The Dirac Equation Reannotated

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Abstract

David Hestenes [4] has developed a theory of the electron based upon the Dirac equation and geometric algebra. I want to understand and duplicate his work.

This particular note first demonstrates the equivalence of the Dirac equations derived from the standard Bjorken and Drell matrices notation and the Dirac equations derived by using a five dimensional geometric algebra with signature (+ - - - +), with the wavefunction being a specialized multivector.

This note then derives the equation set corresponding to the Weyl and Majorana gamma matrix assignments, pointing out the differences in the resulting set of eight equations.

The note then derives the equation set using a (+ + + + -) signature, which is my preferred implementation.

On the one hand, the fact that identical equation sets can be derived using standard matrices and geometric algebra serves to validate the geometric algebra tools.

On the other hand, the zeroes seen in the standard matrix implementations indicate that the Dirac equation occupies a subset of the standard four/five dimensional space. Different choices for the basis matrices result in different equation sets, and different signatures for the geometric algebra. Interpretation of these alternative equation sets is a topic for further work.

Standard Dirac Equation Development

The standard Dirac equation uses the four anti-commutating matrices defined below. I am using $I = \sqrt{-1}$ throughout this section.

$$
\begin{aligned}
\gamma_0 &= e_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
\gamma_1 &= e_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\
\gamma_2 &= e_y = \begin{pmatrix} 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \end{pmatrix} \\
\gamma_3 &= e_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

These matrices anti-commute among themselves, meaning

$$e_t e_x = -e_x e_t \quad e_t e_y = -e_y e_t \quad e_t e_z = -e_z e_t$$

$$e_x e_y = -e_y e_x \quad e_x e_z = -e_z e_x \quad e_y e_z = -e_z e_y$$

These matrice have squares

$$
\begin{aligned}
e_t^2 &= +1 \\
e_x^2 &= -1 \\
e_y^2 &= -1 \\
e_z^2 &= -1
\end{aligned}
$$

Write the relativistic energy-momentum relationship with $c = 1$.

$$
\begin{aligned}
E^2 - \vec{p} \cdot \vec{p} &= m^2 \\
E^2 - p_x^2 - p_y^2 - p_z^2 - m^2 &= 0 \\
(E e_t + p_x e_x + p_y e_y + p_z e_z)^2 - m^2 &= 0
\end{aligned}
$$

This last expression factors into

$$(Ee_t + p_x e_x + p_y e_y + p_z e_z + m) * (Ee_t + p_x e_x + p_y e_y + p_z e_z - m) = 0$$

Either factor being zero is a solution for the equation above. The traditional choice is the right hand side.

$$(Ee_t + p_x e_x + p_y e_y + p_z e_z - m) = 0$$

For quantization, we apply the operator above to a wavefunction ψ.

$$(Ee_t + p_x e_x + p_y e_y + p_z e_z - m)\,\psi = 0$$

The left hand side is a 4x4 complex matrix. Writing the matrix form, we have

$$\begin{pmatrix} E-m & 0 & p_z & p_x - Ip_y \\ 0 & E-m & p_x + Ip_y & -p_z \\ -p_z & -p_x + Ip_y & -E-m & 0 \\ -p_x - Ip_y & p_z & 0 & -E-m \end{pmatrix}$$

The wavefunction ψ is a 4x1 complex column.

$$\begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \psi_{0r} + I\psi_{0i} \\ \psi_{1r} + I\psi_{1i} \\ \psi_{2r} + I\psi_{2i} \\ \psi_{3r} + I\psi_{3i} \end{pmatrix}$$

The matrix format equation is

$$\begin{pmatrix} E-m & 0 & p_z & p_x - Ip_y \\ 0 & E-m & p_x + Ip_y & -p_z \\ -p_z & -p_x + Ip_y & -E-m & 0 \\ -p_x - Ip_y & p_z & 0 & -E-m \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = 0$$

The Four Dirac Complex Wave Function Equations

The matrix equation, using complex numbers for wave function components, is

$$\begin{pmatrix} E-m & 0 & p_z & p_x - Ip_y \\ 0 & E-m & p_x + Ip_y & -p_z \\ -p_z & -p_x + Ip_y & -E-m & 0 \\ -p_x - Ip_y & p_z & 0 & -E-m \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = 0$$

This provides four complex equations.

$$\begin{aligned}
+(E-m)\psi_0 + p_z\psi_2 + (p_x - Ip_y)\psi_3 &= 0\\
+(E-m)\psi_1 + (p_x + Ip_y)\psi_2 - p_z\psi_3 &= 0\\
-p_z\psi_0 - (p_x - Ip_y)\psi_1 - (E+m)\psi_2 &= 0\\
-(p_x + Ip_y)\psi_0 + p_z\psi_1 - (E+m)\psi_3 &= 0
\end{aligned}$$

I like to re-arrange the order of the terms, and the overall sign of the last two equations.

$$\begin{aligned}
+(E-m)\psi_0 + p_x\psi_3 - Ip_y\psi_3 + p_z\psi_2 &= 0\\
+(E-m)\psi_1 + p_x\psi_2 + Ip_y\psi_2 - p_z\psi_3 &= 0\\
+(E+m)\psi_2 + p_x\psi_1 - Ip_y\psi_1 + p_z\psi_0 &= 0\\
+(E+m)\psi_3 + p_x\psi_0 + Ip_y\psi_0 - p_z\psi_1 &= 0
\end{aligned}$$

We replace our energy and momentum terms with the quantum operators.

$$E = \left(I\hbar\frac{\partial}{\partial t} + q\phi\right) \quad p_x = \left(-I\hbar\frac{\partial}{\partial x} + qA_x\right) \quad p_y = \left(-I\hbar\frac{\partial}{\partial y} + qA_y\right) \quad p_z = \left(-I\hbar\frac{\partial}{\partial z} + qA_z\right)$$

The following four lines individually sum to zero.

$$+\left(I\hbar\frac{\partial}{\partial t} + q\phi - m\right)\psi_0 + \left(-I\hbar\frac{\partial}{\partial x} + qA_x\right)\psi_3 - I\left(-I\hbar\frac{\partial}{\partial y} + qA_y\right)\psi_3 + \left(-I\hbar\frac{\partial}{\partial z} + qA_z\right)\psi_2$$

$$+\left(I\hbar\frac{\partial}{\partial t} + q\phi - m\right)\psi_1 + \left(-I\hbar\frac{\partial}{\partial x} + qA_x\right)\psi_2 + I\left(-I\hbar\frac{\partial}{\partial y} + qA_y\right)\psi_2 - \left(-I\hbar\frac{\partial}{\partial z} + qA_z\right)\psi_3$$

$$+\left(I\hbar\frac{\partial}{\partial t} + q\phi + m\right)\psi_2 + \left(-I\hbar\frac{\partial}{\partial x} + qA_x\right)\psi_1 - I\left(-I\hbar\frac{\partial}{\partial y} + qA_y\right)\psi_1 + \left(-I\hbar\frac{\partial}{\partial z} + qA_z\right)\psi_0$$

$$+\left(I\hbar\frac{\partial}{\partial t} + q\phi + m\right)\psi_3 + \left(-I\hbar\frac{\partial}{\partial x} + qA_x\right)\psi_0 + I\left(-I\hbar\frac{\partial}{\partial y} + qA_y\right)\psi_0 - \left(-I\hbar\frac{\partial}{\partial z} + qA_z\right)\psi_1$$

Apply operators and remove parenthesis to get

$$\begin{aligned}
I\hbar\frac{\partial\psi_0}{\partial t} + q\phi\psi_0 - m\psi_0 - I\hbar\frac{\partial\psi_3}{\partial x} + qA_x\psi_3 - \hbar\frac{\partial\psi_3}{\partial y} - IqA_y\psi_3 - I\hbar\frac{\partial\psi_2}{\partial z} + qA_z\psi_2 &= 0\\
I\hbar\frac{\partial\psi_1}{\partial t} + q\phi\psi_1 - m\psi_1 - I\hbar\frac{\partial\psi_2}{\partial x} + qA_x\psi_2 + \hbar\frac{\partial\psi_2}{\partial y} + IqA_y\psi_2 + I\hbar\frac{\partial\psi_3}{\partial z} - qA_z\psi_3 &= 0\\
I\hbar\frac{\partial\psi_2}{\partial t} + q\phi\psi_2 + m\psi_2 - I\hbar\frac{\partial\psi_1}{\partial x} + qA_x\psi_1 - \hbar\frac{\partial\psi_1}{\partial y} - IqA_y\psi_1 - I\hbar\frac{\partial\psi_0}{\partial z} + qA_z\psi_0 &= 0\\
I\hbar\frac{\partial\psi_3}{\partial t} + q\phi\psi_3 + m\psi_3 - I\hbar\frac{\partial\psi_0}{\partial x} + qA_x\psi_0 + \hbar\frac{\partial\psi_0}{\partial y} + IqA_y\psi_0 + I\hbar\frac{\partial\psi_1}{\partial z} - qA_z\psi_1 &= 0
\end{aligned}$$

Group terms in a reasonable fashion,

$$
\begin{aligned}
-m\psi_0 + q\phi\psi_0 + qA_x\psi_3 - IqA_y\psi_3 + qA_z\psi_2 + I\hbar\frac{\partial\psi_0}{\partial t} - I\hbar\frac{\partial\psi_3}{\partial x} - \hbar\frac{\partial\psi_3}{\partial y} - I\hbar\frac{\partial\psi_2}{\partial z} &= 0 \\
-m\psi_1 + q\phi\psi_1 + qA_x\psi_2 + IqA_y\psi_2 - qA_z\psi_3 + I\hbar\frac{\partial\psi_1}{\partial t} - I\hbar\frac{\partial\psi_2}{\partial x} + \hbar\frac{\partial\psi_2}{\partial y} + I\hbar\frac{\partial\psi_3}{\partial z} &= 0 \\
+m\psi_2 + q\phi\psi_2 + qA_x\psi_1 - IqA_y\psi_1 + qA_z\psi_0 + I\hbar\frac{\partial\psi_2}{\partial t} - I\hbar\frac{\partial\psi_1}{\partial x} - \hbar\frac{\partial\psi_1}{\partial y} - I\hbar\frac{\partial\psi_0}{\partial z} &= 0 \\
+m\psi_3 + q\phi\psi_3 + qA_x\psi_0 + IqA_y\psi_0 - qA_z\psi_1 + I\hbar\frac{\partial\psi_3}{\partial t} - I\hbar\frac{\partial\psi_0}{\partial x} + \hbar\frac{\partial\psi_0}{\partial y} + I\hbar\frac{\partial\psi_1}{\partial z} &= 0
\end{aligned}
$$

These four equations are the Dirac equation in complex form. The traditional interpretation is

- ψ_0 electron wavefunction, spin up
- ψ_1 electron wavefunction, spin down
- ψ_2 positron wavefunction, spin up
- ψ_3 positron wavefunction, spin down

These equations should be part of every standard quantum text, in my opinion. Due to the choice of basis matrix, we see a factor of I associated with y. This factor of I obscures the geometrical interpretation of these equations.

As an aside, the relativistic energy momentum equation, re-arranged, shows mass is an eigenvalue problem.

$$(Ee_t + p_xe_x + p_ye_y + p_ze_z)\,\psi = m\psi$$

Convert from Four Complex to Eight Real Equations

The four complex equations above can be recast into eight real component equations. Write the wavefunctions in real and imaginary component format. For example, $\psi_0 = \psi_{0r} + I\psi_{0i}$. Substitute in the four equations above, then separate real and imaginary components.

We find eight coupled equations:

$$
\begin{aligned}
-m\psi_{0r} + q\phi\psi_{0r} + qA_x\psi_{3r} + qA_y\psi_{3i} + qA_z\psi_{2r} - \hbar\frac{\partial\psi_{0i}}{\partial t} + \hbar\frac{\partial\psi_{3i}}{\partial x} - \hbar\frac{\partial\psi_{3r}}{\partial y} + \hbar\frac{\partial\psi_{2i}}{\partial z} &= 0 \\
-m\psi_{0i} + q\phi\psi_{0i} + qA_x\psi_{3i} - qA_y\psi_{3r} + qA_z\psi_{2i} + \hbar\frac{\partial\psi_{0r}}{\partial t} - \hbar\frac{\partial\psi_{3r}}{\partial x} - \hbar\frac{\partial\psi_{3i}}{\partial y} - \hbar\frac{\partial\psi_{2r}}{\partial z} &= 0 \\
-m\psi_{1r} + q\phi\psi_{1r} + qA_x\psi_{2r} - qA_y\psi_{2i} - qA_z\psi_{3r} - \hbar\frac{\partial\psi_{1i}}{\partial t} + \hbar\frac{\partial\psi_{2i}}{\partial x} + \hbar\frac{\partial\psi_{2r}}{\partial y} - \hbar\frac{\partial\psi_{3i}}{\partial z} &= 0 \\
-m\psi_{1i} + q\phi\psi_{1i} + qA_x\psi_{2i} + qA_y\psi_{2r} - qA_z\psi_{3i} + \hbar\frac{\partial\psi_{1r}}{\partial t} - \hbar\frac{\partial\psi_{2r}}{\partial x} + \hbar\frac{\partial\psi_{2i}}{\partial y} + \hbar\frac{\partial\psi_{3r}}{\partial z} &= 0 \\
+m\psi_{2r} + q\phi\psi_{2r} + qA_x\psi_{1r} + qA_y\psi_{1i} + qA_z\psi_{0r} - \hbar\frac{\partial\psi_{2i}}{\partial t} + \hbar\frac{\partial\psi_{1i}}{\partial x} - \hbar\frac{\partial\psi_{1r}}{\partial y} + \hbar\frac{\partial\psi_{0i}}{\partial z} &= 0 \\
+m\psi_{2i} + q\phi\psi_{2i} + qA_x\psi_{1i} - qA_y\psi_{1r} + qA_z\psi_{0i} + \hbar\frac{\partial\psi_{2r}}{\partial t} - \hbar\frac{\partial\psi_{1r}}{\partial x} - \hbar\frac{\partial\psi_{1i}}{\partial y} - \hbar\frac{\partial\psi_{0r}}{\partial z} &= 0 \\
+m\psi_{3r} + q\phi\psi_{3r} + qA_x\psi_{0r} - qA_y\psi_{0i} - qA_z\psi_{1r} - \hbar\frac{\partial\psi_{3i}}{\partial t} + \hbar\frac{\partial\psi_{0i}}{\partial x} + \hbar\frac{\partial\psi_{0r}}{\partial y} - \hbar\frac{\partial\psi_{1i}}{\partial z} &= 0 \\
+m\psi_{3i} + q\phi\psi_{3i} + qA_x\psi_{0i} + qA_y\psi_{0r} - qA_z\psi_{1i} + \hbar\frac{\partial\psi_{3r}}{\partial t} - \hbar\frac{\partial\psi_{0r}}{\partial x} + \hbar\frac{\partial\psi_{0i}}{\partial y} + \hbar\frac{\partial\psi_{1r}}{\partial z} &= 0
\end{aligned}
$$

We expect to recover these same equations using five dimensional geometric algebra in the following sections. Notice that m and q have different signs between the top half and bottom half sets of equation. The top four equations are associated with electrons, while the bottom four are associated with positrons. My naive interpretation, is that electrons have a lower energy, by $2m$, than positrons.

The Dirac Equation using Geometric Algebra

The Dirac gamma matrices span a five dimensional space. Consequently, we want to use a five dimensional geometric algebra. Geometric algebra in odd dimensions, such as five dimensions, possesses a pseudoscalar element which squares to negative one, and commutes with all terms in the algebra. As such, these are imaginary elements mapping to i. Using a five dimensional geometric algebra, we can eliminate factors of i in the Dirac equation.

Five Dimensional Geometric Algebra Dirac Equation

The Dirac 4x4 complex gamma matrices span five dimensional geometric algebra, with many mapping of differing signature possible. Consequently, it is easy to map the Dirac equation to geometric algebra many different ways in five-space. In the previous section, we used the standard Bjorken and Drell matrices, mapping $e_t = \gamma_0$, $e_x = \gamma_1$, $e_y = \gamma_2$, and $e_z = \gamma_3$. The fifth dimension did not explicity show in the derivation of the component equations previously, but is part of the gamma matrix set, commonly called γ_5, and will map to the geometric algebra base e_w. Like their associated Bjorken and Drell matrices, the geometric algebra basis e_w, e_x, e_y, e_z, e_t square to (+1, -1, -1, -1, +1) respectively.

For this algebra, the pseudoscalar is $i = e_{wxyzt}$ which squares to negative one and commutes with all algebra elements. However, since my basis notation follows a permutation counting sequence, rather than a dual product sequence, we will see sign changes on many of the basis upon multiplication by i. As an example $ie_x = e_{wxyzt}e_x = -e_{wyzt}$, while $ie_y = e_{wxyzt}e_y = e_{wxzt}$

Dirac Space Geometric Algebra Implementation Details

This algebra can be implemented using 4x4 complex Dirac gamma matrices. In text format, our thirty-two matrices are

```
      q              wxyzt
[ 1  0  0  0]  [ I  0  0  0]
[ 0  1  0  0]  [ 0  I  0  0]
[ 0  0  1  0]  [ 0  0  I  0]
[ 0  0  0  1]  [ 0  0  0  I]

      w              x              y              z              t
[ 0  0  1  0]  [ 0  0  0  1]  [ 0  0  0 -I]  [ 0  0  1  0]  [ 1  0  0  0]
[ 0  0  0  1]  [ 0  0  1  0]  [ 0  0  I  0]  [ 0  0  0 -1]  [ 0  1  0  0]
[ 1  0  0  0]  [ 0 -1  0  0]  [ 0  I  0  0]  [-1  0  0  0]  [ 0  0 -1  0]
[ 0  1  0  0]  [-1  0  0  0]  [-I  0  0  0]  [ 0  1  0  0]  [ 0  0  0 -1]

      wx             wy             wz             wt             xy
[ 0 -1  0  0]  [ 0  I  0  0]  [-1  0  0  0]  [ 0  0 -1  0]  [-I  0  0  0]
[-1  0  0  0]  [-I  0  0  0]  [ 0  1  0  0]  [ 0  0  0 -1]  [ 0  I  0  0]
[ 0  0  0  1]  [ 0  0  0 -I]  [ 0  0  1  0]  [ 1  0  0  0]  [ 0  0 -I  0]
[ 0  0  1  0]  [ 0  0  I  0]  [ 0  0  0 -1]  [ 0  1  0  0]  [ 0  0  0  I]
```

```
      xz              xt              yz              yt              zt
[ 0  1  0  0]  [ 0  0  0 -1]  [ 0 -I  0  0]  [ 0  0  0  I]  [ 0  0 -1  0]
[-1  0  0  0]  [ 0  0 -1  0]  [-I  0  0  0]  [ 0  0 -I  0]  [ 0  0  0  1]
[ 0  0  0  1]  [ 0 -1  0  0]  [ 0  0  0 -I]  [ 0  I  0  0]  [-1  0  0  0]
[ 0  0 -1  0]  [-1  0  0  0]  [ 0  0 -I  0]  [-I  0  0  0]  [ 0  1  0  0]

     wxy             wxz             wxt             wyz             wyt
[ 0  0 -I  0]  [ 0  0  0  1]  [ 0 -1  0  0]  [ 0  0  0 -I]  [ 0  I  0  0]
[ 0  0  0  I]  [ 0  0 -1  0]  [-1  0  0  0]  [ 0  0 -I  0]  [-I  0  0  0]
[-I  0  0  0]  [ 0  1  0  0]  [ 0  0  0 -1]  [ 0 -I  0  0]  [ 0  0  0  I]
[ 0  I  0  0]  [-1  0  0  0]  [ 0  0 -1  0]  [-I  0  0  0]  [ 0  0 -I  0]

     wzt             xyz             xyt             xzt             yzt
[-1  0  0  0]  [ 0  0 -I  0]  [-I  0  0  0]  [ 0  1  0  0]  [ 0 -I  0  0]
[ 0  1  0  0]  [ 0  0  0 -I]  [ 0  I  0  0]  [-1  0  0  0]  [-I  0  0  0]
[ 0  0 -1  0]  [ I  0  0  0]  [ 0  0  I  0]  [ 0  0  0 -1]  [ 0  0  0  I]
[ 0  0  0  1]  [ 0  I  0  0]  [ 0  0  0 -I]  [ 0  0  1  0]  [ 0  0  I  0]

     wxyz            wxyt            wxzt            wyzt            xyzt
[ I  0  0  0]  [ 0  0  I  0]  [ 0  0  0 -1]  [ 0  0  0  I]  [ 0  0  I  0]
[ 0  I  0  0]  [ 0  0  0 -I]  [ 0  0  1  0]  [ 0  0  I  0]  [ 0  0  0  I]
[ 0  0 -I  0]  [-I  0  0  0]  [ 0  1  0  0]  [ 0 -I  0  0]  [ I  0  0  0]
[ 0  0  0 -I]  [ 0  I  0  0]  [-1  0  0  0]  [-I  0  0  0]  [ 0  I  0  0]
```

In C language format, we can write the Dirac five-space multivector product in component form as 32 equations with 32 product sums. These formula are too long to easily typeset, but are online embedded in the demonstration program at `http://www.kurtnalty.com/Dirac_vs_Ginac.cp`.

The generic Dirac space multivector can be written, with each grade on its own line, as

```
Dirac Spacetime =
+ a*q
+ b*w + c*x + d*y + e*z + f*t
+ g*wx + h*wy + j*wz + k*wt + l*xy + m*xz + n*xt + p*yz + r*yt + s*zt
+ S*wxy + R*wxz + P*wxt + N*wyz + M*wyt + L*wzt + K*xyz + J*xyt + H*xzt + G*yzt
+ F*wxyz + E*wxyt + D*wxzt + C*wyzt + B*xyzt
+ A*wxyzt
```

where the coefficients have been chosen to emphasize the dual (complex) structure of the five space multivector. For example, $a + Ae_{wxyzt}$ mimics the complex number $(a + iA)$, and $de_y + De_{wxzt}$ mimics $e_y(d + iD)$.

Multivector to Matrix Form

Multivectors can be written in the equivalent 4x4 complex matrix form.

```
Matrix format =
[(+a+f-j-L)+I(+A+F-J-l), (-g+H+m-P)+I(-G+h+M-p), (+b+e-k-s)+I(+B+E-K-S), (+c-D-n+R)+I(+C-d-N+r)]
[(-g-H-m-P)+I(-G-h-M-p), (+a+f+j+L)+I(+A+F+J+l), (+c+D-n-R)+I(+C+d-N-r), (+b-e-k+s)+I(+B-E-K+S)]
[(+b-e+k-s)+I(+B-E+K-S), (-c+D-n+R)+I(-C+d-N+r), (+a-f+j-L)+I(+A-F+J-l), (+g-H+m-P)+I(+G-h+M-p)]
[(-c-D-n-R)+I(-C-d-N-r), (+b+e+k+s)+I(+B+E+K+S), (+g+H-m-P)+I(+G+h-M-p), (+a-f-j+L)+I(+A-F-J+l)]
```

Matrix to Multivector Form

To recover the multivector components from a generic complex 4x4 matrix $W[4][4]$, we have the 32 trace-like formulas:

$$
\begin{aligned}
a &= \text{real}(+W[0][0] + W[1][1] + W[2][2] + W[3][3])/4 \\
b &= \text{real}(+W[0][2] + W[1][3] + W[2][0] + W[3][1])/4 \\
c &= \text{real}(+W[0][3] + W[1][2] - W[2][1] - W[3][0])/4 \\
d &= \text{imag}(-W[0][3] + W[1][2] + W[2][1] - W[3][0])/4 \\
e &= \text{real}(+W[0][2] - W[1][3] - W[2][0] + W[3][1])/4 \\
f &= \text{real}(+W[0][0] + W[1][1] - W[2][2] - W[3][3])/4 \\
g &= \text{real}(-W[0][1] - W[1][0] + W[2][3] + W[3][2])/4 \\
h &= \text{imag}(+W[0][1] - W[1][0] - W[2][3] + W[3][2])/4 \\
j &= \text{real}(-W[0][0] + W[1][1] + W[2][2] - W[3][3])/4 \\
k &= \text{real}(-W[0][2] - W[1][3] + W[2][0] + W[3][3])/4 \\
l &= \text{imag}(-W[0][0] + W[1][1] - W[2][2] + W[3][3])/4 \\
m &= \text{real}(+W[0][1] - W[1][0] + W[2][3] - W[3][2])/4 \\
n &= \text{real}(-W[0][3] - W[1][2] - W[2][1] - W[3][0])/4 \\
p &= \text{imag}(-W[0][1] - W[1][0] - W[2][3] - W[3][2])/4 \\
r &= \text{imag}(+W[0][3] - W[1][2] + W[2][1] - W[3][0])/4 \\
s &= \text{real}(-W[0][2] + W[1][3] - W[2][0] + W[3][1])/4
\end{aligned}
$$

$$
\begin{aligned}
A &= \text{imag}(+W[0][0] + W[1][1] + W[2][2] + W[3][3])/4 \\
B &= \text{imag}(+W[0][2] + W[1][3] + W[2][0] + W[3][1])/4 \\
C &= \text{imag}(+W[0][3] + W[1][2] - W[2][1] - W[3][0])/4 \\
D &= \text{real}(-W[0][3] + W[1][2] + W[2][1] - W[3][0])/4 \\
E &= \text{imag}(+W[0][2] - W[1][3] - W[2][0] + W[3][1])/4 \\
F &= \text{imag}(+W[0][0] + W[1][1] - W[2][2] - W[3][3])/4 \\
G &= \text{imag}(-W[0][1] - W[1][0] + W[2][3] + W[3][2])/4 \\
H &= \text{real}(+W[0][1] - W[1][0] - W[2][3] + W[3][2])/4 \\
J &= \text{imag}(-W[0][0] + W[1][1] + W[2][2] - W[3][3])/4 \\
K &= \text{imag}(-W[0][2] - W[1][3] + W[2][0] + W[3][3])/4 \\
L &= \text{real}(-W[0][0] + W[1][1] - W[2][2] + W[3][3])/4 \\
M &= \text{imag}(+W[0][1] - W[1][0] + W[2][3] - W[3][2])/4 \\
N &= \text{imag}(-W[0][3] - W[1][2] - W[2][1] - W[3][0])/4 \\
P &= \text{real}(-W[0][1] - W[1][0] - W[2][3] - W[3][2])/4 \\
R &= \text{real}(+W[0][3] - W[1][2] + W[2][1] - W[3][0])/4 \\
S &= \text{imag}(-W[0][2] + W[1][3] - W[2][0] + W[3][1])/4
\end{aligned}
$$

Column Vector to Multivector via Minimal Left Ideal

Wave functions in Dirac theory are a complex 4x1 column vector. Garret Sobczyk [7] maps these column vectors to a square matrix, where three columns are zero. This, in turn, maps to a multivector using the formulas above.

Using the example of a 4x1 complex column matrix mapping to the left column of a square matrix, we have

$$
\begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \rightarrow \begin{pmatrix} \psi_0 & 0 & 0 & 0 \\ \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \end{pmatrix}
$$

This leads to a multivector representation of the column matrix as

$$\begin{aligned}
\psi &= \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \rightarrow \begin{pmatrix} \psi_0 & 0 & 0 & 0 \\ \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \end{pmatrix} \\
a &= +\psi_{0r}/4 \\
b &= +\psi_{2r}/4 \\
c &= -\psi_{3r}/4 \\
d &= -\psi_{3i}/4 \\
e &= -\psi_{2r}/4 \\
f &= +\psi_{0r}/4 \\
g &= -\psi_{1r}/4 \\
h &= -\psi_{1i}/4 \\
j &= -\psi_{0r}/4 \\
k &= +\psi_{2r}/4 \\
l &= -\psi_{0i}/4 \\
m &= -\psi_{1r}/4 \\
n &= -\psi_{3r}/4 \\
p &= -\psi_{1i}/4 \\
r &= -\psi_{3i}/4 \\
s &= -\psi_{2r}/4
\end{aligned}$$

$$
\begin{aligned}
A &= +\psi_{0i}/4\\
B &= +\psi_{2i}/4\\
C &= -\psi_{3i}/4\\
D &= -\psi_{3r}/4\\
E &= -\psi_{2i}/4\\
F &= +\psi_{0i}/4\\
G &= -\psi_{1i}/4\\
H &= -\psi_{1r}/4\\
J &= -\psi_{0i}/4\\
K &= +\psi_{2i}/4\\
L &= -\psi_{0r}/4\\
M &= -\psi_{1i}/4\\
N &= -\psi_{3i}/4\\
P &= -\psi_{1r}/4\\
R &= -\psi_{3r}/4\\
S &= -\psi_{2i}/4
\end{aligned}
$$

$$
\begin{aligned}
4 * \psi = &+\psi_{0r}\\
&+\psi_{2r}e_w - \psi_{3r}e_x - \psi_{3i}e_y - \psi_{2r}e_z + \psi_{0r}e_t\\
&-\psi_{1r}e_{wx} - \psi_{1i}e_{wy} - \psi_{0r}e_{wz} + \psi_{2r}e_{wt} - \psi_{0i}e_{xy}\\
&-\psi_{1r}e_{xz} - \psi_{3r}e_{xt} - \psi_{1i}e_{yz} - \psi_{3i}e_{yt} - \psi_{2r}e_{zt}\\
&-\psi_{2i}e_{wxy} - \psi_{3r}e_{wxz} - \psi_{1r}e_{wxt} - \psi_{3i}e_{wyz} - \psi_{1i}e_{wyt}\\
&-\psi_{0r}e_{wzt} + \psi_{2i}e_{xyz} - \psi_{0i}e_{xyt} - \psi_{1r}e_{xzt} - \psi_{1i}e_{yzt}\\
&+\psi_{0i}e_{wxyz} - \psi_{2i}e_{wxyt} - \psi_{3r}e_{wxzt} - \psi_{3i}e_{wyzt} + \psi_{2i}e_{xyzt}\\
&+\psi_{0i}e_{wxyzt}
\end{aligned}
$$

Multivector Complex Conjugate

From the matrix representation of a multivector, we can deduce the complex conjugate.

```
Complex Conjugate =
+ a*q
- b*w + c*x + d*y + e*z + f*t
- g*wx - h*wy - j*wz - k*wt + l*xy + m*xz + n*xt + p*yz + r*yt + s*zt
- S*wxy - R*wxz - P*wxt - N*wyz - M*wyt - L*wzt + K*xyz + J*xyt + H*xzt + G*yzt
- F*wxyz - E*wxyt - D*wxzt - C*wyzt + B*xyzt
- A*wxyzt
```

For the complex conjugate, we see the easy rule that any term with a w basis changes sign.

Multivector Transpose

We now deduce the matrix transpose for our multivector.

```
Spacetime.transpose() = [
[(+a+c+J+r)+I(+A-C+j-R), (+d+f-l-n)+I(+D-F+L-N), (+H-K-p+s)+I(-h-k+P+S), (-B-e+G+m)+I(+b+E+g+M)],
[(+d-f+l-n)+I(+D+F-L-N), (+a-c+J-r)+I(+A+C+j+R), (+B+e+G+m)+I(-b-E+g+M), (-H-K+p+s)+I(+h-k-P+S)],
[(+H+K+p+s)+I(-h+k-P+S), (-B+e+G-m)+I(+b-E+g-M), (+a+c-J-r)+I(+A-C-j+R), (+d-f-l+n)+I(+D+F+L+N)],
[(+B-e+G-m)+I(-b+E+g-M), (-H+K-p+s)+I(+h+k+P+S), (+d+f+l+n)+I(+D-F-L+N), (+a-c-J+r)+I(+A+C-j-R)]]
```

```
Dirac Transpose =
+ a*q
- b*w + c*x + d*y + e*z - f*t
+ g*wx + h*wy + j*wz - k*wt - l*xy - m*xz + n*xt - p*yz + r*yt + s*zt
+ S*wxy + R*wxz - P*wxt + N*wyz - M*wyt - L*wzt - K*xyz + J*xyt + H*xzt + G*yzt
- F*wxyz + E*wxyt + D*wxzt + C*wyzt - B*xyzt
+ A*wxyzt
```

Multivector Hermitian Conjugate

The Hermitian conjugate is the combination of matrix transposition and complex component conjugation.

```
Dirac Hermitian =
+ a*q
+ b*w + c*x + d*y + e*z - f*t
- g*wx - h*wy - j*wz + k*wt - l*xy - m*xz + n*xt - p*yz + r*yt + s*zt
- S*wxy - R*wxz + P*wxt - N*wyz + M*wyt + L*wzt - K*xyz + J*xyt + H*xzt + G*yzt
+ F*wxyz - E*wxyt - D*wxzt - C*wyzt - B*xyzt
- A*wxyzt
```

We see that the sign change in the Hermitian corresponds to the square of the multivector basis element. For example, $e_t^2 = -1$, so f changes sign.

Implementation of the Dirac Equation

We use the standard Bjorken and Drell matrices mapped to geometric algebra basis below.

$\gamma_0 \to e_t$, $\gamma_1 \to e_x$, $\gamma_2 \to e_y$ and $\gamma_3 \to e_z$

Start with the relativistic energy momentum relationship.

$$\begin{aligned} E^2 - \vec{p}\cdot\vec{p} &= m^2 \\ E^2 - p_x^2 - p_y^2 - p_z^2 - m^2 &= 0 \\ (Ee_t - p_x e_x - p_y e_y - p_z e_z)^2 - (m)^2 &= 0 \end{aligned}$$

This last expression factors into

$$(Ee_t - p_x e_x - p_y e_y - p_z e_z + m) * (Ee_t - p_x e_x - p_y e_y - p_z e_z - m) = 0$$

Either factor being zero is a solution for the equation above. Choose the right hand side.

$$(Ee_t - p_x e_x - p_y e_y - p_z e_z - m) = 0$$

Substitute our operators, and apply to a multivector wavefunction ψ.

$$E = \left(i\hbar\frac{\partial}{\partial t} + q\phi\right) \quad p_x = \left(-i\hbar\frac{\partial}{\partial x} + qA_x\right) \quad p_y = \left(-i\hbar\frac{\partial}{\partial y} + qA_y\right) \quad p_z = \left(-i\hbar\frac{\partial}{\partial z} + qA_z\right)$$

The equation

$$(Ee_t - p_x e_x - p_y e_y - p_z e_z - m)\,\psi = 0$$

becomes

$$\begin{aligned} &+\left(i\hbar\frac{\partial\psi}{\partial t} + q\phi\psi\right)e_t - \left(-i\hbar\frac{\partial\psi}{\partial x} + qA_x\psi\right)e_x - \left(-i\hbar\frac{\partial\psi}{\partial y} + qA_y\psi\right)e_y \\ &-\left(-i\hbar\frac{\partial\psi}{\partial z} + qA_z\psi\right)e_z - m\psi = 0 \end{aligned}$$

Distribute our products and remove parenthesis

$$\begin{aligned}
&+ie_t\hbar\frac{\partial\psi}{\partial t}+q\phi\psi e_t+ie_x\hbar\frac{\partial\psi}{\partial x}-qA_x\psi e_x+ie_y\hbar\frac{\partial\psi}{\partial y}-qA_y\psi e_y\\
&+ie_z\hbar\frac{\partial\psi}{\partial z}-qA_z\psi e_z-m\psi=0
\end{aligned}$$

Re-arrange in a suggestive order.

$$\begin{aligned}
&-m\psi+q\phi\psi e_t-qA_x\psi e_x-qA_y\psi e_y-qA_z\psi e_z\\
&+ie_t\hbar\frac{\partial\psi}{\partial t}+ie_x\hbar\frac{\partial\psi}{\partial x}+ie_y\hbar\frac{\partial\psi}{\partial y}+ie_z\hbar\frac{\partial\psi}{\partial z}=0
\end{aligned}$$

Substitute $i = e_{wxyzt}$ and apply in products.

$$\begin{aligned}
&-m\psi+q\phi\psi e_t-qA_x\psi e_x-qA_y\psi e_y-qA_z\psi e_z\\
&+e_{wxyz}\hbar\frac{\partial\psi}{\partial t}-e_{wyzt}\hbar\frac{\partial\psi}{\partial x}+e_{wxzy}\hbar\frac{\partial\psi}{\partial y}-e_{wxyt}\hbar\frac{\partial\psi}{\partial z}=0
\end{aligned}$$

The column vector nature of ψ results in a multivector format given by

$$\begin{aligned}
4*\psi \;=\;& +\psi_{0r}\\
&+\psi_{0r}e_w+\psi_{1r}e_x-\psi_{3r}e_y+\psi_{1r}e_z-\psi_{1r}e_t\\
&+\psi_{3r}e_{wx}-\psi_{2r}e_{wy}-\psi_{1r}e_{wz}+\psi_{0r}e_{wt}+\psi_{2r}e_{xy}\\
&-\psi_{2r}e_{xz}+\psi_{0r}e_{xt}+\psi_{2r}e_{yz}+\psi_{3r}e_{yt}-\psi_{3r}e_{zt}\\
&-\psi_{3i}e_{wxy}+\psi_{3i}e_{wxz}+\psi_{2i}e_{wxt}+\psi_{0i}e_{wyz}-\psi_{2i}e_{wyt}\\
&+\psi_{2i}e_{wzt}+\psi_{0i}e_{xyz}-\psi_{1i}e_{xyt}-\psi_{2i}e_{xzt}+\psi_{3i}e_{yzt}\\
&-\psi_{1i}e_{wxyz}+\psi_{1i}e_{wxyt}-\psi_{3i}e_{wxzt}+\psi_{1i}e_{wyzt}+\psi_{0i}e_{xyzt}\\
&+\psi_{0i}e_{wxyzt}
\end{aligned}$$

Substitute ψ in the operator equation, and collect terms by common component. This will result in 32 equations, each of which sums to zero. Computer algebra systems such as GiNaC or Sympy are useful here. For our case, only eight of these equations were unique.

$$
\begin{aligned}
-m\psi_{0r} + q\phi\psi_{0r} + qA_x\psi_{3r} + qA_y\psi_{3i} + qA_z\psi_{2r} - \hbar\frac{\partial\psi_{0i}}{\partial t} + \hbar\frac{\partial\psi_{3i}}{\partial x} - \hbar\frac{\partial\psi_{3r}}{\partial y} + \hbar\frac{\partial\psi_{2i}}{\partial z} &= 0 \\
-m\psi_{0i} + q\phi\psi_{0i} + qA_x\psi_{3i} - qA_y\psi_{3r} + qA_z\psi_{2i} + \hbar\frac{\partial\psi_{0r}}{\partial t} - \hbar\frac{\partial\psi_{3r}}{\partial x} - \hbar\frac{\partial\psi_{3i}}{\partial y} - \hbar\frac{\partial\psi_{2r}}{\partial z} &= 0 \\
-m\psi_{1r} + q\phi\psi_{1r} + qA_x\psi_{2r} - qA_y\psi_{2i} - qA_z\psi_{3r} - \hbar\frac{\partial\psi_{1i}}{\partial t} + \hbar\frac{\partial\psi_{2i}}{\partial x} + \hbar\frac{\partial\psi_{2r}}{\partial y} - \hbar\frac{\partial\psi_{3i}}{\partial z} &= 0 \\
-m\psi_{1i} + q\phi\psi_{1i} + qA_x\psi_{2i} + qA_y\psi_{2r} - qA_z\psi_{3i} + \hbar\frac{\partial\psi_{1r}}{\partial t} - \hbar\frac{\partial\psi_{2r}}{\partial x} + \hbar\frac{\partial\psi_{2i}}{\partial y} + \hbar\frac{\partial\psi_{3r}}{\partial z} &= 0 \\
+m\psi_{2r} + q\phi\psi_{2r} + qA_x\psi_{1r} + qA_y\psi_{1i} + qA_z\psi_{0r} - \hbar\frac{\partial\psi_{2i}}{\partial t} + \hbar\frac{\partial\psi_{1i}}{\partial x} - \hbar\frac{\partial\psi_{1r}}{\partial y} + \hbar\frac{\partial\psi_{0i}}{\partial z} &= 0 \\
+m\psi_{2i} + q\phi\psi_{2i} + qA_x\psi_{1i} - qA_y\psi_{1r} + qA_z\psi_{0i} + \hbar\frac{\partial\psi_{2r}}{\partial t} - \hbar\frac{\partial\psi_{1r}}{\partial x} - \hbar\frac{\partial\psi_{1i}}{\partial y} - \hbar\frac{\partial\psi_{0r}}{\partial z} &= 0 \\
+m\psi_{3r} + q\phi\psi_{3r} + qA_x\psi_{0r} - qA_y\psi_{0i} - qA_z\psi_{1r} - \hbar\frac{\partial\psi_{3i}}{\partial t} + \hbar\frac{\partial\psi_{0i}}{\partial x} + \hbar\frac{\partial\psi_{0r}}{\partial y} - \hbar\frac{\partial\psi_{1i}}{\partial z} &= 0 \\
+m\psi_{3i} + q\phi\psi_{3i} + qA_x\psi_{0i} + qA_y\psi_{0r} - qA_z\psi_{1i} + \hbar\frac{\partial\psi_{3r}}{\partial t} - \hbar\frac{\partial\psi_{0r}}{\partial x} + \hbar\frac{\partial\psi_{0i}}{\partial y} + \hbar\frac{\partial\psi_{1r}}{\partial z} &= 0
\end{aligned}
$$

As should be expected, these equation match the matrix derived Dirac equations.

The groupings of the redundancies is interesting. These groups are listed below. Post-factors of e_t and $-e_{wz}$ have significance in this basis choice.

$$
\begin{gathered}
S.q = S.t = S.wz = S.wzt \\
S.xy = S.xz = S.wxt = S.xzt \\
S.wx = S.xz = S.wxt = S.xzt \\
S.wy = S.yz = S.wyt = S.yzt \\
S.w = S.z = S.wt = S.zt \\
S.wxy = S.xyz = S.wxyt = S.xyzt \\
S.x = S.xt = S.wxz = S.wxzt \\
S.y = S.yt = S.wyz = S.wyzt
\end{gathered}
$$

The Weyl Equation using Geometric Algebra

The Weyl matrices are the Dirac Gamma matrices re-arranged.

```
     q             wxyzt
[ 1  0  0  0]  [ I  0  0  0]
[ 0  1  0  0]  [ 0  I  0  0]
[ 0  0  1  0]  [ 0  0  I  0]
[ 0  0  0  1]  [ 0  0  0  I]

     w               x              y              z              t
[ I  0  0  0]  [ 0  0  0  1]  [ 0  0  0 -I]  [ 0  0  1  0]  [ 0  0  I  0]
[ 0  I  0  0]  [ 0  0  1  0]  [ 0  0  I  0]  [ 0  0  0 -1]  [ 0  0  0  I]
[ 0  0 -I  0]  [ 0 -1  0  0]  [ 0  I  0  0]  [-1  0  0  0]  [ I  0  0  0]
[ 0  0  0 -I]  [-1  0  0  0]  [-I  0  0  0]  [ 0  1  0  0]  [ 0  I  0  0]
 Weyl Gamma5       Gamma1         Gamma2         Gamma3      Weyl Gamma0

    wx             wy             wz             wt             xy
[ 0  0  0  I]  [ 0  0  0  1]  [ 0  0  I  0]  [ 0  0 -1  0]  [-I  0  0  0]
[ 0  0  I  0]  [ 0  0 -1  0]  [ 0  0  0 -I]  [ 0  0  0 -1]  [ 0  I  0  0]
[ 0  I  0  0]  [ 0  1  0  0]  [ I  0  0  0]  [ 1  0  0  0]  [ 0  0 -I  0]
[ I  0  0  0]  [-1  0  0  0]  [ 0 -I  0  0]  [ 0  1  0  0]  [ 0  0  0  I]

    xz             xt             yz             yt             zt
[ 0  1  0  0]  [ 0  I  0  0]  [ 0 -I  0  0]  [ 0  1  0  0]  [ I  0  0  0]
[-1  0  0  0]  [ I  0  0  0]  [-I  0  0  0]  [-1  0  0  0]  [ 0 -I  0  0]
[ 0  0  0  1]  [ 0  0  0 -I]  [ 0  0  0 -I]  [ 0  0  0 -1]  [ 0  0 -I  0]
[ 0  0 -1  0]  [ 0  0 -I  0]  [ 0  0 -I  0]  [ 0  0  1  0]  [ 0  0  0  I]

    wxy            wxz            wxt            wyz            wyt
[ 1  0  0  0]  [ 0  I  0  0]  [ 0 -1  0  0]  [ 0  1  0  0]  [ 0  I  0  0]
[ 0 -1  0  0]  [-I  0  0  0]  [-1  0  0  0]  [ 1  0  0  0]  [-I  0  0  0]
[ 0  0 -1  0]  [ 0  0  0 -I]  [ 0  0  0 -1]  [ 0  0  0 -1]  [ 0  0  0  I]
[ 0  0  0  1]  [ 0  0  I  0]  [ 0  0 -1  0]  [ 0  0 -1  0]  [ 0  0 -I  0]

    wzt            xyz            xyt            xzt            yzt
[-1  0  0  0]  [ 0  0 -I  0]  [ 0  0  1  0]  [ 0  0  0  I]  [ 0  0  0  1]
[ 0  1  0  0]  [ 0  0  0 -I]  [ 0  0  0 -1]  [ 0  0 -I  0]  [ 0  0  1  0]
[ 0  0 -1  0]  [ I  0  0  0]  [ 1  0  0  0]  [ 0  I  0  0]  [ 0  1  0  0]
[ 0  0  0  1]  [ 0  I  0  0]  [ 0 -1  0  0]  [-I  0  0  0]  [ 1  0  0  0]

    wxyz           wxyt           wxzt           wyzt           xyzt
[ 0  0  1  0]  [ 0  0  I  0]  [ 0  0  0 -1]  [ 0  0  0  I]  [ 1  0  0  0]
[ 0  0  0  1]  [ 0  0  0 -I]  [ 0  0  1  0]  [ 0  0  I  0]  [ 0  1  0  0]
[ 1  0  0  0]  [-I  0  0  0]  [ 0  1  0  0]  [ 0 -I  0  0]  [ 0  0 -1  0]
[ 0  1  0  0]  [ 0  I  0  0]  [-1  0  0  0]  [-I  0  0  0]  [ 0  0  0 -1]
```

The five basis matrices square to negative one, meaning that the Clifford signature is (- - - - -). As such, this basis will not factor the Klein Gordon equation.

Proceeding anyway, we can document the different relationships between the generic multivector, the matrix representation, and the left ideal.

The generic Weyl space multivector can be written, with each grade on its own line, as

```
Weyl Spacetime =
+ a*q
+ b*w + c*x + d*y + e*z + f*t
+ g*wx + h*wy + j*wz + k*wt + l*xy + m*xz + n*xt + p*yz + r*yt + s*zt
+ S*wxy + R*wxz + P*wxt + N*wyz + M*wyt + L*wzt + K*xyz + J*xyt + H*xzt + G*yzt
+ F*wxyz + E*wxyt + D*wxzt + C*wyzt + B*xyzt
+ A*wxyzt
```

where the coefficients have been chosen to emphasize the dual (complex) structure of the five space multivector. For example, $a + Ae_{wxyzt}$ mimics the complex number $(a + iA)$, and $de_y + De_{wxzt}$ mimics $e_y(d + iD)$.

Multivector to Weyl Matrix Form

Multivectors can be written in the equivalent 4x4 complex matrix form.

```
Matrix format = (Weyl)
[(+a+B-L+S)+I*(+A+b-l+s),(+m+N-P+r)+I*(+M+n-p+R),(+e+F+J-k)+I*(+E+f+j-K),(+c-D+G+h)+I*(+C-d+g+H)]
[(-m+N-P-r)+I*(-M+n-p-R),(+a+B+L-S)+I*(+A+b+l-s),(+c+D+G-h)+I*(+C+d+g-H),(-e+F-J-k)+I*(-E+f-j-K)]
[(-e+F+J+k)+I*(-E+j+f+K),(-c+D+G+h)+I*(-C+d+g+H),(+a-B-L-S)+I*(+A-b-l-s),(+m-N-P-r)+I*(+M-n-p-R)]
[(-c-D+G-h)+I*(-C-d+g-H),(+e+F-J+k)+I*(+E+f-j+K),(-m-N-P+r)+I*(-M-n-p+R),(+a-B+L+S)+I*(+A-b+l+s)]
```

Weyl Matrix to Multivector Form

To recover the multivector components from a generic complex 4x4 matrix $W[4][4]$, we have the 32 trace-like formulas:

$$
\begin{aligned}
a &= \text{real}(+W[0][0] + W[1][1] + W[2][2] + W[3][3])/4 \\
b &= \text{imag}(+W[0][0] + W[1][1] - W[2][2] - W[3][3])/4 \\
c &= \text{real}(+W[0][3] + W[1][2] - W[2][1] - W[3][0])/4 \\
d &= \text{imag}(-W[0][3] + W[1][2] + W[2][1] - W[3][0])/4 \\
e &= \text{real}(+W[0][2] - W[1][3] - W[2][0] + W[3][1])/4 \\
f &= \text{imag}(+W[0][2] + W[1][3] + W[2][0] + W[3][1])/4 \\
g &= \text{imag}(+W[0][3] + W[1][2] + W[2][1] + W[3][0])/4 \\
h &= \text{real}(+W[0][3] - W[1][2] + W[2][1] - W[3][0])/4 \\
j &= \text{imag}(+W[0][2] - W[1][3] + W[2][0] - W[3][3])/4 \\
k &= \text{real}(-W[0][2] - W[1][3] + W[2][0] + W[3][1])/4 \\
l &= \text{imag}(-W[0][0] + W[1][1] - W[2][2] + W[3][3])/4 \\
m &= \text{real}(+W[0][1] - W[1][0] + W[2][3] - W[3][2])/4 \\
n &= \text{imag}(+W[0][1] + W[1][0] - W[2][3] - W[3][2])/4 \\
p &= \text{imag}(-W[0][1] - W[1][0] - W[2][3] - W[3][2])/4 \\
r &= \text{real}(+W[0][1] - W[1][0] - W[2][3] + W[3][2])/4 \\
s &= \text{imag}(+W[0][0] - W[1][1] - W[2][2] + W[3][3])/4
\end{aligned}
$$

$$
\begin{aligned}
A &= \text{imag}(+W[0][0] + W[1][1] + W[2][2] + W[3][3])/4 \\
B &= \text{real}(+W[0][0] + W[1][1] - W[2][2] - W[3][3])/4 \\
C &= \text{imag}(+W[0][3] + W[1][2] - W[2][1] - W[3][0])/4 \\
D &= \text{real}(-W[0][3] + W[1][2] + W[2][1] - W[3][0])/4 \\
E &= \text{imag}(+W[0][2] - W[1][3] - W[2][0] + W[3][1])/4 \\
F &= \text{real}(+W[0][2] + W[1][3] + W[2][0] + W[3][1])/4 \\
G &= \text{real}(+W[0][3] + W[1][2] + W[2][1] + W[3][0])/4 \\
H &= \text{imag}(+W[0][3] - W[1][2] + W[2][1] - W[3][0])/4 \\
J &= \text{real}(+W[0][2] - W[1][3] + W[2][0] - W[3][3])/4 \\
K &= \text{imag}(-W[0][2] - W[1][3] + W[2][0] + W[3][1])/4 \\
L &= \text{real}(-W[0][0] + W[1][1] - W[2][2] + W[3][3])/4 \\
M &= \text{imag}(+W[0][1] - W[1][0] + W[2][3] - W[3][2])/4 \\
N &= \text{real}(+W[0][1] + W[1][0] - W[2][3] - W[3][2])/4 \\
P &= \text{real}(-W[0][1] - W[1][0] - W[2][3] - W[3][2])/4 \\
R &= \text{imag}(+W[0][1] - W[1][0] - W[2][3] + W[3][2])/4 \\
S &= \text{real}(+W[0][0] - W[1][1] - W[2][2] + W[3][3])/4
\end{aligned}
$$

Column Vector to Multivector via Minimal Left Ideal

We have a different formula for the left ideal as compared to the Bjorken Drell matrices.

$$
\begin{aligned}
4 * \psi = \; & +\psi_{0r} \\
& +\psi_{2r}e_w - \psi_{3r}e_x - \psi_{3i}e_y - \psi_{2r}e_z + \psi_{0r}e_t \\
& -\psi_{1r}e_{wx} - \psi_{1i}e_{wy} - \psi_{0r}e_{wz} + \psi_{2r}e_{wt} - \psi_{0i}e_{xy} \\
& -\psi_{1r}e_{xz} - \psi_{3r}e_{xt} - \psi_{1i}e_{yz} - \psi_{3i}e_{yt} - \psi_{2r}e_{zt} \\
& -\psi_{2i}e_{wxy} - \psi_{3r}e_{wxz} - \psi_{1r}e_{wxt} - \psi_{3i}e_{wyz} - \psi_{1i}e_{wyt} \\
& -\psi_{0r}e_{wzt} + \psi_{2i}e_{xyz} - \psi_{0i}e_{xyt} - \psi_{1r}e_{xzt} - \psi_{1i}e_{yzt} \\
& +\psi_{0i}e_{wxyz} - \psi_{2i}e_{wxyt} - \psi_{3r}e_{wxzt} - \psi_{3i}e_{wyzt} + \psi_{2i}e_{xyzt} \\
& +\psi_{0i}e_{wxyzt}
\end{aligned}
$$

Weyl Factoring the Klein Gordon Equation

The Weyl geometric algebra has signature (- - - - -). As such, we don't have a drop-in replacement for the Bjorken Drell matrices for factoring the Klein-Gordon equation. To accommodate the change in sign for the time metric, we introduce a factor of $i = e_{wxyzt}$ for the energy term, effectively creating a new signature (- - - - +).

Write the relativistic energy-momentum relationship with $c = 1$.

$$\begin{aligned} E^2 - \vec{p} \cdot \vec{p} &= m^2 \\ E^2 - p_x^2 - p_y^2 - p_z^2 - m^2 &= 0 \\ (Eie_t + p_x e_x + p_y e_y + p_z e_z)^2 - m^2 &= 0 \end{aligned}$$

This last expression factors into

$$(Eie_t + p_x e_x + p_y e_y + p_z e_z + m) * (Eie_t + p_x e_x + p_y e_y + p_z e_z - m) = 0$$

Either factor being zero is a solution for the equation above. The traditional choice is the right hand side.

$$(Eie_t + p_x e_x + p_y e_y + p_z e_z - m) = 0$$

For quantization, we apply the operator above to a wavefunction ψ.

$$(Eie_t + p_x e_x + p_y e_y + p_z e_z - m)\,\psi = 0$$

Substitute $i = e_{wxyzt}$,

$$(Ee_{wxyz} + p_x e_x + p_y e_y + p_z e_z - m)\,\psi = 0$$

Substitute our operators, and apply to a multivector wavefunction ψ.

$$E = \left(i\hbar\frac{\partial}{\partial t} + q\phi\right) \quad p_x = \left(-i\hbar\frac{\partial}{\partial x} + qA_x\right) \quad p_y = \left(-i\hbar\frac{\partial}{\partial y} + qA_y\right) \quad p_z = \left(-i\hbar\frac{\partial}{\partial z} + qA_z\right)$$

The equation

$$(Ee_{wxyz} - p_x e_x - p_y e_y - p_z e_z - m)\,\psi = 0$$

becomes

$$+\left(i\hbar\frac{\partial\psi}{\partial t}+q\phi\psi\right)e_{wxyz}-\left(-i\hbar\frac{\partial\psi}{\partial x}+qA_x\psi\right)e_x-\left(-i\hbar\frac{\partial\psi}{\partial y}+qA_y\psi\right)e_y$$
$$-\left(-i\hbar\frac{\partial\psi}{\partial z}+qA_z\psi\right)e_z-m\psi=0$$

Distribute our products and remove parenthesis

$$+ie_{wxyz}\hbar\frac{\partial\psi}{\partial t}+q\phi\psi e_{wxyz}+ie_x\hbar\frac{\partial\psi}{\partial x}-qA_x\psi e_x+ie_y\hbar\frac{\partial\psi}{\partial y}-qA_y\psi e_y$$
$$+ie_z\hbar\frac{\partial\psi}{\partial z}-qA_z\psi e_z-m\psi=0$$

Re-arrange in a suggestive order.

$$-m\psi+q\phi\psi e_{wxyz}-qA_x\psi e_x-qA_y\psi e_y-qA_z\psi e_z$$
$$+ie_{wxyz}\hbar\frac{\partial\psi}{\partial t}+ie_x\hbar\frac{\partial\psi}{\partial x}+ie_y\hbar\frac{\partial\psi}{\partial y}+ie_z\hbar\frac{\partial\psi}{\partial z}=0$$

Substitute $i=e_{wxyzt}$ and apply in products.

$$-m\psi+q\phi\psi e_{wxyz}-qA_x\psi e_x-qA_y\psi e_y-qA_z\psi e_z$$
$$+e_t\hbar\frac{\partial\psi}{\partial t}-e_{wyzt}\hbar\frac{\partial\psi}{\partial x}+e_{wxzy}\hbar\frac{\partial\psi}{\partial y}-e_{wxyt}\hbar\frac{\partial\psi}{\partial z}=0$$

We find our set of eight equations, differing from previous sets, is

$$
\begin{aligned}
-m\psi_{0r} + q\phi\psi_{2r} + qA_x\psi_{3r} + qA_y\psi_{3i} + qA_z\psi_{2r} - \hbar\frac{\partial\psi_{2i}}{\partial t} + \hbar\frac{\partial\psi_{3i}}{\partial x} - \hbar\frac{\partial\psi_{3r}}{\partial y} + \hbar\frac{\partial\psi_{2i}}{\partial z} &= 0 \\
-m\psi_{0i} + q\phi\psi_{2i} + qA_x\psi_{3i} - qA_y\psi_{3r} + qA_z\psi_{2i} + \hbar\frac{\partial\psi_{2r}}{\partial t} - \hbar\frac{\partial\psi_{3r}}{\partial x} - \hbar\frac{\partial\psi_{3i}}{\partial y} - \hbar\frac{\partial\psi_{2r}}{\partial z} &= 0 \\
-m\psi_{1r} + q\phi\psi_{3r} + qA_x\psi_{2r} - qA_y\psi_{2i} - qA_z\psi_{3r} - \hbar\frac{\partial\psi_{3i}}{\partial t} + \hbar\frac{\partial\psi_{2i}}{\partial x} + \hbar\frac{\partial\psi_{2r}}{\partial y} - \hbar\frac{\partial\psi_{3i}}{\partial z} &= 0 \\
-m\psi_{1i} + q\phi\psi_{3i} + qA_x\psi_{2i} + qA_y\psi_{2r} - qA_z\psi_{3i} + \hbar\frac{\partial\psi_{3r}}{\partial t} - \hbar\frac{\partial\psi_{2r}}{\partial x} + \hbar\frac{\partial\psi_{2i}}{\partial y} + \hbar\frac{\partial\psi_{3r}}{\partial z} &= 0 \\
-m\psi_{2r} + q\phi\psi_{0r} - qA_x\psi_{1r} - qA_y\psi_{1i} - qA_z\psi_{0r} - \hbar\frac{\partial\psi_{0i}}{\partial t} - \hbar\frac{\partial\psi_{1i}}{\partial x} + \hbar\frac{\partial\psi_{1r}}{\partial y} - \hbar\frac{\partial\psi_{0i}}{\partial z} &= 0 \\
-m\psi_{2i} + q\phi\psi_{0i} - qA_x\psi_{1i} + qA_y\psi_{1r} - qA_z\psi_{0i} + \hbar\frac{\partial\psi_{0r}}{\partial t} + \hbar\frac{\partial\psi_{1r}}{\partial x} + \hbar\frac{\partial\psi_{1i}}{\partial y} + \hbar\frac{\partial\psi_{0r}}{\partial z} &= 0 \\
-m\psi_{3r} + q\phi\psi_{1r} - qA_x\psi_{0r} + qA_y\psi_{0i} + qA_z\psi_{1r} - \hbar\frac{\partial\psi_{1i}}{\partial t} - \hbar\frac{\partial\psi_{0i}}{\partial x} - \hbar\frac{\partial\psi_{0r}}{\partial y} + \hbar\frac{\partial\psi_{1i}}{\partial z} &= 0 \\
-m\psi_{3i} + q\phi\psi_{1i} - qA_x\psi_{0i} - qA_y\psi_{0r} + qA_z\psi_{1i} + \hbar\frac{\partial\psi_{1r}}{\partial t} + \hbar\frac{\partial\psi_{0r}}{\partial x} - \hbar\frac{\partial\psi_{0i}}{\partial y} - \hbar\frac{\partial\psi_{1r}}{\partial z} &= 0
\end{aligned}
$$

The claim has been made that the Weyl matrices model massless fermions. I need to chase down the primary references to see if I accept such claims. The use of chiral projectors ψ_l and ψ_r is associated with Weyl matrices, but only requires a γ_5, and does not require Weyl matrices.

Majorana Matrices

Majorana matrices are used to describe fermions which are their own anti-particle. The Majorana representation is a five dimensional Clifford algebra with signature (+ - - - +). As such, it is compatible with the Bjorken and Drell space.

```
*********** Majorana Basis Matrices ************
      w              x              y              z              t
[ 0 -I  0  0]  [ I  0  0  0]  [ 0  0  0  I]  [ 0 -I  0  0]  [ 0  0  0 -I]
[ I  0  0  0]  [ 0 -I  0  0]  [ 0  0 -I  0]  [-I  0  0  0]  [ 0  0  I  0]
[ 0  0  0  I]  [ 0  0  I  0]  [ 0 -I  0  0]  [ 0  0  0 -I]  [ 0 -I  0  0]
[ 0  0 -I  0]  [ 0  0  0 -I]  [ I  0  0  0]  [ 0  0 -I  0]  [ I  0  0  0]
    Gamma5         Gamma1         Gamma2         Gamma3         Gamma0
```

The mapping between multivectors and matrices is different from Bjorken and Drell.

```
Generic MV = (a, b,c,d,e,f, g,h,j,k,l,m,n,p,r,s, S,R,P,N,M,L,K,J,H,G, F,E,D,C,B, A)

Matrix form for Generic MV =

[[+a-C-j-r+I*(+A+c-J+R),+B+E-g+m+I*(-b-e-G+M),-h+k+p+s+I*(+H+K-P+S),+D+F-l+n+I(+d-f+L+N)],
 [-B+E-g-m+I*(+b-e-G-M),+a+C+j-r+I*(+A-c+J+R),-D-F-l+n+I*(-d+f+L+N),-h+k-p-s+I(+H+K+P-S)],
 [-h-k-p+s+I*(+H-K+P+S),-D+F+l+n+I*(-d-f-L+N),+a-C+j+r+I*(+A+c+J-R),-B+E+g+m+I(+b-e+G+M)],
 [+D-F+l+n+I*(+d+f-L+N),-h-k+p-s+I*(+H-K-P-S),+B+E+g-m+I*(-b-e+G-M),+a+C-j+r+I(+A-c-J-R)]]
```

The Klein Gordon equation factors as

$$
\begin{aligned}
(E^2 - p_x^2 - p_y^2 - p_z^2 - m^2) &= 0 \\
((E\gamma_0 + p_x\gamma_1 + p_y\gamma_2 + p_z\gamma_3) + m) * ((E\gamma_0 + p_x\gamma_1 + p_y\gamma_2 + p_z\gamma_3) - m) &= 0
\end{aligned}
$$

This again matches Bjorken and Drell. The eight equations resulting from substituting operators in the left factor above, are

$$
\begin{aligned}
+m\psi_{0r} + q\phi\psi_{3i} - qA_x\psi_{0i} - qA_y\psi_{3i} + qA_z\psi_{1i} + \hbar\frac{\partial\psi_{3r}}{\partial t} + \hbar\frac{\partial\psi_{0r}}{\partial x} + \hbar\frac{\partial\psi_{3r}}{\partial y} - \hbar\frac{\partial\psi_{1r}}{\partial z} &= 0 \\
+m\psi_{0i} - q\phi\psi_{3r} + qA_x\psi_{0r} + qA_y\psi_{3r} - qA_z\psi_{1r} + \hbar\frac{\partial\psi_{3i}}{\partial t} + \hbar\frac{\partial\psi_{0i}}{\partial x} + \hbar\frac{\partial\psi_{3i}}{\partial y} - \hbar\frac{\partial\psi_{1i}}{\partial z} &= 0 \\
+m\psi_{1r} - q\phi\psi_{2i} + qA_x\psi_{1i} + qA_y\psi_{2i} + qA_z\psi_{0i} - \hbar\frac{\partial\psi_{2r}}{\partial t} - \hbar\frac{\partial\psi_{1r}}{\partial x} - \hbar\frac{\partial\psi_{2r}}{\partial y} - \hbar\frac{\partial\psi_{0r}}{\partial z} &= 0 \\
+m\psi_{1i} + q\phi\psi_{2r} - qA_x\psi_{1r} - qA_y\psi_{2r} - qA_z\psi_{0r} - \hbar\frac{\partial\psi_{2i}}{\partial t} - \hbar\frac{\partial\psi_{1i}}{\partial x} - \hbar\frac{\partial\psi_{2i}}{\partial y} - \hbar\frac{\partial\psi_{0i}}{\partial z} &= 0 \\
+m\psi_{2r} + q\phi\psi_{1i} - qA_x\psi_{2i} + qA_y\psi_{1i} + qA_z\psi_{3i} + \hbar\frac{\partial\psi_{1r}}{\partial t} + \hbar\frac{\partial\psi_{2r}}{\partial x} - \hbar\frac{\partial\psi_{1r}}{\partial y} - \hbar\frac{\partial\psi_{3r}}{\partial z} &= 0 \\
+m\psi_{2i} - q\phi\psi_{1r} + qA_x\psi_{2r} - qA_y\psi_{1r} - qA_z\psi_{3r} + \hbar\frac{\partial\psi_{1i}}{\partial t} + \hbar\frac{\partial\psi_{2i}}{\partial x} - \hbar\frac{\partial\psi_{1i}}{\partial y} - \hbar\frac{\partial\psi_{3i}}{\partial z} &= 0 \\
+m\psi_{3r} - q\phi\psi_{0i} + qA_x\psi_{3i} - qA_y\psi_{0i} + qA_z\psi_{2i} - \hbar\frac{\partial\psi_{0r}}{\partial t} - \hbar\frac{\partial\psi_{3r}}{\partial x} + \hbar\frac{\partial\psi_{0r}}{\partial y} - \hbar\frac{\partial\psi_{2r}}{\partial z} &= 0 \\
+m\psi_{3i} + q\phi\psi_{0r} - qA_x\psi_{3r} + qA_y\psi_{0r} - qA_z\psi_{2r} - \hbar\frac{\partial\psi_{0i}}{\partial t} - \hbar\frac{\partial\psi_{3i}}{\partial x} + \hbar\frac{\partial\psi_{0i}}{\partial y} - \hbar\frac{\partial\psi_{2i}}{\partial z} &= 0
\end{aligned}
$$

Comparing against the Bjorken and Drell equations, we see differences. I believe these difference are due to freedom of implementation choices, where the same geometric algebra signature can be satisfied by different matrix choices.

Perfect Square Factoring

My preferred implementation has wxyzt signature of (+ + + + -).

```
      q              wxyzt
[ 1  0  0  0]  [ I  0  0  0]
[ 0  1  0  0]  [ 0  I  0  0]
[ 0  0  1  0]  [ 0  0  I  0]
[ 0  0  0  1]  [ 0  0  0  I]

      w              x              y              z              t
[ 0  0  0 -I]  [ 1  0  0  0]  [ 0  1  0  0]  [ 0  0  0 -1]  [ 0 -1  0  0]
[ 0  0  I  0]  [ 0 -1  0  0]  [ 1  0  0  0]  [ 0  0  1  0]  [ 1  0  0  0]
[ 0 -I  0  0]  [ 0  0  1  0]  [ 0  0  0  1]  [ 0  1  0  0]  [ 0  0  0  1]
[ I  0  0  0]  [ 0  0  0 -1]  [ 0  0  1  0]  [-1  0  0  0]  [ 0  0 -1  0]

      wx             wy             wz             wt             xy
[ 0  0  0  I]  [ 0  0 -I  0]  [ I  0  0  0]  [ 0  0  I  0]  [ 0  1  0  0]
[ 0  0  I  0]  [ 0  0  0  I]  [ 0  I  0  0]  [ 0  0  0  I]  [-1  0  0  0]
[ 0  I  0  0]  [-I  0  0  0]  [ 0  0 -I  0]  [-I  0  0  0]  [ 0  0  0  1]
[ I  0  0  0]  [ 0  I  0  0]  [ 0  0  0 -I]  [ 0 -I  0  0]  [ 0  0 -1  0]

      xz             xt             yz             yt             zt
[ 0  0  0 -1]  [ 0 -1  0  0]  [ 0  0  1  0]  [ 1  0  0  0]  [ 0  0  1  0]
[ 0  0 -1  0]  [-1  0  0  0]  [ 0  0  0 -1]  [ 0 -1  0  0]  [ 0  0  0  1]
[ 0  1  0  0]  [ 0  0  0  1]  [-1  0  0  0]  [ 0  0 -1  0]  [ 1  0  0  0]
[ 1  0  0  0]  [ 0  0  1  0]  [ 0  1  0  0]  [ 0  0  0  1]  [ 0  1  0  0]

     wxy            wxz            wxt            wyz            wyt
[ 0  0  I  0]  [-I  0  0  0]  [ 0  0 -I  0]  [ 0 -I  0  0]  [ 0  0  0 -I]
[ 0  0  0  I]  [ 0  I  0  0]  [ 0  0  0  I]  [-I  0  0  0]  [ 0  0 -I  0]
[ I  0  0  0]  [ 0  0  I  0]  [ I  0  0  0]  [ 0  0  0  I]  [ 0  I  0  0]
[ 0  I  0  0]  [ 0  0  0 -I]  [ 0 -I  0  0]  [ 0  0  I  0]  [ I  0  0  0]

     wzt            xyz            xyt            xzt            yzt
[ 0 -I  0  0]  [ 0  0  1  0]  [ 1  0  0  0]  [ 0  0  1  0]  [ 0  0  0  1]
[ I  0  0  0]  [ 0  0  0  1]  [ 0  1  0  0]  [ 0  0  0 -1]  [ 0  0  1  0]
[ 0  0  0 -I]  [-1  0  0  0]  [ 0  0 -1  0]  [ 1  0  0  0]  [ 0  1  0  0]
[ 0  0  I  0]  [ 0 -1  0  0]  [ 0  0  0 -1]  [ 0 -1  0  0]  [ 1  0  0  0]

     wxyz           wxyt           wxzt           wyzt           xyzt
[ 0  I  0  0]  [ 0  0  0  I]  [ 0  I  0  0]  [-I  0  0  0]  [ 0  0  0  1]
[-I  0  0  0]  [ 0  0 -I  0]  [ I  0  0  0]  [ 0  I  0  0]  [ 0  0 -1  0]
[ 0  0  0 -I]  [ 0 -I  0  0]  [ 0  0  0  I]  [ 0  0 -I  0]  [ 0  1  0  0]
[ 0  0  I  0]  [ I  0  0  0]  [ 0  0  I  0]  [ 0  0  0  I]  [-1  0  0  0]
```

The generic Dirac space multivector can be written, with each grade on its own line, as

```
Dirac Spacetime =
+ a*q
+ b*w + c*x + d*y + e*z + f*t
+ g*wx + h*wy + j*wz + k*wt + l*xy + m*xz + n*xt + p*yz + r*yt + s*zt
+ S*wxy + R*wxz + P*wxt + N*wyz + M*wyt + L*wzt + K*xyz + J*xyt + H*xzt + G*yzt
+ F*wxyz + E*wxyt + D*wxzt + C*wyzt + B*xyzt
+ A*wxyzt
```

where the coefficients have been chosen to emphasize the dual structure of the five space multivector.

We can convert this to the equivalent 4x4 complex matrix representation.

```
Matrix format =
[(+a+c+J+r)+I(+A-C+j-R), (+d-f+l-n)+I(+D+F-L-N), (+H+K+p+s)+I(-h+k-P+S), (+B-e+G-m)+I(-b+E+g-M)],
[(+d+f-l-n)+I(+D-F+L-N), (+a-c+J-r)+I(+A+C+j+R), (-B+e+G-m)+I(+b-E+g-M), (-H+K-p+s)+I(+h+k+P+S)],
[(+H-K-p+s)+I(-h-k+P+S), (+B+e+G+m)+I(-b-E+g+M), (+a+c-J-r)+I(+A-C-j+R), (+d+f+l+n)+I(+D-F-L+N)],
[(-B-e+G+m)+I(+b+E+g+M), (-H-K+p+s)+I(+h-k-P+S), (+d-f-l+n)+I(+D+F+L+N), (+a-c-J+r)+I(+A+C-j-R)]
```

We can likewise convert from matrix to multivector

```
a = real( + W[0][0] + W[1][1] + W[2][2] + W[3][3])/4;
b = imag( - W[0][3] + W[1][2] - W[2][1] + W[3][0])/4;
c = real( + W[0][0] - W[1][1] + W[2][2] - W[3][3])/4;
d = real( + W[0][1] + W[1][0] + W[2][3] + W[3][2])/4;

e = real( - W[0][3] + W[1][2] + W[2][1] - W[3][0])/4;
f = real( - W[0][1] + W[1][0] + W[2][3] - W[3][2])/4;
g = imag( + W[0][3] + W[1][2] + W[2][1] + W[3][0])/4;
h = imag( - W[0][2] + W[1][3] - W[2][0] + W[3][1])/4;

j = imag( + W[0][0] + W[1][1] - W[2][2] - W[3][3])/4;
k = imag( + W[0][2] + W[1][3] - W[2][0] - W[3][1])/4;
l = real( + W[0][1] - W[1][0] + W[2][3] - W[3][2])/4;
m = real( - W[0][3] - W[1][2] + W[2][1] + W[3][0])/4;

n = real( - W[0][1] - W[1][0] + W[2][3] + W[3][2])/4;
p = real( + W[0][2] - W[1][3] - W[2][0] + W[3][1])/4;
r = real( + W[0][0] - W[1][1] - W[2][2] + W[3][3])/4;
s = real( + W[0][2] + W[1][3] + W[2][0] + W[3][1])/4;

S = imag( + W[0][2] + W[1][3] + W[2][0] + W[3][1])/4;
R = imag( - W[0][0] + W[1][1] + W[2][2] - W[3][3])/4;
P = imag( - W[0][2] + W[1][3] + W[2][0] - W[3][1])/4;
N = imag( - W[0][1] - W[1][0] + W[2][3] + W[3][2])/4;
```

```
M = imag( - W[0][3] - W[1][2] + W[2][1] + W[3][0])/4;
L = imag( - W[0][1] + W[1][0] - W[2][3] + W[3][2])/4;
K = real( + W[0][2] + W[1][3] - W[2][0] - W[3][1])/4;
J = real( + W[0][0] + W[1][1] - W[2][2] - W[3][3])/4;

H = real( + W[0][2] - W[1][3] + W[2][0] - W[3][1])/4;
G = real( + W[0][3] + W[1][2] + W[2][1] + W[3][0])/4;
F = imag( + W[0][1] - W[1][0] - W[2][3] + W[3][2])/4;
E = imag( + W[0][3] - W[1][2] - W[2][1] + W[3][0])/4;

D = imag( + W[0][1] + W[1][0] + W[2][3] + W[3][2])/4;
C = imag( - W[0][0] + W[1][1] - W[2][2] + W[3][3])/4;
B = real( + W[0][3] - W[1][2] + W[2][1] - W[3][0])/4;
A = imag( + W[0][0] + W[1][1] + W[2][2] + W[3][3])/4;
```

We can map column vectors to multivectors.

$$
\begin{aligned}
4 * \psi \;=\; & +\psi_{0r} \\
& +\psi_{3i}e_w + \psi_{0r}e_x + \psi_{1r}e_y - \psi_{3r}e_z + \psi_{1r}e_t \\
& +\psi_{3i}e_{wx} - \psi_{2i}e_{wy} + \psi_{0i}e_{wz} - \psi_{2i}e_{wt} - \psi_{1r}e_{xy} \\
& +\psi_{3r}e_{xz} - \psi_{1r}e_{xt} - \psi_{2r}e_{yz} + \psi_{0r}e_{yt} + \psi_{2r}e_{zt} \\
& +\psi_{2i}e_{wxy} - \psi_{0i}e_{wxz} + \psi_{2i}e_{wxt} - \psi_{1i}e_{wyz} + \psi_{3i}e_{wyt} \\
& +\psi_{1i}e_{wzt} - \psi_{2r}e_{xyz} + \psi_{0r}e_{xyt} + \psi_{2r}e_{xzt} + \psi_{3r}e_{yzt} \\
& -\psi_{1i}e_{wxyz} + \psi_{3i}e_{wxyt} + \psi_{1i}e_{wxzt} - \psi_{0i}e_{wyzt} - \psi_{3r}e_{xyzt} \\
& +\psi_{0i}e_{wxyzt}
\end{aligned}
$$

The relativistic energy-momentum equation is

$$-E^2 + p_x^2 + p_y^2 + p_z^2 + m^2 \;=\; 0 \quad \text{Original Equation}$$

Rather than using Dirac's scalar plus multivector factoring, I want to start with a simple square form.

$$(Ee_t + p_xe_x + p_ye_y + p_ze_z + me_w)^2 = -E^2 + p_x^2 + p_y^2 + p_z^2 + m^2 = 0$$

Our geometric, linear form is

$$Ee_t + p_xe_x + p_ye_y + p_ze_z + me_w = 0$$

We substitute operators for energy and momentum

$$\begin{aligned} E &= i\hbar\frac{\partial}{\partial t} + q\phi \\ \vec{p} &= -i\hbar\vec{\nabla} + q\mathbf{A} \end{aligned}$$

$$p_x = -i\hbar\frac{\partial}{\partial x} + qA_x \quad p_y = -i\hbar\frac{\partial}{\partial y} + qA_y \quad p_z = -i\hbar\frac{\partial}{\partial z} + qA_z$$

We obtain

$$\left(e_t\left(i\hbar\frac{\partial}{\partial t} + q\phi\right) - e_x\left(i\hbar\frac{\partial}{\partial x} - qA_x\right) - e_y\left(i\hbar\frac{\partial}{\partial y} - qA_y\right) - e_z\left(i\hbar\frac{\partial}{\partial z} - qA_z\right) + e_w m\right)\psi = 0$$

Substituting $i = e_{wxyzt}$, resolving basis products, and grouping by basis, we obtain the re-annotated Dirac equation.

$$\begin{aligned} &\left(+e_{wyzt}\hbar\frac{\partial\psi}{\partial x} - e_{wxzt}\hbar\frac{\partial\psi}{\partial y} + e_{wxyt}\hbar\frac{\partial\psi}{\partial z} + e_{wxyz}\hbar\frac{\partial\psi}{\partial t}\right) + \\ &(+e_w m + e_x qA_x + e_y qA_y + e_z qA_z + e_t q\phi)\,\psi = 0 \end{aligned}$$

This leads to set of eight real equations.

$$\begin{aligned}
+m\psi_{0r} - q\phi\psi_{2i} - qA_x\psi_{3i} + qA_y\psi_{2i} - qA_z\psi_{0i} - \hbar\frac{\partial\psi_{2r}}{\partial t} + \hbar\frac{\partial\psi_{3r}}{\partial x} - \hbar\frac{\partial\psi_{2r}}{\partial y} + \hbar\frac{\partial\psi_{0r}}{\partial z} &= 0 \\
+m\psi_{0i} + q\phi\psi_{2r} + qA_x\psi_{3r} - qA_y\psi_{2r} + qA_z\psi_{0r} - \hbar\frac{\partial\psi_{2i}}{\partial t} + \hbar\frac{\partial\psi_{3i}}{\partial x} - \hbar\frac{\partial\psi_{2i}}{\partial y} + \hbar\frac{\partial\psi_{0i}}{\partial z} &= 0 \\
+m\psi_{1r} - q\phi\psi_{3i} - qA_x\psi_{2i} - qA_y\psi_{3i} - qA_z\psi_{1i} - \hbar\frac{\partial\psi_{3r}}{\partial t} + \hbar\frac{\partial\psi_{2r}}{\partial x} + \hbar\frac{\partial\psi_{3r}}{\partial y} + \hbar\frac{\partial\psi_{1r}}{\partial z} &= 0 \\
+m\psi_{1i} + q\phi\psi_{3r} + qA_x\psi_{2r} + qA_y\psi_{3r} + qA_z\psi_{1r} - \hbar\frac{\partial\psi_{3i}}{\partial t} + \hbar\frac{\partial\psi_{2i}}{\partial x} + \hbar\frac{\partial\psi_{3i}}{\partial y} + \hbar\frac{\partial\psi_{1i}}{\partial z} &= 0 \\
+m\psi_{2r} + q\phi\psi_{0i} - qA_x\psi_{1i} + qA_z\psi_{2i} + qA_y\psi_{0i} + \hbar\frac{\partial\psi_{0r}}{\partial t} + \hbar\frac{\partial\psi_{1r}}{\partial x} - \hbar\frac{\partial\psi_{0r}}{\partial y} - \hbar\frac{\partial\psi_{2r}}{\partial z} &= 0 \\
+m\psi_{2i} - q\phi\psi_{0r} + qA_x\psi_{1r} - qA_y\psi_{0r} - qA_z\psi_{2r} + \hbar\frac{\partial\psi_{0i}}{\partial t} + \hbar\frac{\partial\psi_{1i}}{\partial x} - \hbar\frac{\partial\psi_{0i}}{\partial y} - \hbar\frac{\partial\psi_{2i}}{\partial z} &= 0 \\
+m\psi_{3r} + q\phi\psi_{1i} - qA_x\psi_{0i} - qA_y\psi_{1i} + qA_z\psi_{3i} + \hbar\frac{\partial\psi_{1r}}{\partial t} + \hbar\frac{\partial\psi_{0r}}{\partial x} + \hbar\frac{\partial\psi_{1r}}{\partial y} - \hbar\frac{\partial\psi_{3r}}{\partial z} &= 0 \\
+m\psi_{3i} - q\phi\psi_{1r} + qA_x\psi_{0r} + qA_y\psi_{1r} - qA_z\psi_{3r} + \hbar\frac{\partial\psi_{1i}}{\partial t} + \hbar\frac{\partial\psi_{0i}}{\partial x} + \hbar\frac{\partial\psi_{1i}}{\partial y} - \hbar\frac{\partial\psi_{3i}}{\partial z} &= 0
\end{aligned}$$

The details of these equations differ from the standard Dirac format.

Discussion of These Variants

At a top level, each of these variants has included products of equations, where I select one of the two factors. What is the other factor? In the Bjorken-Drell factorization, the second factor seems to be the positron solution, which corresponds to a well-known particle. In the case of the Dirac alpha matrix implementation, this seems to correspond to negative energy solutions, which are again, anti-matter.

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