Dirac's Radial Fields from Curled Potentials

Kurt Nalty

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Abstract

P. A. M. Dirac's monopoles mimic radial charges, with the addition of a singular string attached. This note presents such fields, using the lecture note EDC.pdf of Professor Jose Figueroa as a starting point. I suspect that the electric field, and Faraday's electric field lines, may have a Dirac basis, as opposed to the radial Coulomb basis.

Radial Fields from Curled Vector Potentials

Jose Figueroa presents a nice discussion of Dirac's monopoles in his lecture notes "EDC.pdf". Using spherical coordinates (r, θ, ϕ), he presents two vector potentials which which curl into radial, inverse square fields, albeit with a small singuality along the half z axis.

In spherical coordinates, the three dimensional curl of a vector potential $\vec{A} = (A_r, A_\theta, A_\phi)$ is

$$\begin{aligned}
\vec{\nabla} \times \vec{A} &= \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial \theta} \left(A_\phi \sin\theta \right) - \frac{\partial A_\theta}{\partial \phi} \right] \vec{a}_r \\
&+ \frac{1}{r} \left[\frac{1}{\sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} \left(r A_\phi \right) \right] \vec{a}_\theta \\
&+ \frac{1}{r} \left[\frac{\partial}{\partial r} \left(r A_\theta \right) - \frac{\partial A_r}{\partial \theta} \right] \vec{a}_\phi
\end{aligned}$$

Figueroa's example potentials (ignoring constant factors) are

$$
\begin{aligned}
\vec{A}_+ &= \frac{1 - \cos\theta}{r\sin\theta}\vec{a}_\phi \\
\vec{A}_- &= \frac{-1 - \cos\theta}{r\sin\theta}\vec{a}_\phi
\end{aligned}
$$

As these potential have no radial or θ component, the curl simplifies to

$$
\vec{\nabla} \times (A_\phi \vec{a}_\phi) = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi \sin\theta)\right]\vec{a}_r + \frac{1}{r}\left[-\frac{\partial}{\partial r}(rA_\phi)\right]\vec{a}_\theta
$$

so,

$$
\begin{aligned}
\vec{\nabla} \times \vec{A}_+ &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{1 - \cos\theta}{r}\right)\right]\vec{a}_r + \frac{1}{r}\left[-\frac{\partial}{\partial r}\left(\frac{1 - \cos\theta}{\sin\theta}\right)\right]\vec{a}_\theta \\
&= \frac{\vec{a}_r}{r^2} \\
\vec{\nabla} \times \vec{A}_- &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{-1 - \cos\theta}{r}\right)\right]\vec{a}_r + \frac{1}{r}\left[-\frac{\partial}{\partial r}\left(\frac{-1 - \cos\theta}{\sin\theta}\right)\right]\vec{a}_\theta \\
&= \frac{\vec{a}_r}{r^2}
\end{aligned}
$$

We see that the factor of r is eliminated in the second term, zeroing out the $\vec{a}_\theta$ component in the derivative. We also note that $\vec{A}_+$ has a singularity at $\theta = 180°$, and $\vec{A}_-$ has a singularity at $\theta = 0°$.

Speculation about Electrons as Rotational Particles

I like to speculate that the gravitational field, as a unipolar field, will be a gradient of a scalar potential, while the electron, as a bipolar charge element, will instead be rotational analog, with the true potential being a vector potential curled with a string singularity as the true description. I find it ironic that Michael Faraday's visualization of the electric field as filament elements joining opposite polarity charges might be correct in detail.

More General Curl to Radial Potential

Having seen Figueroa's specific examples, we now want to look at the more general case of vector potentials curling into radial fields. Returning to the spherical curl formula, and remembering the examples above,

$$\begin{aligned}\vec{\nabla} \times \vec{A} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(A_\phi \sin\theta\right) - \frac{\partial A_\theta}{\partial\phi}\right]\vec{a}_r \\ &+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rA_\phi\right)\right]\vec{a}_\theta \\ &+ \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right]\vec{a}_\phi\end{aligned}$$

We see that if $A_r = A_r(r)$, and if $A_\theta = f_\theta(\theta,\phi)/r$ and $A_\phi = g_\phi(\theta,\phi)/r$, then the $\vec{a}_\theta$ and $\vec{a}_\phi$ terms will be zero. Likewise, $f_\theta = (k - \cos\theta)/\sin\theta$ and $g_\phi = M - \phi\sin\theta$ result in purely radial, inverse square terms. Thus we can say a more general potential is

$$\vec{A} = \left(A_r(r), \alpha\frac{k - \cos\theta}{r\sin\theta}, \beta\frac{m - \phi\sin\theta}{r}\right)$$

Most General Form for Radial Field from Curl

To find our most general form for an inverse square, radial field from curl, we start by examining the radial term of the curl. We see that the desired result of an inverse square in radius term, given that the partial derivatives are in θ and ϕ, requires that A_θ and A_ϕ must be inverse in r. This previous observation turns out to be a requirement. Given the inverse radial nature of these components, the $\vec{a}_\theta$ and $\vec{a}_\phi$ expressions force no angular dependence in the A_r term. Again, an observation proves to be a requirement.

We now return to the radial component. Using our previous notations,

$$\begin{aligned}A_\theta &= f_\theta(\theta,\phi)/r \\ A_\phi &= g_\phi(\theta,\phi)/r\end{aligned}$$

We have

$$
\begin{aligned}
\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(A_\phi\sin\theta\right)-\frac{\partial A_\theta}{\partial\phi}\right]\vec{a}_r &= \vec{a}_r \\
\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(g_\phi\left(\theta,\phi\right)\sin\theta\right)-\frac{\partial f_\theta(\theta,\phi)}{\partial\phi}\right] &= 1 \\
\frac{\partial}{\partial\theta}\left(g_\phi\left(\theta,\phi\right)\sin\theta\right)-\frac{\partial f_\theta(\theta,\phi)}{\partial\phi} &= \sin\theta
\end{aligned}
$$

We see that if we take $f_\theta = 0$, we recover the solution $g_\phi = (k - \cos\theta)/\sin\theta$, as used above. If we take $g_\phi = 0$, we find $f_\theta = m - \phi\sin\theta$, as noted above.

For a more general case, however, let g be declared. and find f.

$$
\begin{aligned}
\frac{\partial f_\theta\left(\theta,\phi\right)}{\partial\phi} &= \frac{\partial}{\partial\theta}\left(g_\phi\left(\theta,\phi\right)\sin\theta\right)-\sin\theta \\
&= \sin\theta\left[\frac{\partial g_\phi\left(\theta,\phi\right)}{\partial\theta}-1\right]+\cos\theta\left[g_\phi\left(\theta,\phi\right)\right]
\end{aligned}
$$

Various Forms and Field Visualizations

In Figure 1, I show the potential using $\vec{A} = \vec{a}_\theta(1-\cos\theta)/(r\sin\theta)$ for $r = 2.0$. Notice the flare as we approach the singularity on the z half axis. Clearly, a vane placed in this field does rotate with the axis in the radial direction.

Figure 2 illustrates a variation of the above formula, where I have eliminated the constant 1. Eliminating the constant 1 of the formula, we now have singularities on the entire z axis, as well as a change of direction of spin in top versus bottom hemispheres. Figure 2 also satisfies the spinometer sanity check.

Finally, in Figure 3, we show the potential for $\vec{A} = \vec{a}_\phi(-\phi\sin\theta/r)$ with $r = 3$. There is quite a bit of similarity to this function, and the shape of a nautilus shell. As ϕ is continuously increasing, the discontinuity at one revolution will likely not be an issue. I can see winding numbers being significant with this turbine or windscoop field.

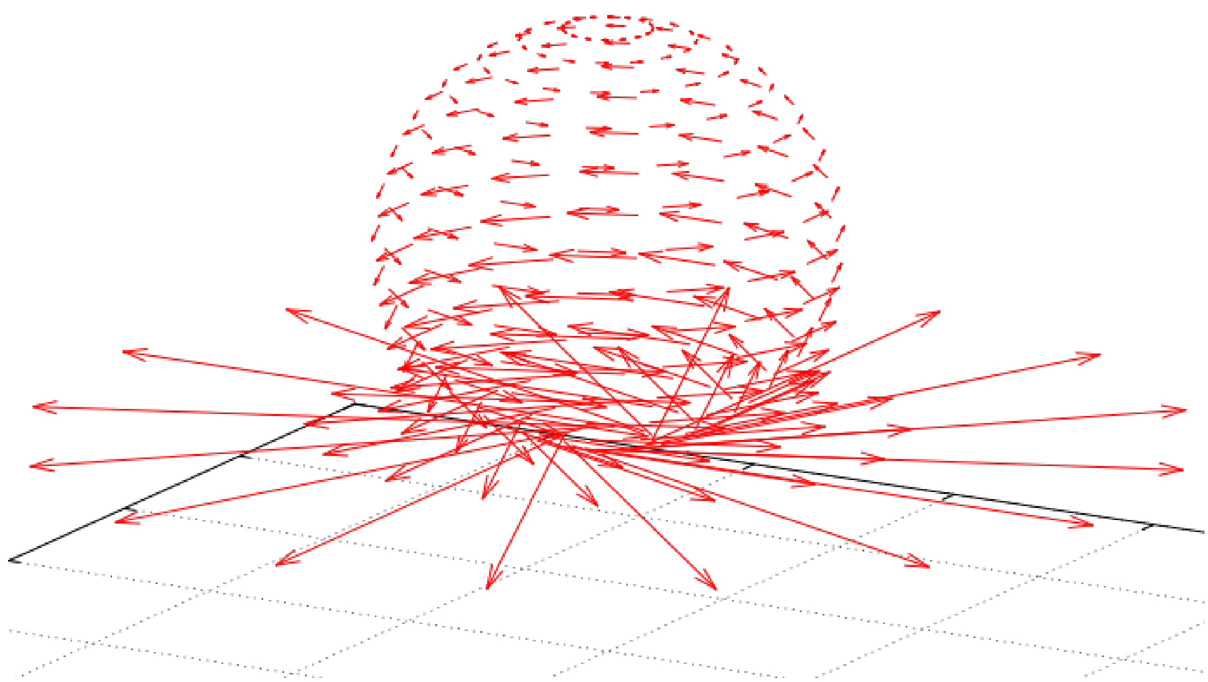

Figure 1: $\vec{A} = \vec{a}_\theta (1 - \cos\theta)/(r \sin\theta)$

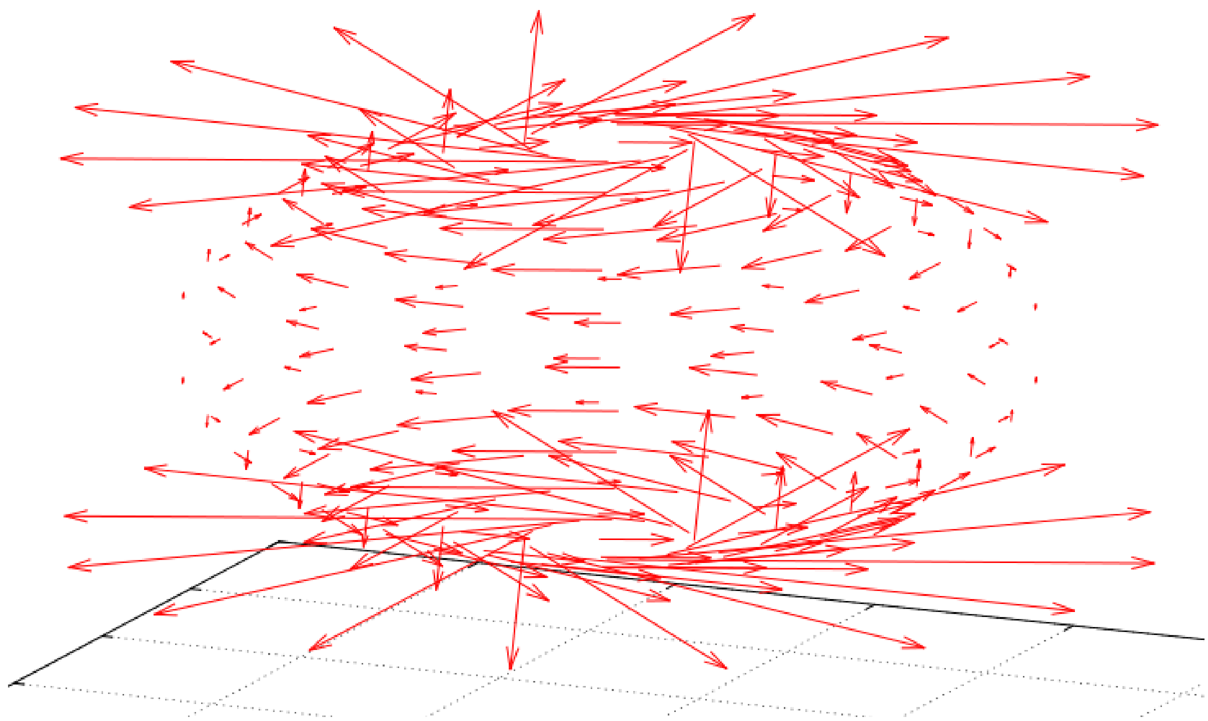

Figure 2: $\vec{A} = \vec{a}_\theta(-\cos\theta)/(r\sin\theta)$

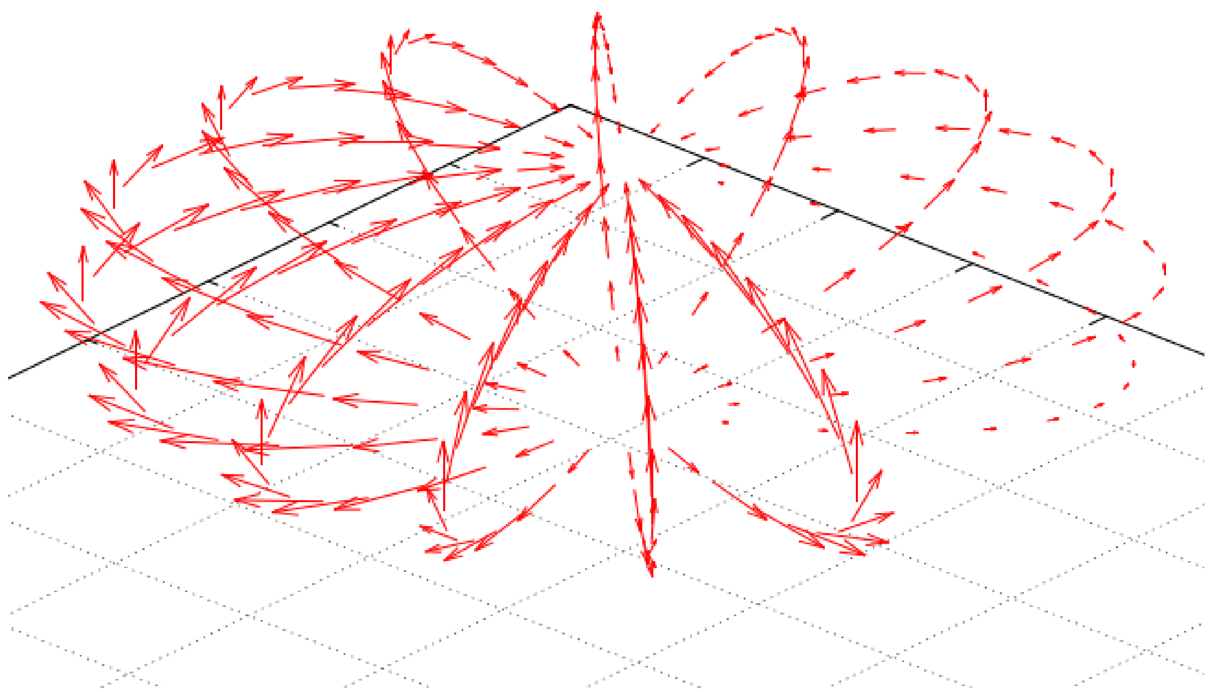

Figure 3: $\vec{A} = \vec{a}_\phi(-\phi \sin\theta / r)$

Further Observations on these Expressions

The divergence in spherical coordinates is

$$\vec{\nabla} \cdot \vec{G} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 G_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta G_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial G_\phi}{\partial\phi}$$

If we let $\vec{G}$ be the more general form of the previous form, and take the divergence, we find

$$\begin{aligned}
\vec{G} &= \vec{a}_\theta \frac{k-\cos\theta}{r\sin\theta} + \vec{a}_\phi \frac{m-\phi\sin\theta}{r} \\
\vec{\nabla}\cdot\vec{G} &= 0 + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{k-\cos\theta}{r\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{m-\phi\sin\theta}{r}\right) \\
&= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{k-\cos\theta}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{m-\phi\sin\theta}{r}\right) \\
&= \frac{1}{r^2\sin\theta}\sin\theta - \frac{1}{r^2\sin\theta}\sin\theta = \frac{1}{r^2} - \frac{1}{r^2} \\
&= 0
\end{aligned}$$

Consequently, this function $\vec{G}$ can be written as the curl of another function, plus arbitrary gradients. A little inspection provides two fine functions, which curl into $\vec{G}$.

$$\begin{aligned}
\vec{E} &= \vec{a}_r\left(\phi\left(k-\cos\theta\right)\right) + \vec{a}_\theta m + \vec{a}_\phi 0 \\
\vec{F} &= \vec{a}_r\left(\phi k\right) + \vec{a}_\theta\left(m-\phi\sin\theta\right) + \vec{a}_\phi\left(\frac{\cos\theta}{\sin\theta}\right)
\end{aligned}$$

We demonstrate . . .

$$
\begin{aligned}
\vec{\nabla} \times \vec{E} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(E_\phi \sin\theta\right) - \frac{\partial E_\theta}{\partial\phi}\right]\vec{a}_r \\
&\quad + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rE_\phi\right)\right]\vec{a}_\theta \\
&\quad + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rE_\theta\right) - \frac{\partial E_r}{\partial\theta}\right]\vec{a}_\phi \\
&= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(0\sin\theta\right) - \frac{\partial m}{\partial\phi}\right]\vec{a}_r \\
&\quad + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\phi\left(k - \cos\theta\right)\right) - \frac{\partial}{\partial r}\left(r0\right)\right]\vec{a}_\theta \\
&\quad + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rm\right) - \frac{\partial}{\partial\theta}\left(\phi\left(k - \cos\theta\right)\right)\right]\vec{a}_\phi \\
&= \vec{a}_r(0) + \vec{a}_\theta\left(\frac{k - \cos\theta}{r\sin\theta}\right) + \vec{a}_\phi\left(\frac{m - \phi\sin\theta}{r}\right)
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
\vec{\nabla} \times \vec{F} &= \frac{1}{r \sin\theta}\left[\frac{\partial}{\partial \theta}\left(F_\phi \sin\theta\right) - \frac{\partial F_\theta}{\partial \phi}\right]\vec{a}_r \\
&+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \phi} - \frac{\partial}{\partial r}\left(rF_\phi\right)\right]\vec{a}_\theta \\
&+\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rF_\theta\right) - \frac{\partial F_r}{\partial \theta}\right]\vec{a}_\phi \\
&= \frac{1}{r \sin\theta}\left[\frac{\partial}{\partial \theta}\left(\left(\frac{\cos\theta}{\sin\theta}\right)\sin\theta\right) - \frac{\partial}{\partial \phi}\left(m - \phi\sin\theta\right)\right]\vec{a}_r \\
&+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\left(\phi k\right) - \frac{\partial}{\partial r}\left(r\left(\frac{\cos\theta}{\sin\theta}\right)\right)\right]\vec{a}_\theta \\
&+\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\left(m - \phi\sin\theta\right)\right) - \frac{\partial}{\partial \theta}\left(\phi k\right)\right]\vec{a}_\phi \\
&= \frac{1}{r \sin\theta}\left[\frac{\partial}{\partial \theta}\left(\cos\theta\right) + \sin\theta\right]\vec{a}_r \\
&+\frac{1}{r}\left[\frac{k}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right]\vec{a}_\theta \\
&+\frac{1}{r}\left[m - \phi\sin\theta\right]\vec{a}_\phi \\
&= \vec{a}_r(0) + \vec{a}_\theta\left(\frac{k - \cos\theta}{r\sin\theta}\right) + \vec{a}_\phi\left(\frac{m - \phi\sin\theta}{r}\right)
\end{aligned}
$$

Now the divergence of both expressions is

$$
\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \vec{F} = \frac{2\phi\left(k - \cos\theta\right)}{r} + \frac{m\cos\theta}{r\sin\theta}
$$

And the difference between these two expressions

$$
\begin{aligned}
\vec{E} &= \vec{a}_r\left(\phi\left(k - \cos\theta\right)\right) + \vec{a}_\theta m + \vec{a}_\phi 0 \\
\vec{F} &= \vec{a}_r\left(\phi k\right) + \vec{a}_\theta\left(m - \phi\sin\theta\right) + \vec{a}_\phi\left(\frac{\cos\theta}{\sin\theta}\right) \\
\vec{E} - \vec{F} &= \vec{a}_r\left(-\phi\cos\theta\right) + \vec{a}_\theta\left(\phi\sin\theta\right) + \vec{a}_\phi\left(-\frac{\cos\theta}{\sin\theta}\right) \\
&= \vec{\nabla}\left(-r\phi\cos\theta\right)
\end{aligned}
$$