Quaternions and Dual Coupled Orthogonal Rotations in Four-Space

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Abstract

Quaternion multiplication causes tensor (stretching) and versor (turning) operations. Multiplying by unit quaternions results in simple versor operations. This paper shows the geometric interpretation of unit quaternion multiplication as coupled rotations in two orthogonal planes for both prefactor and postfactor multiplications. This paper then shows prefactor and postfactor interpretations for the differential angular conjugated quaternion products.

Quaternion Definitions

Quaternions are a combination of a scalar and three-vector, resulting in an effective four-vector. I use a tilde overmark over capital letters to identify quaternions, and then use a lower case letter to identify the scalar (time) component, and a capital letter with vector overmark to identify the vector component. As an example, $\tilde{Q} = q + \vec{Q}$. While I will commonly use matched lower and upper case letters, this is not always the case. For example, with electromagnetic potential, I will often use $\tilde{\Phi} = \phi + \vec{A}$.

Given the symbology above, the right-handed quaternion product is

$$\begin{aligned} \tilde{A}\tilde{B} &= (a + \vec{A})(b + \vec{B}) \\ &= (ab - \vec{A} \cdot \vec{B}) + (a\vec{B} + b\vec{A} + \vec{A} \times \vec{B}) \end{aligned}$$

This product is associative, but not commutative. I mention in passing that there also exists a left-handed quaternion product, where the sign of the cross product is changed. This left handed product is also associative, but not commutative.

We define the magnitude of a quaternion as

$$|\tilde{A}| \;=\; \sqrt{a^2 + \vec{A}\cdot\vec{A}}$$

We define the dot product of two quaternions as

$$\tilde{A}\cdot\tilde{B} \;=\; ab + \vec{A}\cdot\vec{B}$$

The product has the fine identity that

$$|\tilde{A}\tilde{B}| \;=\; |\tilde{A}|\;|\tilde{B}|$$

We define conjugation as

$$\tilde{A}^* \;=\; a - \vec{A}$$

Combining the previous two equation, we can define unit division as

$$\frac{1}{\tilde{A}} \;=\; \frac{\tilde{A}^*}{|\tilde{A}|^2}$$

We have two forms of division: pre-division and post-division.

$$\begin{aligned}
\frac{1}{\tilde{B}}\tilde{A} &= \frac{\tilde{B}^*\tilde{A}}{|\tilde{B}|^2} \\
\tilde{A}\frac{1}{\tilde{B}} &= \frac{\tilde{A}\tilde{B}^*}{|\tilde{B}|^2}
\end{aligned}$$

We can explicitly separate the tensor and versor forms of the quaternion product.

$$\begin{aligned}
\tilde{A} &= \sqrt{a^2 + \vec{A}\cdot\vec{A}}\frac{a + \vec{A}}{\sqrt{a^2 + \vec{A}\cdot\vec{A}}} \\
&= \sqrt{a^2 + \vec{A}\cdot\vec{A}}\,(\cos\theta + \vec{u}\sin\theta) \\
\vec{u} &= \frac{\vec{A}}{\sqrt{\vec{A}\cdot\vec{A}}} \\
\sin\theta &= \frac{|\vec{A}|}{\sqrt{a^2 + \vec{A}\cdot\vec{A}}} \\
\cos\theta &= \frac{a}{\sqrt{a^2 + \vec{A}\cdot\vec{A}}}
\end{aligned}$$

The generic form for a unit quaternion is $\cos\theta + \vec{u}\sin\theta$.

Influenced by four-vectors in physics, I call the scalar portion the time component. Comparing this with a generic complex number, we see that the unit quaternion traces out a circle in the $(t, \vec{u})$ plane as θ runs from 0 to 2π.

Geometric Interpretation of Unit Quaternion Post-Multiplication

We look at the generic product of a quaternion $\tilde{A}$ post-multiplied by a unit quaternion.

$$\begin{aligned}
\tilde{A}\tilde{U} &= (a + \vec{A})(\cos\theta + \vec{u}\sin\theta) \\
&= \left(a\cos\theta - \vec{A}\cdot\vec{u}\sin\theta\right) + a\vec{u}\sin\theta + \vec{A}\cos\theta + \vec{A}\times\vec{u}\sin\theta
\end{aligned}$$

We now express $\vec{A}$ in terms of perpendicular and parallel components with respect to $\vec{u}$.

$$\vec{A} = \vec{u}\times\left(\vec{A}\times\vec{u}\right) + \vec{u}\left(\vec{A}\cdot\vec{u}\right)$$

While we are at it, express $\tilde{A}$ in terms of coplanar with $\tilde{U}$ and normal components.

$$\begin{aligned}
\tilde{A} &= a + \vec{A} \\
&= \left(a + \vec{u}\left(\vec{A}\cdot\vec{u}\right)\right) + \left(0 + \vec{u}\times\left(\vec{A}\times\vec{u}\right)\right)
\end{aligned}$$

Substituting this back in our product, and re-arranging, we have

$$\begin{aligned}
\tilde{A}\tilde{U} &= (a + \vec{A})(\cos\theta + \vec{u}\sin\theta) \\
&= \left(a\cos\theta - \left(\vec{A}\cdot\vec{u}\right)\sin\theta\right) + a\vec{u}\sin\theta + \vec{A}\cos\theta + \left(\vec{A}\times\vec{u}\right)\sin\theta \\
&= \left(a\cos\theta - \left(\vec{A}\cdot\vec{u}\right)\sin\theta\right) + a\vec{u}\sin\theta + \left(\vec{u}\times\left(\vec{A}\times\vec{u}\right) + \vec{u}\left(\vec{A}\cdot\vec{u}\right)\right)\cos\theta + \left(\vec{A}\times\vec{u}\right)\sin\theta \\
&= \left(a\cos\theta - \left(\vec{A}\cdot\vec{u}\right)\sin\theta\right) + \vec{u}\left(a\sin\theta + \left(\vec{A}\cdot\vec{u}\right)\cos\theta\right) \\
&\quad + \left(\vec{u}\times\left(\vec{A}\times\vec{u}\right)\right)\cos\theta + \left(\vec{A}\times\vec{u}\right)\sin\theta
\end{aligned}$$

For reference, a vector (x, y) rotated by θ in the xy plane has new coordinates

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

We see that with post-factor multiplication, the coplanar component of $\tilde{A}$ with $\tilde{U}$ has been rotated by θ in the $(t, \vec{u})$ plane, while the normal component has been rotated by θ in the normal plane $(\vec{u} \times (\vec{A} \times \vec{u}), \vec{A} \times u)$.

Geometric Interpretation of Unit Quaternion Pre-Multiplication

We now repeat this exercise with pre-factor multiplication.

$$\begin{aligned}
\tilde{U}\tilde{A} &= (\cos\theta + \vec{u}\sin\theta)(a + \vec{A}) \\
&= (\cos\theta + \vec{u}\sin\theta)\left[\left(a + \vec{u}\left(\vec{A}\cdot\vec{u}\right)\right) + \left(0 + \vec{u}\times\left(\vec{A}\times\vec{u}\right)\right)\right] \\
&= \left(a\cos\theta - \left(\vec{u}\cdot\vec{A}\right)\sin\theta\right) + \vec{A}\cos\theta + \vec{u}a\sin\theta + \left(\vec{u}\times\vec{A}\right)\sin\theta \\
&= \left(a\cos\theta - \left(\vec{u}\cdot\vec{A}\right)\sin\theta\right) + \left(\vec{u}\left(\vec{A}\cdot\vec{u}\right) + \vec{u}\times\left(\vec{A}\times\vec{u}\right)\right)\cos\theta \\
&\quad +\vec{u}a\sin\theta + \left(\vec{u}\times\vec{A}\right)\sin\theta \\
&= \left(a\cos\theta - \left(\vec{u}\cdot\vec{A}\right)\sin\theta\right) + \vec{u}\left(\left(\vec{A}\cdot\vec{u}\right)\cos\theta + a\sin\theta\right) \\
&\quad +\vec{u}\times\left(\vec{A}\times\vec{u}\right)\cos\theta - \left(\vec{A}\times\vec{u}\right)\sin\theta
\end{aligned}$$

Comparing with the post-multiplication formula, we see that the $t, \vec{u}$ plane rotated by θ as before. However, we see that the normal plane is rotated θ counter-clockwise, or reverse from above.

Conjugated Post-Factor Post-Multiplication

The conjugated post-factor product provides an angular differential product.

$$\begin{aligned}
\tilde{A}\tilde{U}^* &= (a + \vec{A})(\cos\theta - \vec{u}\sin\theta) \\
&= (a + \vec{A})\left(\cos(-\theta) + \vec{u}\sin(-\theta)\right)
\end{aligned}$$

This is the same as the post-product, except that the angle has been reversed for both planes. The coplanar component of $\tilde{A}$ with $\tilde{U}$ has been rotated by $-\theta$ in the $(t, \vec{u})$ plane, as has the normal component, which has been rotated by $-\theta$ in the normal plane $(\vec{u} \times (\vec{A} \times \vec{u}), \vec{A} \times u)$.

I want to point out that the time component of this product is the dot product of the two quaternions, while the space component is a four-space cross product.

$$
\begin{aligned}
\tilde{A}\tilde{U}^* &= (a + \vec{A})(\cos\theta - \vec{u}\sin\theta) \\
&= \left(a\cos\theta + \left(\vec{A} \cdot \vec{u}\right)\sin\theta\right) + \vec{A}\cos\theta - \vec{u}a\sin\theta + \left(\vec{u} \times \vec{A}\right)\sin\theta
\end{aligned}
$$

Conjugated Pre-Factor Pre-Multiplication

$$
\begin{aligned}
\tilde{U}^*\tilde{A} &= (\cos\theta - \vec{u}\sin\theta)(a + \vec{A}) \\
&= (a + \vec{A})\left(\cos(-\theta) + \vec{u}\sin(-\theta)\right)
\end{aligned}
$$

For the conjugated pre-factor pre-multiplication, the $t, \vec{u}$ plane is rotated by $-\theta$ while the normal plane is rotated by θ. Again, the time component is a dot product of the two quaternions, while the space component is a four-space cross product.

Sandwich Quaternion Formula for Spatial Rotations

The sandwich product for spatial rotations about an arbitrary axis is

$$
\text{rotate by } \theta \quad = \quad e^{-\vec{u}\theta/2}\vec{v}e^{\vec{u}\theta/2}
$$

The first product does half the spatial rotation, coupled with a negative rotation in the $(t, \vec{u})$ plane. The second product completes the spatial rotation, and undoes the rotation in the $(r, \vec{u})$ plane.