Electromagnetic Duality in 3D Geometric Algebra

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Abstract

This is a paper looking at duality in three dimensional electromagnetism, from the point of view of Geometric Algebra. I present the standard Maxwell Equations, followed by a brief discussion of parity, and the axial versus polar vector problems. I then present the standard dualities of electromagnetism, leading to the complex number format for the Maxwell equations. Next is a presentation of geometric algebra in three dimensional Euclidean space. I then finish with the Maxwell equations translated into multivector format.

Standard Maxwell Equations

Conventional Maxwell equations [1] in SI units [6] are

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \frac{\rho_e}{\epsilon} \\
\vec{\nabla} \cdot \vec{B} &= 0 \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\vec{\nabla} \times \vec{B} &= \mu\epsilon \frac{\partial \vec{E}}{\partial t} + \mu \vec{j}_e
\end{aligned}$$

where μ is the magnetic permeability of space, ϵ is the electric permittivity of space, and $\mu\epsilon = 1/c^2$, where c is the speed of light.

Extending the Maxwell equations to include magnetic monopoles, we have

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \frac{\rho_e}{\epsilon} \\
\vec{\nabla} \cdot \vec{B} &= \mu \rho_m \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} - \mu \vec{j}_m \\
\vec{\nabla} \times \vec{B} &= \mu \epsilon \frac{\partial \vec{E}}{\partial t} + \mu \vec{j}_e
\end{aligned}$$

An irritating feature seen in classical mechanics and the Maxwell equations as shown above, is the requirement of two incompatible vector types, being polar versus axial vectors.

Polar Versus Axial Vectors

Under a parity transformation, where all spatial coordinates change sign, polar vectors change sign. Vectors, and electric fields are examples of polar vectors.

By contrast, under a parity transformation, axial vectors, also known as pseudovectors, remain unchanged. Cross products and magnetic fields are examples of axial vectors.

This potentially confusing situation, where we have instrinsically incompatible vector types, is resolved in geometric algebra by the distinction between vector versus bivector elements.

Parity Properties of EM Elements

From Jackson [5], Table 6.1, we see that coordinates, velocity, linear momentum, force, current density, electric field and the Poynting vector are polar vectors.

We see that angular momentum, torque, and magnetic field are axial vectors, due to the use of a cross product in their definitions.

In a similar fashion, we see electric charge density is a scalar, with no sign change under parity transformation. By contrast, magnetic monopole charge density is a pseudoscalar, changing sign with under parity transformation.

The Cross Product is Not Recommended

We will find that the vector cross product, which takes two vectors and returns a third vector normal to the two factors, to be a source of confusion. In three dimensions, we can associate a vector as a unique (within a sign) normal to a plane. In higher dimensions, we do not have this luxury. Instead, in higher dimensions, the product of two vectors will generally have scalar (colinear component) and planar (transverse component) components. In hindsight, the result of the cross product is a planar, two dimensional entity, called a bivector in geometric algebra. Cross products are not recommended. Instead, we can replace cross products with wedge products with an associated trivector factor.

Duality in EM

Oliver Heaviside [2] noticed that the Maxwell equations retain their form when suitably scaled magnetic and electric fields are interchanged.

Following is the extended Maxwell Equations, re-written to emphasis the dualities $c\vec{B} \rightarrow \vec{E}$ and $g \rightarrow cq$, with $c = 1/\sqrt{\mu\epsilon}$ and $z = \sqrt{\mu/\epsilon}$.

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= z\left(c\rho_e\right) \\
\vec{\nabla} \cdot \left(c\vec{B}\right) &= z\rho_m \\
\vec{\nabla} \times \vec{E} &= -\frac{1}{c}\frac{\partial(c\vec{B})}{\partial t} - \mu\vec{j}_m \\
\vec{\nabla} \times \left(c\vec{B}\right) &= \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \mu\left(c\vec{j}_e\right)
\end{aligned}
$$

In this format, we see clearly see that ct is a better fit than t, and $c\vec{B}$ is a better fit than $\vec{B}$.

Sir Joseph Larmor noted that Heaviside's transformation matched a ninety degree rotation of a complex number, and showed that a continous rotation based upon a complex number format for the electromagnetic field also maintained the form of Maxwell equations, and allowed us to eliminate either electric or magnetic charge by suitable choice of phase angle.

Larmor's format defined $\vec{F} = \vec{E} + ic\vec{B}$. The generalized Maxwell equations then become

$$\begin{aligned}\vec{\nabla} \cdot \vec{F} &= z\,(c\rho_e + i\rho_m) \\ \vec{\nabla} \times \vec{F} &= i\left(\frac{1}{c}\frac{\partial \vec{F}}{\partial t} + \mu\left(c\vec{j}_e + i\vec{j}_m\right)\right)\end{aligned}$$

Geometric Algebra in 3D Euclidean Space

Three dimensional Euclidean geometrical algebra has a scalar (1), three vectors (e_x, e_y and e_z), three bivectors (e_xe_y, e_ze_x, and e_ye_z), and one trivector ($e_xe_ye_z$) defining the geometry. Multivector multiplication is associative, but not necessarily commutative.

In 3D Euclidean space, by definition, the three vector elements individually square to +1.

$$\begin{aligned} e_x * e_x &= e_xe_x = 1 \\ e_y * e_y &= e_ye_y = 1 \\ e_z * e_z &= e_ze_z = 1 \end{aligned}$$

In contrast to the cross product, the product of different vector basis is an anti-commutating bivector.

$$\begin{aligned} e_x * e_y &= e_xe_y = -e_ye_x \\ e_y * e_z &= e_ye_z = -e_ze_y \\ e_z * e_x &= e_ze_x = -e_xe_z \end{aligned}$$

These bivectors square to -1, as illustrated by

$$\begin{aligned} (e_xe_y) * (e_xe_y) &= e_xe_ye_xe_y \\ &= -e_ye_xe_xe_y \\ &= -e_y1e_y = -e_ye_y \\ &= -1 \end{aligned}$$

The trivector $e_xe_ye_z$ squares to negative one, and commutes will all multivector components. This trivector, mimicing the behavior of i, is commonly written as I, sometimes as i, sometimes as j in the literature. In our case, whenever I see an i in a parent equation prior geometric algebra, I will suspect this to translate into the trivector in the post geometric algebra format.

When I want to emphasize the correlation to older equations, I will use the capital $I = e_x e_y e_z$.

With our bivectors, I have a preference to use $e_x e_y, e_y e_z, e_z e_x$ as the preferred order of products, which leads to component equations with obvious dot product and couple terms.

In multiplication table format, the order-sensitive multiplication among these elements, with prefactors on the left column and postfactors on top row, is

	1	e_x	e_y	e_z	$e_x e_y$	$e_z e_x$	$e_y e_z$	$e_x e_y e_z$
1	1	e_x	e_y	e_z	$e_x e_y$	$e_z e_x$	$e_y e_z$	$e_x e_y e_z$
e_x	e_x	1	$e_x e_y$	-$e_z e_x$	e_y	$-e_z$	$e_x e_y e_z$	$e_y e_z$
e_y	e_y	$-e_x e_y$	1	$e_y e_z$	$-e_x$	$e_x e_y e_z$	e_z	$e_z e_x$
e_z	e_z	$e_z e_x$	$-e_y e_z$	1	$e_x e_y e_z$	e_x	$-e_y$	$e_x e_y$
$e_x e_y$	$e_x e_y$	$-e_y$	e_x	$e_x e_y e_z$	-1	$e_y e_z$	$-e_z e_x$	$-e_z$
$e_z e_x$	$e_z e_x$	e_z	$e_x e_y e_z$	$-e_x$	$-e_y e_z$	-1	$e_x e_y$	$-e_y$
$e_y e_z$	$e_y e_z$	$e_x e_y e_z$	$-e_z$	e_y	$e_z e_x$	$-e_x e_y$	-1	$-e_x$
$e_x e_y e_z$	$e_x e_y e_z$	$e_y e_z$	$e_z e_x$	$e_x e_y$	$-e_z$	$-e_y$	$-e_x$	-1

In this algebra, scalar multiplication is commutative and associative, basis vectors square to scalar one, and the product of two vectors resulting in a bivector is anti-commutative, associative, squares to negative one, and trivector basis commute with everything, yet square to negative one.

Much of the time, rather than work on a component level basis, I will use a higher level notation representing a generic multivector as the sum of a scalar, vector, bivector and trivector. This is no different conceptually than representing a complex number as a sum of a real and imaginary component. My generic multivector is

$$MV = q + \vec{v} + \mathbf{B} + T$$

where q is the scalar portion (such as charge), $\vec{v}$ (lower case, overarrow) is the vector portion, $\mathbf{B}$ (bold uppercase) is the bivector portion, and T (regular uppercase) is the trivector portion.

Parity of Multivector Elements

For our three dimensional, Euclidean space, we have scalars as parity invariant, meaning even parity like electric charges, vectors as odd parity (like electric fields), bivectors with even parity (like magnetic fields), and trivectors with odd parity (like magnetic monopoles).

Wedge Product

The wedge product is the anti-symmetric product as seen in the product of different vector basii above.

$$\begin{aligned} e_x \wedge e_y &= -e_y \wedge e_x \\ e_x \wedge e_y \wedge e_z &= -e_x \wedge e_z \wedge e_y \end{aligned}$$

Being anti-symmetric, any squared terms will result in a zero product.

$$e_x \wedge e_x = 0$$

Historically, Clifford defined the geometric product of two vectors as a combination of the dot product and wedge.

$$\vec{a}\vec{b} = \vec{a} \cdot \vec{b} + \vec{a} \wedge \vec{b}$$

In the following paragraphs, we will see the wedge product as the superior replacement for the cross product.

Duality and the Pseudoscalar

We have a built-in duality in 3D Euclidean geometric algebra. We notice one scalar component, one trivector component, three vector components, and three bivector components. In essence, we have an axis of symmetry between the vectors and bivectors, and can make a one to one association between components on either side.

$$\begin{aligned} q &\leftrightarrow T \\ e_x &\leftrightarrow e_y e_z \\ e_y &\leftrightarrow e_z e_x \\ e_z &\leftrightarrow e_x e_y \end{aligned}$$

As we examine this in more detail, we notice that multiplying a multivector by $e_x e_y e_z$ effects just this transformation. Under a parity transformation, where we invert the sign of each our our spatial basii, the trivector $e_x e_y e_z$ changes sign. Consequently, we associate the trivector with pseudoscalars, and often call the trivector as the pseudoscalar. In the extended Maxwell equations, electric charge is a scalar, while magnetic charge is a pseudoscalar.

We now introduce our replacement for the cross product in three dimensions. The standard cross product and wedge are

$$\begin{aligned}
\vec{a} &= a_x e_x + a_y e_y + a_z e_z \\
\vec{b} &= b_x e_x + b_y e_y + b_z e_z \\
\vec{a} \times \vec{b} &= e_x(a_y b_z - a_z b_y) + e_y(a_z b_x - a_x b_z) + e_z(a_x b_y - a_y b_x) \\
\vec{a} \wedge \vec{b} &= e_y e_z(a_y b_z - a_z b_y) + e_z e_x(a_z b_x - a_x b_z) + e_x e_y(a_x b_y - a_y b_x) \\
(e_x e_y e_z)\vec{a} \wedge \vec{b} &= (e_x e_y e_z) e_y e_z(a_y b_z - a_z b_y) + \\
&\quad (e_x e_y e_z) e_z e_x(a_z b_x - a_x b_z) + \\
&\quad (e_x e_y e_z) e_x e_y(a_x b_y - a_y b_x) \\
&= -e_x(a_y b_z - a_z b_y) - e_y(a_z b_x - a_x b_z) - e_z(a_x b_y - a_y b_x) \\
\vec{a} \times \vec{b} &= -(e_x e_y e_z)\vec{a} \wedge \vec{b}
\end{aligned}$$

Using $I = e_x e_y e_z$, we see by inspection

$$\vec{a} \times \vec{b} = -I \vec{a} \wedge \vec{b}$$

The wedge product creates a bivector, while the product with the pseudoscalar converts the bivector to a vector. That said, a word of caution is in order. Many times, the use of the cross product caused an ambiguity between vector and bivector terms. Literally applying the formula above will preserve that confusion, which usually is *not* what we want. Most often, we really want to replace the cross product with the wedge, and thus separate out the bivectors from the vectors.

Nilpotents and Idempotents

Nilpotents and idempotents are structures in geometric algebra which can lead to amazing simplications in calculations.

Nilpotents are defined as multivectors which square to zero.

$$N * N = 0$$

. In multivector format, nilpotents are characterized by zero values for scalar and pseudoscalar, with vector and bivector components of equal magnitude, yet orthogonal to each other. With unit vector $\vec{u}$ and $\vec{v}$ orthogonal, and k an arbitrary scale factor, a generic nilpotent has the form

$$z = k(\vec{u} + \vec{u}\vec{v})$$

.

Idempotents are defined as multivectors which square to themselves.

$$P * P = P$$

. As an example, the simple scalars, 0 and 1 are idempotents.

Multivector idempotents (other than the 0 and 1 above) are characterized by a zero value for the trivector component, a value of 1/2 for the scalar component, orthogonality between the vector and bivector components, and the magnitudes of the vector and bivector component matching the hyperbolic function scaling shown below.

Multivector idempotents come in pairs. The two unit vectors $\vec{u}$ and $\vec{v}$ in the definitions below must be orthogonal, meaning $\vec{u}\vec{v} = -\vec{v}\vec{u}$. Notice also that the product of these idempotent pairs is zero.

$$\begin{aligned} P_+ &= \frac{1}{2}(1 + (\vec{u}\cosh(\alpha) + \vec{u}\vec{v}\sinh(\alpha))) = P \\ P_- &= \frac{1}{2}(1 - (\vec{u}\cosh(\alpha) + \vec{u}\vec{v}\sinh(\alpha))) = (1 - P) \\ P_+P_- &= 0 \end{aligned}$$

Nilpotents and idempotents are closely related.

$$\begin{aligned} P_\pm &= \vec{v}z_\pm \\ z_\pm &= \vec{v}P_\pm \\ &= \frac{1}{2}(\vec{v} \pm \vec{v}\vec{u}\cosh(\alpha) \mp \vec{u}\sinh(\alpha)) \end{aligned}$$

Maxwell Equations in 3D Geometric Algebra

With this all said and done, we now return to our complexified, extended Maxwell equations.

$$
\begin{aligned}
\vec{F} &= \vec{E} + ic\vec{B} \\
\vec{\nabla} \cdot \vec{F} &= z\left(c\rho_e + i\rho_m\right) \\
\vec{\nabla} \times \vec{F} &= i\left(\frac{1}{c}\frac{\partial \vec{F}}{\partial t} + \mu\left(c\vec{j}_e + i\vec{j}_m\right)\right)
\end{aligned}
$$

We see the imaginary element i, and get excited. Clearly, the field $\vec{F}$ should become a multivector field F, with vector component $\vec{E}$ and bivector component $c\mathbf{B}$.

$$F = \vec{E} + c\mathbf{B}$$

We note in passing, that F definitely can have a nilpotent component, and even be totally nilpotent, under the right conditions.

In the dot product term, we see scalar components for electric charge, and pseudoscalar components for magnetic charge. Here we simply replace i with our trivector $I = e_x e_y e_z$.

$$\vec{\nabla} \cdot F = z\left(c\rho_e + (e_x e_y e_z)\rho_m\right)$$

Finally, we look at the cross product term. Blindly substituting for the cross product using $\vec{\nabla} \times \vec{F} = -I\vec{\nabla} \wedge \vec{F}$, and replacing i with I yields

$$
\begin{aligned}
\vec{\nabla} \times \vec{F} &= I\left(\frac{1}{c}\frac{\partial \vec{F}}{\partial t} + \mu\left(c\vec{j}_e + I\vec{j}_m\right)\right) \\
-I\nabla \wedge F &= I\left(\frac{1}{c}\frac{\partial F}{\partial t} + \mu\left(c\vec{j}_e + I\vec{j}_m\right)\right) \\
\nabla \wedge F &= -\left(\frac{1}{c}\frac{\partial F}{\partial t} + \mu\left(c\vec{j}_e + I\vec{j}_m\right)\right) \\
\frac{1}{c}\frac{\partial F}{\partial t} + \nabla \wedge F &= -\mu\left(c\vec{j}_e + (e_x e_y e_z)\vec{j}_m\right)
\end{aligned}
$$

This interesting equation states that current creates a field gradient in the time direction, and twists the field in the spatial directions.

While this equation is interesting, it does not match standard form, as it only uses a wedge product, rather than the full geometric product. To get standard form, we exploit $\nabla F = \nabla \cdot F + \nabla \wedge F$, and simply add $\nabla \cdot F$ to both sides of the equation.

$$\begin{aligned}
\frac{1}{c}\frac{\partial F}{\partial t} + \nabla \wedge F &= -\mu\left(c\vec{j}_e + (e_x e_y e_z)\vec{j}_m\right) \\
\frac{1}{c}\frac{\partial F}{\partial t} + \nabla \wedge F + \nabla \cdot F &= \nabla \cdot F - \mu\left(c\vec{j}_e + (e_x e_y e_z)\vec{j}_m\right) \\
\frac{1}{c}\frac{\partial F}{\partial t} + \nabla F &= \nabla \cdot F - \mu\left(c\vec{j}_e + (e_x e_y e_z)\vec{j}_m\right)
\end{aligned}$$

Now,

$$\begin{aligned}
\nabla \cdot F &= \nabla \cdot (E + IcB) \\
&= \nabla \cdot E + Ic\nabla \cdot B \\
&= zc\rho_e + Iz\rho_m
\end{aligned}$$

Consequently, we retrieve the standard form as found in Chappell [7]

$$\begin{aligned}
\frac{1}{c}\frac{\partial F}{\partial t} + \nabla F &= \nabla \cdot F - \mu\left(c\vec{j}_e + (e_x e_y e_z)\vec{j}_m\right) \\
\left(\frac{1}{c}\frac{\partial}{\partial t} + \nabla\right) F &= (zc\rho_e + Iz\rho_m) - \mu\left(c\vec{j}_e + (e_x e_y e_z)\vec{j}_m\right)
\end{aligned}$$

We now follow the derivation of Chappell and company [7], and then find agreement with the formula above. Repeating Maxwell's extended equations,

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= z\left(c\rho_e\right) \\
\vec{\nabla} \cdot \left(c\vec{B}\right) &= z\rho_m \\
\vec{\nabla} \times \vec{E} &= -\frac{1}{c}\frac{\partial (c\vec{B})}{\partial t} - \mu\vec{j}_m \\
\vec{\nabla} \times \left(c\vec{B}\right) &= \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \mu\left(c\vec{j}_e\right)
\end{aligned}$$

We multiply equations 3 and 4 by I to convert cross product to wedge.

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= z\left(c\rho_e\right) \\
\vec{\nabla} \cdot \left(c\vec{B}\right) &= z\rho_m \\
I\vec{\nabla} \times \vec{E} &= -\frac{I}{c}\frac{\partial(c\vec{B})}{\partial t} - I\mu\vec{j}_m \\
I\vec{\nabla} \times \left(c\vec{B}\right) &= \frac{I}{c}\frac{\partial\vec{E}}{\partial t} + I\mu\left(c\vec{j}_e\right)
\end{aligned}$$

Notice we are still treating $\vec{B}$ as a vector. Our wedged set of equations are

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= z\left(c\rho_e\right) \\
\vec{\nabla} \cdot \left(c\vec{B}\right) &= z\rho_m \\
\nabla \wedge \vec{E} &= -\frac{I}{c}\frac{\partial(c\vec{B})}{\partial t} - I\mu\vec{j}_m \\
\nabla \wedge \left(c\vec{B}\right) &= \frac{I}{c}\frac{\partial\vec{E}}{\partial t} + I\mu\left(c\vec{j}_e\right)
\end{aligned}$$

The geometric product of two vectors is $\vec{a}\vec{b} = \vec{a} \cdot \vec{b} + \vec{a} \wedge \vec{b}$. Applying this to our differential operators

$$\nabla\vec{E} = \nabla \cdot \vec{E} + \nabla \wedge E$$

We notice that adding equations 1 and 3, 2 and 4 exploits this relationship to eliminate the dot and wedge products.

$$\begin{aligned}
\nabla\vec{E} &= z\left(c\rho_e\right) - \frac{I}{c}\frac{\partial(c\vec{B})}{\partial t} - I\mu\vec{j}_m \\
\nabla\left(c\vec{B}\right) &= z\rho_m + \frac{I}{c}\frac{\partial\vec{E}}{\partial t} + I\mu\left(c\vec{j}_e\right)
\end{aligned}$$

In the above, we are treating $\vec{B}$ as a vector. We now shift B to a bivector by absorbing a factor of I in the first equation, and multiplying the second equation by I.

$$\begin{aligned}
\nabla\vec{E} &= z\left(c\rho_e\right) - \frac{1}{c}\frac{\partial(Ic\vec{B})}{\partial t} - I\mu\vec{j}_m \\
\nabla\left(Ic\vec{B}\right) &= Iz\rho_m - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} - \mu\left(c\vec{j}_e\right)
\end{aligned}$$

Adding these two equations, we have

$$\begin{aligned}\nabla\left(\vec{E}+Ic\vec{B}\right) &= z\left(c\rho_e\right)-\frac{1}{c}\frac{\partial(Ic\vec{B})}{\partial t}-I\mu\vec{j}_m+Iz\rho_m-\frac{1}{c}\frac{\partial\vec{E}}{\partial t}-\mu\left(c\vec{j}_e\right)\\ &= -\frac{1}{c}\frac{\partial}{\partial t}\left(\vec{E}+Ic\vec{B}\right)+\left(zc\rho_e-\mu c\vec{j}_e\right)+I\left(z\rho_m-\mu\vec{j}_m\right)\end{aligned}$$

Transposing the partial time derivative, we obtain

$$\left(\frac{1}{c}\frac{\partial}{\partial t}+\nabla\right)\left(\vec{E}+Ic\vec{B}\right) = \left(zc\rho_e-\mu c\vec{j}_e\right)+I\left(z\rho_m-\mu\vec{j}_m\right)$$

In the absense of magnetic monopoles ($\rho_m=0,\vec{j}_m=0$), we have the same expression as in the Chappell paper.

Duality Using Geometric Algebra on Maxwell Equations

We are now in a position to repeat the observations of Heaviside and Larmor. If we multiply our equation on both sides by I, we obtain

$$\begin{aligned}\left(\frac{1}{c}\frac{\partial}{\partial t}+\nabla\right)\left(\vec{E}+Ic\vec{B}\right) &= \left(zc\rho_e-\mu c\vec{j}_e\right)+I\left(z\rho_m-\mu\vec{j}_m\right)\\ \left(\frac{1}{c}\frac{\partial}{\partial t}+\nabla\right)\left(I\vec{E}-c\vec{B}\right) &= I\left(zc\rho_e-\mu c\vec{j}_e\right)-\left(z\rho_m-\mu\vec{j}_m\right)\end{aligned}$$

We have the same basic structure of the Maxwell equations, but we have interchange E and B as far as vector versus bivector. Likewise, we have changed the source charges nature, per scalar versus pseudoscalar.

Following the observation of Joseph Larmor, we can implement a continous mixing of electric and magnetic fields. In 3D Euclidean geometric algebra, both scalar and trivector terms commute with all other algebra elements. We can thus apply a complex scale factor of $(\cos\theta+I\sin\theta)$ to both sides of our equation.

$$
\begin{aligned}
\left(\frac{1}{c}\frac{\partial}{\partial t} + \nabla\right)\left(\vec{E} + Ic\vec{B}\right) &= \left(zc\rho_e - \mu c\vec{j}_e\right) + I\left(z\rho_m - \mu\vec{j}_m\right) \\
F' &= \left((\cos\theta + I\sin\theta)\left(\vec{E} + Ic\vec{B}\right)\right) \\
&= (\cos\theta\vec{E} - c\sin\theta\vec{B}) + I(\sin\theta\vec{E} + c\cos\theta\vec{B}) \\
\left(\frac{1}{c}\frac{\partial}{\partial t} + \nabla\right)F' &= \left(zc\rho_e - \mu c\vec{j}_e\right)(\cos\theta + I\sin\theta) \\
&\quad + I\left(z\rho_m - \mu\vec{j}_m\right)(\cos\theta + I\sin\theta) \\
\left(\frac{1}{c}\frac{\partial}{\partial t} + \nabla\right)F' &= \left(zc\rho_e - \mu c\vec{j}_e\right)\cos\theta - \left(z\rho_m - \mu\vec{j}_m\right)\sin\theta \\
&\quad + I\left[\left(zc\rho_e - \mu c\vec{j}_e\right)\sin\theta + \left(z\rho_m - \mu\vec{j}_m\right)\cos\theta\right]
\end{aligned}
$$

Once again, we maintain the form of the Maxwell equation, and we allow continuous mixing of electric and magnetic charge.

Julian Schwinger [4] took this concept of mixing angle one step further. Postulating coexisting magnetic and electric charge in a fundamental particle, he points out that as long as the ratio of magnetic versus electric charge does not change, the magnetic and electric current densities will likewise be proportional, and that we can choose a convention for the mixing angle such that magnetic charge, corresponding to the trivector term on the right hand side, seems to disappear.

References

[1] James Clerk Maxwell, *Electricity and Magnetism, Vol II, Article 701*

[2] Oliver Heaviside, Phil. Trans. Roy. Soc. (London) A 183, 423 (1893)

[3] Joseph Larmor, *Collected Papers*, London, 1928.

[4] Julian Schwinger, *Classical Electrodynamics*, Perseus Books, Reading Massachusetts, 1998.

[5] John David Jackson, *Classical Electrodynamics*, Wiley, New York 1975.

[6] David J. Griffiths, *Introduction to Electromechanics*, Prentice-Hall, 1981

[7] James M. Chappell, Samuel P. Drake, Cameron L. Seidel, Lachlan J. Gunn, Azhar Iqbal, Andrew Allison, and Derek Abbott, *Geometric Algebra for Electrical and Electronic Engineers* Proceedings of the IEEE, Vol. 102, No. 9, September 2014

[8] Anthony Lasenby, Chris Doran and E. Arcaute, *Applications of Geometic Algebra in Electromagnetism, Quantum Mechanics and Gravity* http://www.mrao.cam.ac.uk/~clifford/publications/ps/lda_tenn.pdf

[9] John Denker, *Electromagnetism using Geometric Algebra versus Components*, http://www.av8n.com/physics/maxwell-ga.pdf

[10] Chris Doran and Anthony Lasenby, *Geometric Algebra for Physicists* Cambridge University Press, ISBN 978-0-521-71595-9

[11] Leo Dorst, Daniel Fontune and Stephen Mann, *Geometric Algebra for Computer Science* Morgan Kaufmann Publishers, ISBN 978-0-12-374942-0

[12] David Hestenes and Garret Sobczyk, *Clifford Algebra to Geometric Calculus* D. Reidal Publishing Company, ISBN 978-90-277-2581-5

[13] Anthony Lasenby and Chris Doran, *Lectures and Handouts 1999* http://www.mrao.cam.ac.uk/clifford/ptIIIcourse/