k\partial^2}{\partial z\partial x},\frac{k\partial^2}{\partial y\partial z},\frac{m\partial^3}{\partial x\partial y\partial z},\right)(0,0,0,0,cA_{xy},cA_{zx},cA_{yz},cA_{xyz})$$

The resulting scalar portion is

$$K_q = -\frac{k\partial^2 cA_{xy}}{\partial x\partial y} - \frac{k\partial^2 cA_{zx}}{\partial z\partial x} - \frac{k\partial^2 cA_{yz}}{\partial y\partial z} - \frac{m\partial^3 cA_{xyz}}{\partial x\partial y\partial z}$$

Next, we have the vector portion.

$$K_x = \left(-\frac{k\partial^2 cA_{xyz}}{\partial y\partial z} - \frac{m\partial^3 cA_{yz}}{\partial x\partial y\partial z}\right)$$
$$K_y = \left(-\frac{k\partial^2 cA_{xyz}}{\partial z\partial x} - \frac{m\partial^3 cA_{zx}}{\partial x\partial y\partial z}\right)$$
$$K_z = \left(-\frac{k\partial^2 cA_{xyz}}{\partial x\partial y} - \frac{m\partial^3 cA_{xy}}{\partial x\partial y\partial z}\right)$$

Now we have the bivector portion.

$$K_{xy} = \left(\frac{k\partial^2 cA_{yz}}{\partial z\partial x} - \frac{k\partial^2 cA_{zx}}{\partial y\partial z}\right)$$
$$K_{zx} = \left(\frac{k\partial^2 cA_{xy}}{\partial y\partial z} - \frac{k\partial^2 cA_{yz}}{\partial x\partial y}\right)$$
$$K_{yz} = \left(\frac{k\partial^2 cA_{zx}}{\partial x\partial y} - \frac{k\partial^2 cA_{xy}}{\partial z\partial x}\right)$$

We end with a trivector of zero.

$$K_{xyz} = 0$$

Full Product, Color Coded

We now assemble the full product of terms, keeping the color coding from above. The product of interest is shown in sidewise Table 1.

$$
\begin{aligned}
S &= \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{k\partial^2}{\partial x\partial y}, \frac{k\partial^2}{\partial z\partial x}, \frac{k\partial^2}{\partial y\partial z}, \frac{m\partial^3}{\partial x\partial y\partial z}\right)(\phi, cA_x, cA_y, cA_z, cA_{xy}, cA_{zx}, cA_{yz}, cA_{xyz}) \\
S_q &= \frac{\partial\phi}{\partial(ct)} + \frac{\partial cA_x}{\partial x} + \frac{\partial cA_y}{\partial y} + \frac{\partial cA_z}{\partial z} - \frac{k\partial^2 cA_{xy}}{\partial x\partial y} - \frac{k\partial^2 cA_{zx}}{\partial z\partial x} - \frac{k\partial^2 cA_{yz}}{\partial y\partial z} - \frac{m\partial^3 cA_{xyz}}{\partial x\partial y\partial z} \\
S_x &= \left(\frac{\partial\phi}{\partial x} + \frac{\partial cA_x}{\partial(ct)}\right) + \left(\frac{\partial cA_{zx}}{\partial z} - \frac{\partial cA_{xy}}{\partial y}\right) + \left(\frac{k\partial^2 cA_y}{\partial x\partial y} - \frac{k\partial^2 cA_z}{\partial z\partial x}\right) + \left(-\frac{k\partial^2 cA_{xyz}}{\partial y\partial z} - \frac{m\partial^3 cA_{yz}}{\partial x\partial y\partial z}\right) \\
S_y &= \left(\frac{\partial\phi}{\partial y} + \frac{\partial cA_y}{\partial(ct)}\right) + \left(\frac{\partial cA_{xy}}{\partial x} - \frac{\partial cA_{yz}}{\partial z}\right) + \left(\frac{k\partial^2 cA_z}{\partial y\partial z} - \frac{k\partial^2 cA_x}{\partial x\partial y}\right) + \left(-\frac{k\partial^2 cA_{xyz}}{\partial z\partial x} - \frac{m\partial^3 cA_{zx}}{\partial x\partial y\partial z}\right) \\
S_z &= \left(\frac{\partial\phi}{\partial z} + \frac{\partial cA_z}{\partial(ct)}\right) + \left(\frac{\partial cA_{yz}}{\partial y} - \frac{\partial cA_{zx}}{\partial x}\right) + \left(\frac{k\partial^2 cA_x}{\partial z\partial x} - \frac{k\partial^2 cA_y}{\partial y\partial z}\right) + \left(-\frac{k\partial^2 cA_{xyz}}{\partial x\partial y} - \frac{m\partial^3 cA_{xy}}{\partial x\partial y\partial z}\right) \\
S_{xy} &= \left(\frac{\partial cA_y}{\partial x} - \frac{\partial cA_x}{\partial y}\right) + \left(\frac{\partial cA_{xy}}{\partial(ct)} + \frac{\partial cA_{xyz}}{\partial z}\right) + \left(\frac{k\partial^2\phi}{\partial x\partial y} + \frac{m\partial^3 cA_z}{\partial x\partial y\partial z}\right) + \left(\frac{k\partial^2 cA_{yz}}{\partial z\partial x} - \frac{k\partial^2 cA_{zx}}{\partial y\partial z}\right) \\
S_{zx} &= \left(\frac{\partial cA_x}{\partial z} - \frac{\partial cA_z}{\partial x}\right) + \left(\frac{\partial cA_{zx}}{\partial(ct)} + \frac{\partial cA_{xyz}}{\partial y}\right) + \left(\frac{k\partial^2\phi}{\partial z\partial x} + \frac{m\partial^3 cA_y}{\partial x\partial y\partial z}\right) + \left(\frac{k\partial^2 cA_{xy}}{\partial y\partial z} - \frac{k\partial^2 cA_{yz}}{\partial x\partial y}\right) \\
S_{yz} &= \left(\frac{\partial cA_z}{\partial y} - \frac{\partial cA_y}{\partial z}\right) + \left(\frac{\partial cA_{yz}}{\partial(ct)} + \frac{\partial cA_{xyz}}{\partial x}\right) + \left(\frac{k\partial^2\phi}{\partial y\partial z} + \frac{m\partial^3 cA_x}{\partial x\partial y\partial z}\right) + \left(\frac{k\partial^2 cA_{zx}}{\partial x\partial y} - \frac{k\partial^2 cA_{xy}}{\partial z\partial x}\right) \\
S_{xyz} &= \frac{\partial cA_{xyz}}{\partial(ct)} + \frac{\partial cA_{yz}}{\partial x} + \frac{\partial cA_{zx}}{\partial y} + \frac{\partial cA_{xy}}{\partial z} + \frac{k\partial^2 cA_x}{\partial y\partial z} + \frac{k\partial^2 cA_y}{\partial z\partial x} + \frac{k\partial^2 cA_z}{\partial x\partial y} + \frac{m\partial^3\phi}{\partial x\partial y\partial z}
\end{aligned}
$$

Table 1: Full Equation Set

Extended Equation Set Using Wedge Product

The wedge product is a subset of the geometric product. How does the extended derivative and potential look under the wedge product, as opposed to the geometric product? Table 2 provides the colorized equation set found under the wedge product. This is a subset of the larger geometric product of Table 1.

Wedge Product with Mixed Unit Multivectors

The geometric product, in my opinion, requires similar units of measure in every component for the forefactor, and perhaps different, but uniform units of measure in the postfactor to prevent mismatched units in the product.

The wedge product, however, can maintain different units, such as no units for the scalar, meters for the vector, square meters for the bivector and cubic meters for the trivector in both the forefactor and postfactor, and have the product with the same units in the same components, as long as the units maintain simple power factors for forefactor and postfactor, with the scalar portion a pure number. I use the positive units such as the m, m^2, and m^3 sequence in integration. Inverse units, such as m^{-1} for the vector, m^{-2} for the bivector and m^{-3} for the trivector naturally arise during gradient operations.

Wedge mixed unit operators and potentials are totally different from the operators and potentials above. My differential operator of interest, and favorite multipotential for the mixed unit wedge product is

$$\begin{aligned} S &= \left(T\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial^2}{\partial x \partial y}, \frac{\partial^2}{\partial z \partial x}, \frac{\partial^2}{\partial y \partial z}, \frac{\partial^3}{\partial x \partial y \partial z}\right) \\ Q &= (\Phi, A_x, A_y, A_z, B_{xy}, B_{zx}, B_{yz}, G_{xyz}) \\ W &= S \wedge Q \end{aligned}$$

In this setup, T has units of time, allowing our first term to be unit-free. The derivative vector portion has units of inverse meters, the bivector portion has units of inverse meters squared, and the trivector portion has units of inverse meters cubed.

For the multipotential, Φ has units of magnetic flux, being Vs. The vector potential A has units of $(Vs)/m$, the bivector portion, like the magnetic field, has units of $(Vs)/m^2$, but is not necessary the magnetic field, as such.

$$
\begin{aligned}
W &= \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{k\partial^2}{\partial x\partial y}, \frac{k\partial^2}{\partial z\partial x}, \frac{k\partial^2}{\partial y\partial z}, \frac{m\partial^3}{\partial x\partial y\partial z}\right) \wedge (\phi, cA_x, cA_y, cA_z, cA_{xy}, cA_{zx}, cA_{yz}, cA_{xyz}) \\
W_q &= \frac{\partial\phi}{\partial(ct)} \\
W_x &= \left(\frac{\partial\phi}{\partial x} + \frac{\partial cA_x}{\partial(ct)}\right) \\
W_y &= \left(\frac{\partial\phi}{\partial y} + \frac{\partial cA_y}{\partial(ct)}\right) \\
W_z &= \left(\frac{\partial\phi}{\partial z} + \frac{\partial cA_z}{\partial(ct)}\right) \\
W_{xy} &= \left(\frac{\partial cA_y}{\partial x} - \frac{\partial cA_x}{\partial y}\right) + \left(\frac{\partial cA_{xy}}{\partial(ct)}\right) + \left(\frac{k\partial^2\phi}{\partial x\partial y}\right) \\
W_{zx} &= \left(\frac{\partial cA_x}{\partial z} - \frac{\partial cA_z}{\partial x}\right) + \left(\frac{\partial cA_{zx}}{\partial(ct)}\right) + \left(\frac{k\partial^2\phi}{\partial z\partial x}\right) \\
W_{yz} &= \left(\frac{\partial cA_z}{\partial y} - \frac{\partial cA_y}{\partial z}\right) + \left(\frac{\partial cA_{yz}}{\partial(ct)}\right) + \left(\frac{k\partial^2\phi}{\partial y\partial z}\right) \\
W_{xyz} &= \frac{\partial cA_{xyz}}{\partial(ct)} + \frac{\partial cA_{yz}}{\partial x} + \frac{\partial cA_{zx}}{\partial y} + \frac{\partial cA_{xy}}{\partial z} + \frac{k\partial^2 cA_x}{\partial y\partial z} + \frac{k\partial^2 cA_y}{\partial z\partial x} + \frac{k\partial^2 cA_z}{\partial x\partial y} + \frac{m\partial^3\phi}{\partial x\partial y\partial z}
\end{aligned}
$$

Table 2: Extended Derivative Wedge Extended Potential

Finally, the trivector has units of flux density. I like this format quite a bit. Multiplying by q results in units of action (Js), for the scalar part. These are the same units I use for wavefunctions.

This multipotential I call the magnet potential chain. I intend to spend a bit more time examining this chain from the point of view of the wedge product.

Magnetic Flux Wedge Chain

We now apply the wedge differential multivector operator to Q to see the result. This is different from the previous sections, both in the operator and operand. Notice I've also dropped the factor of c from the above, as well as the factors k and m. I'm also changing colorization and order.

$$
\begin{aligned}
W &= \left(T\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial^2}{\partial x \partial y}, \frac{\partial^2}{\partial z \partial x}, \frac{\partial^2}{\partial y \partial z}, \frac{\partial^3}{\partial x \partial y \partial z}\right) \\
&\quad \wedge(\Phi, A_x, A_y, A_z, B_{xy}, B_{zx}, B_{yz}, G_{xyz}) \\
W_q &= T\frac{\partial \Phi}{\partial t} \\
W_x &= T\frac{\partial A_x}{\partial t} + \frac{\partial \Phi}{\partial x} \\
W_y &= T\frac{\partial A_y}{\partial t} + \frac{\partial \Phi}{\partial y} \\
W_z &= T\frac{\partial A_z}{\partial t} + \frac{\partial \Phi}{\partial z} \\
W_{xy} &= T\frac{\partial B_{xy}}{\partial t} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + \frac{\partial^2 \Phi}{\partial x \partial y} \\
W_{zx} &= T\frac{\partial B_{zx}}{\partial t} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial^2 \Phi}{\partial z \partial x} \\
W_{yz} &= T\frac{\partial B_{yz}}{\partial t} + \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial^2 \Phi}{\partial y \partial z} \\
W_{xyz} &= T\frac{\partial G_{xyz}}{\partial t} + \frac{\partial B_{yz}}{\partial x} + \frac{\partial B_{zx}}{\partial y} + \frac{\partial B_{xy}}{\partial z} \\
&\quad + \frac{\partial^2 A_x}{\partial y \partial z} + \frac{\partial^2 A_y}{\partial z \partial x} + \frac{\partial^2 A_z}{\partial x \partial y} + \frac{\partial^3 \Phi}{\partial x \partial y \partial z}
\end{aligned}
$$

This format is very interesting. The time derivative terms in green are obvious, and disappear in steady state. The gradient terms of Φ in the vector portion are obvious, as are the curl terms of A in the bivector portion. The divergence of B is seen in the trivector portion. Our new terms here, are the antigradient, being double derivatives of Ψ in red in the bivector portion, and the antidivergence of A, also in red, in the trivector component.

Repeated Derivatives

The pure wedge derivative, where the scalar component is zero, has the fine property of nilpotentcy of the order of the dimensional space, in our case, three. This means that all cascade derivatives three or higher are zero.

Using the previous section notation, we can explicitly write out the non-zero derivatives. I use capital D for this pure wedge derivative operator. I will also borrow F, S and T.

$$
\begin{aligned}
D &= \left(0, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial^2}{\partial x \partial y}, \frac{\partial^2}{\partial z \partial x}, \frac{\partial^2}{\partial y \partial z}, \frac{\partial^3}{\partial x \partial y \partial z}\right) \\
Q &= (\Phi, A_x, A_y, A_z, B_{xy}, B_{zx}, B_{yz}, G_{xyz}) \\
F &= D \wedge Q \qquad \text{(First Wedge Derivative)} \\
F_q &= 0 \\
F_x &= \frac{\partial \Phi}{\partial x} \\
F_y &= \frac{\partial \Phi}{\partial y} \\
F_z &= \frac{\partial \Phi}{\partial z} \\
F_{xy} &= \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + \frac{\partial^2 \Phi}{\partial x \partial y} \\
F_{zx} &= \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial^2 \Phi}{\partial z \partial x} \\
F_{yz} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial^2 \Phi}{\partial y \partial z} \\
F_{xyz} &= \frac{\partial B_{yz}}{\partial x} + \frac{\partial B_{zx}}{\partial y} + \frac{\partial B_{xy}}{\partial z} \\
&\quad + \frac{\partial^2 A_x}{\partial y \partial z} + \frac{\partial^2 A_y}{\partial z \partial x} + \frac{\partial^2 A_z}{\partial x \partial y} + \frac{\partial^3 \Phi}{\partial x \partial y \partial z}
\end{aligned}
$$

The second wedge derivative has only a trivector component.

$$
\begin{aligned}
S &= D \wedge (D \wedge Q) \qquad \text{(Second Wedge Derivative)} \\
&= \left(0,0,0,0,0,0,0,6\frac{\partial^3 \Phi}{\partial x \partial y \partial z}\right) \\
&= 6\frac{\partial^3 \Phi}{\partial x \partial y \partial z} e_x e_y e_z \\
T &= D \wedge (D \wedge (D \wedge Q)) = 0 \qquad \text{(Third Wedge Derivative)}
\end{aligned}
$$

As the trivector in three dimensional Euclidean space wedged with any non-scalar yields zero, all higher pure wedge derivatives are zero.

It is very instructive and pleasant to calculate the second wedge derivative by hand. Each individual vector term creates a bivector couple, as well as a trivector couple and trivector density term. The sum of the three vector components leads to massive cancellation, leaving three density terms. Next, the bivector derivative terms acting on the vector terms lead to three more density terms, being our final result above for the square wedge.

Given this finite extent of the pure wedge differential product, we can express combinations of scalar and wedge differential products as polynomials, truncating the wedge product terms above squared. Take the example from earlier.

$$
\begin{aligned}
W &= \left(T\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial^2}{\partial x \partial y}, \frac{\partial^2}{\partial z \partial x}, \frac{\partial^2}{\partial y \partial z}, \frac{\partial^3}{\partial x \partial y \partial z}\right) \\
&= T\frac{\partial}{\partial t} + D
\end{aligned}
$$

Repeated application of this operator, with slight abuse of notation, to a multivector simplifies as follows.

$$
\begin{aligned}
\left(T\frac{\partial}{\partial t} + D\wedge\right)^2 M &= T^2\frac{\partial^2}{\partial t^2}M + 2T\frac{\partial}{\partial t}D \wedge M + D \wedge D \wedge M \\
\left(T\frac{\partial}{\partial t} + D\wedge\right)^3 M &= T^3\frac{\partial^3}{\partial t^3}M + 3T^2\frac{\partial^2}{\partial t^2}D \wedge M + 3T\frac{\partial}{\partial t}D \wedge D \wedge M \\
\left(T\frac{\partial}{\partial t} + D\wedge\right)^n M &= T^n\frac{\partial^n}{\partial t^n}M + nT^{n-1}\frac{\partial^{n-1}}{\partial t^{n-1}}D \wedge M + nT^{n-2}\frac{\partial^{n-2}}{\partial t^{n-2}}D \wedge D \wedge M
\end{aligned}
$$

Repeating, we get simple expressions for higher order derivatives due to the finite truncation of the pure wedge product.

Electric Potential Wedge Chain

In a similar fashion to the magnetic flux chain, we can make an electric field potential chain.

$$
\begin{aligned}
W &= \left(T\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial^2}{\partial x\partial y}, \frac{\partial^2}{\partial z\partial x}, \frac{\partial^2}{\partial y\partial z}, \frac{\partial^3}{\partial x\partial y\partial z}\right) \\
&\quad \wedge(\phi, E_x, E_y, E_z, E_{xy}, E_{zx}, E_{yz}, g_{xyz}) \\
W_q &= T\frac{\partial \phi}{\partial t} \\
W_x &= T\frac{\partial E_x}{\partial t} + \frac{\partial \phi}{\partial x} \\
W_y &= T\frac{\partial E_y}{\partial t} + \frac{\partial \phi}{\partial y} \\
W_z &= T\frac{\partial E_z}{\partial t} + \frac{\partial \phi}{\partial z} \\
W_{xy} &= T\frac{\partial E_{xy}}{\partial t} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) + \frac{\partial^2 \phi}{\partial x\partial y} \\
W_{zx} &= T\frac{\partial E_{zx}}{\partial t} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \frac{\partial^2 \phi}{\partial z\partial x} \\
W_{yz} &= T\frac{\partial E_{yz}}{\partial t} + \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \frac{\partial^2 \phi}{\partial y\partial z} \\
W_{xyz} &= T\frac{\partial g_{xyz}}{\partial t} + \frac{\partial E_{yz}}{\partial x} + \frac{\partial E_{zx}}{\partial y} + \frac{\partial E_{xy}}{\partial z} \\
&\quad + \frac{\partial^2 E_x}{\partial y\partial z} + \frac{\partial^2 E_y}{\partial z\partial x} + \frac{\partial^2 E_z}{\partial x\partial y} + \frac{\partial^3 \phi}{\partial x\partial y\partial z}
\end{aligned}
$$

The bivector terms, if this corresponds to electromagnetics, will be related to the dB/dt terms via Maxwell.

Time Free Potential Wedge Formulas

At this point, I am playing with symbols which happen to have some similarity to electromagnetics. In the formulas above, the only requirement was that the scalar portion of potential be unit-free, and independent of the positional derivatives. In the following formulas, I will initially restrict the scalar portion to be a constant, then further restrict that constant to be zero. Both of these formulas belong to the 'crank the simulation' model, where time is replaced by a simulation sequence number.

Constant Scalar Scenario

In this speculative form, the scalar portion of the differential operator is replaced by a constant, in this case c. In this scenario, the W are the next value for the multipotentials. Each tick of the clock is equivalent to reapplication of the wedge operator on the multipotentials.

Following is the wedge product with the differential operator having a fixed scalar component.

$$
\begin{aligned}
S &= \left(c, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial^2}{\partial x \partial y}, \frac{\partial^2}{\partial z \partial x}, \frac{\partial^2}{\partial y \partial z}, \frac{\partial^3}{\partial x \partial y \partial z}\right) \\
Q &= (\Phi, A_x, A_y, A_z, B_{xy}, B_{zx}, B_{yz}, G_{xyz}) \\
W &= S \wedge Q \\
W_q &= c\Phi \\
W_x &= cA_x + \frac{\partial \Phi}{\partial x} \\
W_y &= cA_y + \frac{\partial \Phi}{\partial y} \\
W_z &= cA_z + \frac{\partial \Phi}{\partial z} \\
W_{xy} &= cB_{xy} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + \left(\frac{\partial^2 \Phi}{\partial x \partial y}\right) \\
W_{zx} &= cB_{zx} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \left(\frac{\partial^2 \Phi}{\partial z \partial x}\right) \\
W_{yz} &= cB_{yz} + \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \left(\frac{\partial^2 \Phi}{\partial y \partial z}\right) \\
W_{xyz} &= cG_{xyz} + \frac{\partial B_{yz}}{\partial x} + \frac{\partial B_{zx}}{\partial y} + \frac{\partial B_{xy}}{\partial z} \\
&\quad + \frac{\partial^2 A_x}{\partial y \partial z} + \frac{\partial^2 A_y}{\partial z \partial x} + \frac{\partial^2 A_z}{\partial x \partial y} + \frac{\partial^3 \Phi}{\partial x \partial y \partial z}
\end{aligned}
$$

In our expression above, $S = c + D$. The repeated wedge operators are given by the simple formulas

$$
\begin{aligned}
(c + D\wedge)\, M &= cM + D \wedge M \\
(c + D\wedge)^2\, M &= c^2 M + 2cD \wedge M + D \wedge D \wedge M \\
(c + D\wedge)^3\, M &= c^3 M + 3c^2 D \wedge M + 3cD \wedge D \wedge M \\
(c + D\wedge)^n\, M &= c^n M + nc^{n-1} D \wedge M + nc^{n-2} D \wedge D \wedge M
\end{aligned}
$$

For the important case of $c = 0$, being the pure wedge product, only the first and second wedge cascades are non-zero.

Conclusion

Geometric algebra has a nice fit for the conventional potential formulation of electrodynamics, and suggests extensions of electromagnetics which I suspect will lead to a single geometric electro-quantum description. The wedge product suggests a different formulation of potentials as multivector potentials, which I need to further investigate. My current plan is to shelve this work for a few days, and work backwards from Dirac → Pauli → Schrödinger equations toward electromagnetism, as Doran, Hestenes and others have a fairly clear geometric interpretation of Dirac's equations and quantum mechanics.

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