Electromagnetic Duality in SI Units

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Abstract

This note presents the continuous electromagnetic duality of I. Larmor [4] and J. Schwinger [5], also referencing notes by Steven Errede [6], using SI units.

Maxwell Equations and Monopole Extensions

Conventional Maxwell equations [1] in SI units [2] are

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \frac{\rho_e}{\epsilon} \\
\vec{\nabla} \cdot \vec{B} &= 0 \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\vec{\nabla} \times \vec{B} &= \mu\epsilon \frac{\partial \vec{E}}{\partial t} + \mu \vec{j}_e
\end{aligned}
$$

where μ is the magnetic permeability of space, ϵ is the electric permittivity of space, and $\mu\epsilon = 1/c^2$, where c is the speed of light.

In the absense of sources, the free space equation become

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= 0 \\
\vec{\nabla} \cdot \vec{B} &= 0 \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\vec{\nabla} \times \vec{B} &= \mu\epsilon \frac{\partial \vec{E}}{\partial t}
\end{aligned}
$$

In 1893, Oliver Heaviside [3] noted the substitutions $\vec{E} \to -c\vec{B}$ and $\vec{B} \to \vec{E}/c$ leave the free space Maxwell equations unchanged.

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{E} = 0 \quad &\to \quad \vec{\nabla} \cdot \left(-c\vec{B}\right) = 0 \to \vec{\nabla} \cdot \vec{B} = 0 \\
\vec{\nabla} \cdot \vec{B} = 0 \quad &\to \quad \vec{\nabla} \cdot \left(\vec{E}/c\right) = 0 \to \vec{\nabla} \cdot \vec{E} = 0 \\
\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad &\to \quad \vec{\nabla} \times \left(-c\vec{B}\right) = -\frac{\partial \left(\vec{E}/c\right)}{\partial t} \to \vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu\epsilon\frac{\partial \vec{E}}{\partial t} \\
\vec{\nabla} \times \vec{B} = \mu\epsilon\frac{\partial \vec{E}}{\partial t} \quad &\to \quad \vec{\nabla} \times \left(\vec{E}/c\right) = -c\mu\epsilon\frac{\partial \vec{B}}{\partial t} \to \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}
\end{aligned}
$$

Applying the same transformation to the full Maxwell equations requires the introduction of magnetic charge, analogous to electric charge. First, we substitute into the standard Maxwell equations.

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{E} = \frac{\rho_e}{\epsilon} \quad &\to \quad \vec{\nabla} \cdot \left(-c\vec{B}\right) = \frac{\rho_e}{\epsilon} \\
&\to \quad \vec{\nabla} \cdot \vec{B} = -\frac{\rho_e}{c\epsilon} = -c\mu\rho_e \\
\vec{\nabla} \cdot \vec{B} = 0 \quad &\to \quad \vec{\nabla} \cdot \left(\vec{E}/c\right) = 0 \to \vec{\nabla} \cdot \vec{E} = 0 \\
\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad &\to \quad \vec{\nabla} \times \left(-c\vec{B}\right) = -\frac{\partial \vec{E}/c}{\partial t} \\
&\to \quad \vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu\epsilon\frac{\partial \vec{E}}{\partial t} \\
\vec{\nabla} \times \vec{B} = \mu\epsilon\frac{\partial \vec{E}}{\partial t} + \mu\vec{j}_e \quad &\to \quad \vec{\nabla} \times \left(\vec{E}/c\right) = \mu\epsilon\frac{\partial \left(-c\vec{B}\right)}{\partial t} + \mu\vec{j}_e \\
&\to \quad \vec{\nabla} \times \vec{E} = -c^2\mu\epsilon\frac{\partial \vec{B}}{\partial t} + c\mu\vec{j}_e \\
&\to \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} + c\mu\vec{j}_e
\end{aligned}
$$

We see that the free space symmetry is lost. However, introduction of a magnetic charge, which transforms into electric charge similar to the electric and magnetic field transformation restores symmetry.

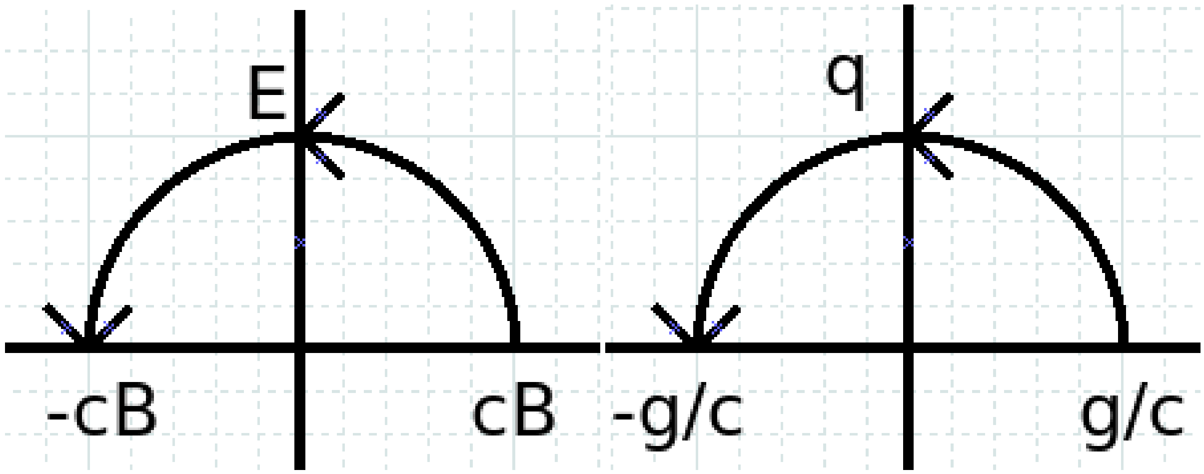

Figure 1: Rotation of Fields and Charges

Extending the Maxwell equations to include magnetic monopoles, we have

$$\begin{aligned}\vec{\nabla}\cdot\vec{E} &= \frac{\rho_e}{\epsilon}\\ \vec{\nabla}\cdot\vec{B} &= \mu\rho_m\\ \vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t}-\mu\vec{j}_m\\ \vec{\nabla}\times\vec{B} &= \mu\epsilon\frac{\partial\vec{E}}{\partial t}+\mu\vec{j}_e\end{aligned}$$

We now concurrently rotate fields and charges where magnetic charge is g and electric charge is q. These rotations are shown in Figure 1.

$$\begin{aligned}\vec{E}\rightarrow -c\vec{B} \quad &, \quad \vec{B}\rightarrow \vec{E}/c\\ q\rightarrow -g/c \quad &, \quad g\rightarrow cq\\ \rho_e\rightarrow -\rho_m/c \quad &, \quad \rho_m\rightarrow c\rho_e\\ \vec{j}_e\rightarrow -\vec{j}_m/c \quad &, \quad \vec{j}_m\rightarrow c\vec{j}_e\end{aligned}$$

Explicitly demonstrating the equivalence of the rotated forms, we have

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{E} = \frac{\rho_e}{\epsilon} \quad &\rightarrow \quad \vec{\nabla}\cdot\left(-c\vec{B}\right) = \frac{(-\rho_m/c)}{\epsilon} \\
&\rightarrow \quad \vec{\nabla}\cdot\vec{B} = \frac{\rho_m}{c^2\epsilon} = \mu\rho_m \\
\vec{\nabla}\cdot\vec{B} = \mu\rho_m \quad &\rightarrow \quad \vec{\nabla}\cdot\left(\vec{E}/c\right) = \mu\left(c\rho_e\right) \\
&\rightarrow \quad \vec{\nabla}\cdot\vec{E} = \mu c^2\rho_e = \frac{\rho_e}{\epsilon} \\
\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} - \mu\vec{j}_m \quad &\rightarrow \quad \vec{\nabla}\times\left(-c\vec{B}\right) = -\frac{\partial\left(\vec{E}/c\right)}{\partial t} - \mu\left(c\vec{j}_e\right) \\
&\rightarrow \quad \vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} + \mu\vec{j}_e = \mu\epsilon\frac{\partial\vec{E}}{\partial t} + \mu\vec{j}_e \\
\vec{\nabla}\times\vec{B} = \mu\epsilon\frac{\partial\vec{E}}{\partial t} + \mu\vec{j}_e \quad &\rightarrow \quad \vec{\nabla}\times\left(\vec{E}/c\right) = \mu\epsilon\frac{\partial\left(-c\vec{B}\right)}{\partial t} + \mu\left(-\vec{j}_m/c\right) \\
&\rightarrow \quad \vec{\nabla}\times\vec{E} = -c^2\mu\epsilon\frac{\partial\vec{B}}{\partial t} - \mu\vec{j}_m = -\frac{\partial\vec{B}}{\partial t} - \mu\vec{j}_m
\end{aligned}
$$

One advantage of using relativistic units, where $c = 1$, is that the rotation involved in the substitution above is easily apparent. However, since I am using SI units, I need to include factors of c explicitly. Essentially, whenever the units of measure include seconds, I have a factor of c applied for each instance.

Following is the extended Maxwell Equations, re-written to emphasis the dualities $c\vec{B} \rightarrow \vec{E}$ and $g \rightarrow cq$, with $c = 1/\sqrt{\mu\epsilon}$ and $z = \sqrt{\mu/\epsilon}$.

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{E} &= z\left(c\rho_e\right) \\
\vec{\nabla}\cdot\left(c\vec{B}\right) &= z\rho_m \\
\vec{\nabla}\times\vec{E} &= -\frac{1}{c}\frac{\partial(c\vec{B})}{\partial t} - \mu\vec{j}_m \\
\vec{\nabla}\times\left(c\vec{B}\right) &= \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \mu\left(c\vec{j}_e\right)
\end{aligned}
$$

In this format, we see clearly see that ct is a better fit than t in the partial derivative terms. In a sense, fourspace is more natural than spacetime.

Discrete and Continuous Duality

Oliver Heaviside's [3] discrete transformations $\vec{E} \to -c\vec{B}$ and $c\vec{B} \to \vec{E}$ were generalized by Larmor to be continous rotations of a complex field.

Define $\vec{F} = \vec{E} + ic\vec{B}$. The generalized Maxwell equations then become

$$\begin{aligned}
\vec{\nabla} \cdot \vec{F} &= z\left(c\rho_e + i\rho_m\right) \\
\vec{\nabla} \times \vec{F} &= -\frac{1}{c}\frac{\partial(c\vec{B})}{\partial t} - \mu\vec{j}_m + i\left(\frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \mu\left(c\vec{j}_e\right)\right) \\
&= i\left(\frac{1}{c}\frac{\partial\vec{E} + ic\vec{B}}{\partial t} + \mu\left(c\vec{j}_e + i\vec{j}_m\right)\right) \\
\vec{\nabla} \times \vec{F} &= i\left(\frac{1}{c}\frac{\partial\vec{F}}{\partial t} + \mu\left(c\vec{j}_e + i\vec{j}_m\right)\right)
\end{aligned}$$

When Larmor's phase factor $e^{i\phi}$ is applied to both charges and fields, we maintain the form of the complexified Maxwell Equations.

$$\begin{aligned}
\vec{\nabla} \cdot \vec{F} &= z\left(c\rho_e + i\rho_m\right) \\
\vec{\nabla} \times \vec{F} &= i\left(\frac{1}{c}\frac{\partial\vec{F}}{\partial t} + \mu\left(c\vec{j}_e + i\vec{j}_m\right)\right) \\
\vec{G} &= e^{i\phi}\vec{F} \\
\rho_r &= e^{i\phi}\left(c\rho_e + i\rho_m\right) \\
\vec{j}_r &= e^{i\phi}\left(c\vec{j}_e + i\vec{j}_m\right) \\
\vec{\nabla} \cdot \vec{G} &= z\rho_r \\
\vec{\nabla} \times \vec{G} &= i\left(\frac{1}{c}\frac{\partial\vec{G}}{\partial t} + \mu\vec{j}_r\right)
\end{aligned}$$

As an aside, just as a factor of c above provides encourage for a fourspace concept, the present of i here provides encouragement for the Minkowski spacetime, with ict for the fourth dimension.

References

[1] J. C. Maxwell, *Electricity and Magnetism, Vol II, Article 701*

[2] David J. Griffiths, *Introduction to Electromechanics*, Prentice-Hall, 1981

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[5] Julian Schwinger, *Classical Electrodynamics*, Perseus Books, Reading Massachusetts, 1998.

[6] Steven Errede, *UIUC Physics 436 EM Fields & Sources II, Fall Semester, 2011*

[7] John David Jackson, *Classical Electrodynamics*, Wiley, New York 1975.