Direct Product Factors for Minkowski Matrices

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Abstract

Sixteen 4x4 matrices can be used to represent the basis for Minkowski geometric algebra. These matrices are direct products of a set of 2x2 planar Euclidean geometric algebra matrices. This note lists the 2x2 matrices, and their direct product 4x4 matrices.

Minkowski Basis Matrices

Sixteen 4x4 matrices provide a basis for geometric algebra in a Minkowski space with metric signature of (+,+,+,-).

A preformatted list of the these matrices, organized by rank, follows on the next page.

On inspection, these matrices are seen to be the direct or Kronecker product of four different 2x2 matrices which define geometric algebra in the plane. I will use a_x to define the 2x2 matrix set, and e_x for the 4x4 matrix.

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad a_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad a_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad a_{xy} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Illustrating the direct product by example,

$$a \otimes b \quad = \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$$

```
      Unity
[  1  0  0  0]
[  0  1  0  0]
[  0  0  1  0]
[  0  0  0  1]


         x              y              z              t
[  0  0  1  0] [  1  0  0  0] [  0  0  0  1] [  0  0 -1  0]
[  0  0  0  1] [  0  1  0  0] [  0  0 -1  0] [  0  0  0  1]
[  1  0  0  0] [  0  0 -1  0] [  0 -1  0  0] [  1  0  0  0]
[  0  1  0  0] [  0  0  0 -1] [  1  0  0  0] [  0 -1  0  0]


        xy             xz             yz             xt             yt             zt
[  0  0 -1  0] [  0 -1  0  0] [  0  0  0  1] [  1  0  0  0] [  0  0 -1  0] [  0 -1  0  0]
[  0  0  0 -1] [  1  0  0  0] [  0  0 -1  0] [  0 -1  0  0] [  0  0  0  1] [ -1  0  0  0]
[  1  0  0  0] [  0  0  0  1] [  0  1  0  0] [  0  0 -1  0] [ -1  0  0  0] [  0  0  0 -1]
[  0  1  0  0] [  0  0 -1  0] [ -1  0  0  0] [  0  0  0  1] [  0  1  0  0] [  0  0 -1  0]


        xyz            xyt            xzt            yzt
[  0  1  0  0] [ -1  0  0  0] [  0  0  0 -1] [  0 -1  0  0]
[ -1  0  0  0] [  0  1  0  0] [  0  0 -1  0] [ -1  0  0  0]
[  0  0  0  1] [  0  0 -1  0] [  0 -1  0  0] [  0  0  0  1]
[  0  0 -1  0] [  0  0  0  1] [ -1  0  0  0] [  0  0  1  0]


       xyzt
[  0  0  0  1]
[  0  0  1  0]
[  0 -1  0  0]
[ -1  0  0  0]
```

Table 1, on the next page, generates a set of 16 basis 4x4 matrices from the direct product of the 2x2 geometric algebra basis. As an example of the process, here are two sample calculations used in the table.

$$a_1 \otimes a_1 \quad = \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\otimes$	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}$	$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}$
$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix}$	$\begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{matrix}$
$\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{matrix}$
$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{matrix}$
$\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}$

Table 1: Minkowski Basis Via Direct Product

$$a_1 \otimes a_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Direct and Matrix Product Identity

If A, B, C, and D are square matrices, it is known that

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

This identity is useful for evaluating 4x4 basis and their products. It is also valuable for explaining why only six of the Minkowski basis square to negative one, with the other basis squaring to one.

Each of the Minkowski basis is the direct product of two of the planar basis. Look at the squares of the products.

$$\begin{aligned}(A \otimes B)^2 &= (A \otimes B)(A \otimes B) \\ &= (AA) \otimes (BB)\end{aligned}$$

Of the planar basis, we have scalar and vectors square to one, but the bivector square to negative one.

$$\begin{aligned}(a_1)^2 &= 1 \\ (a_x)^2 &= 1 \\ (a_y)^2 &= 1 \\ (a_{xy})^2 &= -1\end{aligned}$$

Consequently, as we transition to the direct product space, only the basis which have a planar factor of a_{xy} will square to negative one.

Preferred Minkowski Implementation

From the direct product of the two dimensional geometric algebra, we have a set of sixteen matrices implementing the Minkowski geometric algebra. We have choices of sign conventions associated with choosing a_x versus a_y, which I will acknowledge and ignore at this time, using the choice presented above. What I wish to do at this time, is to see if the direct product matrix identity provides a preference for associating the Minkowski matrices with particular vectors and bivectors.

Previously, I had identified a set of symmetries in Minkowsky geometric algebra, which when written in a four by four grid, placed the nine Minkowski basis multivectors which square to one in the upper right, and which made trivector-like products across and down in such a fashion as to generate six matrices which square to negative one.

With these tables, we can rotate top three rows, or the left three columns in cyclic order without invalidating the table. Table 3 shows a permutated version of Table 2.

We have a similar table made from the 2x2 basis with direct products.

e_x	e_y	e_z	e_{xyz}
e_{yzt}	e_{xzt}	e_{xyt}	e_t
e_{xt}	e_{yt}	e_{zt}	e_{xyzt}
e_{yz}	e_{xz}	e_{xy}	-1

Table 2: Sets of Products Across and Down

e_{yzt}	e_{xzt}	e_{xyt}	e_t
e_{xt}	e_{yt}	e_{zt}	e_{xyzt}
e_x	e_y	e_z	e_{xyz}
e_{yz}	e_{xz}	e_{xy}	-1

Table 3: Cyclic Permutated Table

$((a_x a_y) \otimes (a_x a_y))$	$(a_y \otimes a_1)$	$(a_x \otimes a_1)$	$(a_1 \otimes (a_x a_y))$
$(a_1 \otimes a_y)$	$(a_x \otimes a_x)$	$(a_y \otimes a_x)$	$((a_x a_y) \otimes a_y)$
$(a_1 \otimes a_x)$	$(a_x \otimes a_y)$	$(a_y \otimes a_y)$	$((a_x a_y) \otimes a_x)$
$((a_x a_y) \otimes a_1)$	$(a_y \otimes (a_x a_y))$	$(a_x \otimes (a_x a_y))$	$(-a_1 \otimes a_1)$

Table 4: Minkowski Products Using Direct Product

$(a_y \otimes a_y)$	$(a_1 \otimes a_x)$	$(a_x \otimes a_y)$	$((a_x a_y) \otimes a_x)$
$(a_x \otimes a_1)$	$((a_x a_y) \otimes (a_x a_y))$	$(a_y \otimes a_1)$	$(a_1 \otimes (a_x a_y))$
$(a_y \otimes a_x)$	$(a_1 \otimes a_y)$	$(a_x \otimes a_x)$	$((a_x a_y) \otimes a_y)$
$(a_x \otimes (a_x a_y))$	$((a_x a_y) \otimes a_1)$	$(a_y \otimes (a_x a_y))$	$(-a_1 \otimes a_1)$

Table 5: Permutated Minkowski Products Using Direct Product

With these tables, we can rotate top three rows, or the left three columns in cyclic order without invalidating the table. (If we swap rows, or swap columns out of cyclic order, negative signs in the products are introduced.)

Table 5 illustrates this re-ordering, where the $((a_x a_y) \otimes (a_x a_y))$ is moved to the center of the table.

I started this particular exercise to identify a preferred mapping of matrices to geometric algebra axis. I find instead, that this format clearly illustrates the symmetries amoung the bivector/trivector candidates. I like the idea of six candidates for a time axis. I like the idea of a z axis being the product of self interactions of a time axis. However, whether these speculation have merit remains to be seen.

Multiple implementations exist. Following on the next page is the one I currently like best.

$$
\begin{aligned}
e_1 &= a_1 \otimes a_1 \\
\\
e_x &= a_x \otimes a_1 \\
e_y &= a_y \otimes a_1 \\
e_z &= (a_x a_y) \otimes (a_x a_y) \\
e_t &= (a_x a_y) \otimes (a_y) \\
\\
e_x e_y &= (a_x \otimes a_1)(a_y \otimes a_1) = (a_x a_y) \otimes a_1 \\
e_x e_z &= (a_x \otimes a_1)((a_x a_y) \otimes (a_x a_y)) = (a_y) \otimes (a_x a_y) \\
e_y e_z &= (a_y \otimes a_1)((a_x a_y) \otimes (a_x a_y)) = -(a_x) \otimes (a_x a_y) \\
\\
e_x e_t &= (a_x \otimes a_1)((a_x a_y \otimes (a_y)) = a_y \otimes a_y \\
e_y e_t &= (a_y \otimes a_1)((a_x a_y \otimes (a_y)) = -a_x \otimes a_y \\
e_z e_t &= ((a_x a_y) \otimes (a_x a_y))((a_x a_y \otimes (a_y)) = -a_1 \otimes a_x \\
\\
e_x e_y e_z &= ((a_x a_y) \otimes a_1)((a_x a_y) \otimes (a_x a_y)) = -a_1 \otimes (a_x a_y) \\
e_x e_y e_t &= ((a_x a_y) \otimes a_1)((a_x a_y \otimes (a_y)) = -a_1 \otimes a_y \\
e_x e_z e_t &= ((a_y) \otimes (a_x a_y))((a_x a_y \otimes (a_y)) = -a_x \otimes a_x \\
e_y e_z e_t &= (-(a_x) \otimes (a_x a_y))((a_x a_y \otimes (a_y)) = -a_y \otimes a_x \\
\\
e_x e_y e_z e_t &= ((a_x a_y) \otimes a_1)(-a_1 \otimes a_x) = -(a_x a_y) \otimes a_x
\end{aligned}
$$

References

[1] George Arfken, *Mathematical Methods for Physicist* 3rd ed. 1985 Academic Press, ISBN 0-12-059820-5

[2] Alan Bromborsky, Geometric Algebra Module for Sympy online

[3] Alan Bromborsky, An Introduction to Geometric Algebra and Calculus online

[4] John Browne, *Grassman Algebra* Barnard Publishing, ISBN 978-1479197637