The Dirac Equation and Geometric Algebra

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January 8, 2016

Abstract

David Hestenes [5] , Chris Doran [4], Anthony Lasenby, and others have implemented the Dirac equation for the the relativistic electron in geometric algebra. David Hestenes, in particular, has shown how the geometric algebra view provides insight into electron motion.

This note starts with Dirac's development, then repeats the geometric algebra translation using a (-,-,-,+) signature. Finally, the same translation is carried out using a (+,+,+,-) signature. The result is a purely real multivector Dirac Equation with no imaginaries.

Classical Mechanics to Quantum Mechanics

In quantum mechanics, energy and momentum become operators acting on a wavefunction ψ

$$\begin{aligned} E &= i\hbar\frac{\partial}{\partial t} \\ \vec{p} &= -i\hbar\vec{\nabla} \end{aligned}$$

$$p_x = -i\hbar\frac{\partial}{\partial x} \quad p_y = -i\hbar\frac{\partial}{\partial y} \quad p_z = -i\hbar\frac{\partial}{\partial z}$$

Using the classical mechanics relationship between energy and momentum, we obtain the Schrodinger equation.

$$\begin{aligned} E &= \frac{p^2}{2m} + V \\ i\hbar\frac{\partial\psi}{\partial t} &= -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \end{aligned}$$

The Schrodinger equation did not provide electron spin, nor was the equation relativistically invariant. Pauli extended the Schrodinger equation to include spin, but the model was still non-relativistic.

Schrodinger, Gordon and Klein developed an approach which used the relativistic energy momentum equation and the quantum operators as a model. The resulting Klein-Gordon equation, using ϕ instead of ψ, is

$$\begin{aligned} E^2 - c^2p^2 &= m^2c^4 \\ -\hbar^2\frac{\partial^2\phi}{\partial t^2} + c^2\hbar^2\nabla^2\phi &= m^2c^4\phi \\ \frac{\partial^2\phi}{c^2\partial t^2} - \nabla^2\phi + \frac{m^2c^2}{\hbar^2}\phi &= 0 \end{aligned}$$

This equation, when ϕ is interpreted as probability density rather than charge density, suffered the defects of negative values for probability and energy, and was dismissed as unsatisfactory for describing the electron by Schrodinger and Dirac.

Dirac expected time and space would appear in relativistic quantum mechanics as first order linear differential operators. The Klein-Gordon equation, being second order, was not the correct form. Schrodinger's equation, in a sense, is a linear approximation to a square root of the Klein-Gordon equation, but is not relativistic. The correct approach, Dirac reasoned, would be to write the Klein-Gordon equation as the square of the correct linear operator form. This linear form, rather than the Klein-Gordon equation, would then be the correct basis for relativistic quantum mechanics.

The prototype, using the energy momentum operator expression, for free space is

$$\begin{aligned} E^2 - c^2p^2 &= m^2c^4 \\ \frac{E^2}{c^2} - p^2 &= m^2c^2 \end{aligned}$$

By measuring time in meters, we reduce clutter by setting $c = 1$. We also write our momentum square term as the sum of the components in cartesian coordinates.

$$E^2 - p_x^2 - p_y^2 - p_z^2 = m^2$$

Dirac now introduced four scale factors for our four spacetime components, and developed the square.

$$(AE + Bp_x + Cp_y + Dp_z)^2 = E^2 - p_x^2 - p_y^2 - p_z^2 = m^2$$

Knowing that rotations in three dimensional space and higher are order sensitive, Dirac maintained the order of products as he developed the square.

$$\begin{aligned}
(AE + Bp_x + Cp_y + Dp_z)^2 &= AEAE + AEBp_x + AECp_y + AEDP_z \\
&\quad + Bp_xAE + Bp_xBp_x + Bp_xCp_y + Bp_xDp_z \\
&\quad + Cp_yAE + Cp_yBp_x + Cp_yCp_y + Cp_yDp_z \\
&\quad + Dp_zAE + Dp_zBp_x + Dp_zCp_y + Dp_zDp_z \\
(AE + Bp_x + Cp_y + Dp_z)^2 &= AEAE + Bp_xBp_x + Cp_yCp_y + Dp_zDp_z \\
&\quad + (AEBp_x + Bp_xAE) + (AECp_y + Cp_yAE) \\
&\quad + (AEDp_z + Dp_zAE) + (Bp_xCp_y + Cp_yBp_x) \\
&\quad + (Bp_xDp_z + Dp_zBp_x) + (Cp_yDp_z + Dp_zCp_y)
\end{aligned}$$

In the expression above, E, p_x, p_y and p_z are pure numbers, and commute with all basis. However, A, B, C and D are directional elements. Moving scalar terms to the front of each expression, Dirac wrote

$$\begin{aligned}
(AE + Bp_x + Cp_y + Dp_z)^2 &= E^2AA + p_x^2BB + p_y^2CC + p_z^2DD \\
&\quad + Ep_x(AB + BA) + Ep_y(AC + CA) \\
&\quad + Ep_z(AD + DA) + p_xp_y(BC + CB) \\
&\quad + p_xp_z(BD + DB) + p_yp_z(CD + DC)
\end{aligned}$$

To achieve his square, Dirac required

$$\begin{aligned}
AA &= 1 \\
BB &= -1 \\
CC &= -1 \\
DD &= -1 \\
AB &= -BA \\
AC &= -CA \\
AD &= -DA \\
BC &= -CB \\
BD &= -DB \\
CD &= -DC
\end{aligned}$$

Dirac Matrices

Dirac, at this point, assigned matrices to the quantities A through D, without speculating on the meaning of these terms.

$$
\begin{aligned}
A = \gamma^0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
B = \gamma^1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\
C = \gamma^2 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \\
D = \gamma^3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

Using the gamma notation, the Klein Gordon, then Dirac equations can be written

$$
\begin{aligned}
\left(\gamma^0 E + \gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z\right)^2 &= m^2 \\
\left(\gamma^0 E + \gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z\right) &= \pm m
\end{aligned}
$$

Implementing the quantum operators, we have the free space Dirac equation.

$$
\left[\gamma^0 \left(i\hbar \frac{\partial}{\partial t}\right) + \gamma^1 \left(-i\hbar \frac{\partial}{\partial x}\right) + \gamma^2 \left(-i\hbar \frac{\partial}{\partial y}\right) + \gamma^3 \left(-i\hbar \frac{\partial}{\partial z}\right)\right] \psi = \pm m\psi
$$

Matrix and Component Level Equations

At this point, I like to write out the matrix form, then component level form for the Dirac equation.

$$i\hbar \begin{pmatrix} \left(\frac{\partial}{\partial t}\right) & (0) & \left(-\frac{\partial}{\partial z}\right) & \left(-\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \\ (0) & \left(\frac{\partial}{\partial t}\right) & \left(-\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) & \left(\frac{\partial}{\partial z}\right) \\ \left(\frac{\partial}{\partial z}\right) & \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) & \left(-\frac{\partial}{\partial t}\right) & (0) \\ \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) & \left(-\frac{\partial}{\partial z}\right) & (0) & \left(-\frac{\partial}{\partial t}\right) \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \pm m \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

The straight forward equation transcription is

$$\begin{aligned}
i\hbar\left(\frac{\partial\psi_0}{\partial t} - \frac{\partial\psi_2}{\partial z} - \frac{\partial\psi_3}{\partial x} + i\frac{\partial\psi_3}{\partial y}\right) &= \pm m\psi_0 \\
i\hbar\left(\frac{\partial\psi_1}{\partial t} - \frac{\partial\psi_2}{\partial x} - i\frac{\partial\psi_2}{\partial y} + \frac{\partial\psi_3}{\partial z}\right) &= \pm m\psi_1 \\
i\hbar\left(\frac{\partial\psi_0}{\partial z} + \frac{\partial\psi_1}{\partial x} - i\frac{\partial\psi_1}{\partial y} - \frac{\partial\psi_2}{\partial t}\right) &= \pm m\psi_2 \\
i\hbar\left(\frac{\partial\psi_0}{\partial x} + i\frac{\partial\psi_0}{\partial y} - \frac{\partial\psi_1}{\partial z} - \frac{\partial\psi_3}{\partial t}\right) &= \pm m\psi_3
\end{aligned}$$

Re-ordering terms in sequence $txyz$ of partial derivaties, we have

$$\begin{aligned}
i\hbar\left(\frac{\partial\psi_0}{\partial t} - \frac{\partial\psi_3}{\partial x} + i\frac{\partial\psi_3}{\partial y} - \frac{\partial\psi_2}{\partial z}\right) &= \pm m\psi_0 \\
i\hbar\left(\frac{\partial\psi_1}{\partial t} - \frac{\partial\psi_2}{\partial x} - i\frac{\partial\psi_2}{\partial y} + \frac{\partial\psi_3}{\partial z}\right) &= \pm m\psi_1 \\
i\hbar\left(-\frac{\partial\psi_2}{\partial t} + \frac{\partial\psi_1}{\partial x} - i\frac{\partial\psi_1}{\partial y} + \frac{\partial\psi_0}{\partial z}\right) &= \pm m\psi_2 \\
i\hbar\left(-\frac{\partial\psi_3}{\partial t} + \frac{\partial\psi_0}{\partial x} + i\frac{\partial\psi_0}{\partial y} - \frac{\partial\psi_1}{\partial z}\right) &= \pm m\psi_3
\end{aligned}$$

The imaginary terms in γ^2 and $i\partial/\partial y$ have always annoyed me. They seem caprious at best, or wrong with other compensating errors at worst. In reality, this seems to be just one of many workable representations. I believe I have a better representation shown in a later section.

Minkowski Space-Time Algebra (-,-,-,+)

Ignoring the matrix representation, and looking purely at the Dirac squaring requirements, David Hestenes and company see a four dimensional Minkowsli Space Time Algebra with signature (-,-,-,+), where time has a positive signature, and the three space axii have a negative signature. The wavefunction ψ, traditionally a four component complex vector, maps to the even rank components of the space time algebra, per Doran [4], page 282. Doran makes the mapping explicit in [6], page 102.

The beauty of this assignment is complete compatiblity with existing quantum mechanical literature. The algebra is the same, the symbols are the same. All that has changed is a replacement of a matrix implementation by a geometric algebra implementation, which allows a more geometrical investigation of quantum models. This approach reduces the likelihood of immediate rejection by scholars invested in standard quantum notation.

Minkowski Space-Time Algebra (+,+,+,-)

I, personally, have no strong attachment to the established quantum notation. Given freedom to choose notation, I greatly prefer the (+,+,+,-) metric for two reasons. First, standard three dimensional Euclidean space has a metric of (+,+,+). I prefer to carry this signature into spacetime, and allow the negative metric to be assigned to the time axis. The second reason deals with the nature of 4x4 matrix representations of geometric algebra. I have a set of sixteen real matrices, which orthogonally span the 4x4 matrix space, which are twelve way isomorphic to Minkowsi algebra with the (+,+,+,-) signature. However, I have not found a set of 4x4 matrices for (-,-,-,+) signature or (+,+,+,+) signature. This strongly encourages me to use the (+,+,+,-) signature.

The use of the classic Minkowski signature is going to result in a change of notation for quantum mechanics. This will cause resistance in the established notation community. However, in this case, I believe change is good.

Start by revisiting the free space, no EM field Dirac equation. We slide the i from the operator to be a factor associated with the γ basis.

$$\begin{aligned}
\left[\gamma^0\left(i\hbar\frac{\partial}{\partial t}\right)+\gamma^1\left(-i\hbar\frac{\partial}{\partial x}\right)+\gamma^2\left(-i\hbar\frac{\partial}{\partial y}\right)+\gamma^3\left(-i\hbar\frac{\partial}{\partial z}\right)\right]\psi &= \pm m\psi \\
\left[\left(i\gamma^0\right)\left(\hbar\frac{\partial}{\partial t}\right)+\left(i\gamma^1\right)\left(-\hbar\frac{\partial}{\partial x}\right)+\left(i\gamma^2\right)\left(-\hbar\frac{\partial}{\partial y}\right)+\left(i\gamma^3\right)\left(-\hbar\frac{\partial}{\partial z}\right)\right]\psi &= \pm m\psi
\end{aligned}$$

In one stroke, we have accomplished two great things. We have defined a set of basis vectors which have the desired (+,+,+,-) signature, and we have eliminated a spurious imaginary from the Dirac equation. I am sure my notation will change over time, but I currently suggest using upper case Γ for the new system of basis.

$$\begin{aligned}
\Gamma^0 &= i\gamma^0 \\
\Gamma^1 &= i\gamma^1 \\
\Gamma^2 &= i\gamma^2 \\
\Gamma^3 &= i\gamma^3
\end{aligned}$$

Using this notation, we can re-arrange the Dirac equation a bit.

$$\begin{aligned}
\left[\gamma^0\left(i\hbar\frac{\partial}{\partial t}\right)+\gamma^1\left(-i\hbar\frac{\partial}{\partial x}\right)+\gamma^2\left(-i\hbar\frac{\partial}{\partial y}\right)+\gamma^3\left(-i\hbar\frac{\partial}{\partial z}\right)\right]\psi &= \pm m\psi \\
\left[\Gamma^0\left(\hbar\frac{\partial}{\partial t}\right)+\Gamma^1\left(-\hbar\frac{\partial}{\partial x}\right)+\Gamma^2\left(-\hbar\frac{\partial}{\partial y}\right)+\Gamma^3\left(-\hbar\frac{\partial}{\partial z}\right)\right]\psi &= \pm m\psi \\
\left[\Gamma^0\left(\frac{\partial}{\partial t}\right)+\Gamma^1\left(-\frac{\partial}{\partial x}\right)+\Gamma^2\left(-\frac{\partial}{\partial y}\right)+\Gamma^3\left(-\frac{\partial}{\partial z}\right)\right]\psi &= \pm\frac{m}{\hbar}\psi
\end{aligned}$$

This last equation is a real multivector version of the Dirac equation, showing significant resemblance to a continuity equation in four space.

Explicit Γ Matrices

As mentioned earlier, I have twelve implementations of the Minkowski geometric algebra using 4x4 real matrices. One implementation of these matrices is shown below. I have not formed opinions on the canonical ordering of the terms in the bivector, trivector and quadvector products. The details of sign for these matrices vary depending upon factor order. As long as one is consistent, results of calculations should match despite convention differences.

Nicely Typeset Gamma Matrices

$$e_x = \Gamma^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad e_y = \Gamma^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad e_z = \Gamma^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$e_t = \Gamma^0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

The higher order multivector basis are formed by order sensitive matrix multiplication. The resulting sixteen matrices, shown later, form a complete set for 4x4 real matrices. For any 4x4 matrix, I can collect the set of sixteen scale factors for each multivector component by dotting each basis to the matrix of interest, and dividing by four. The full list of sixteen matrices with multiplication table is shown on the next page.

Matrix and Component Level Equations in New Basis

We now write out the matrix and component equations for this new notation.

$$\left[\Gamma^0\left(\frac{\partial}{\partial t}\right) + \Gamma^1\left(-\frac{\partial}{\partial x}\right) + \Gamma^2\left(-\frac{\partial}{\partial y}\right) + \Gamma^3\left(-\frac{\partial}{\partial z}\right)\right]\psi = \pm\frac{m}{\hbar}\psi$$

$$\begin{pmatrix} \left(\frac{\partial}{\partial y}\right) & \left(-\frac{\partial}{\partial x}\right) & (0) & \left(-\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right) \\ \left(-\frac{\partial}{\partial x}\right) & \left(-\frac{\partial}{\partial y}\right) & \left(-\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right) & (0) \\ (0) & \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right) & \left(\frac{\partial}{\partial y}\right) & \left(\frac{\partial}{\partial x}\right) \\ \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right) & (0) & \left(\frac{\partial}{\partial x}\right) & \left(-\frac{\partial}{\partial y}\right) \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \pm\frac{m}{\hbar} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

$$\begin{aligned} -\frac{\partial\psi_3}{\partial t} - \frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_0}{\partial y} - \frac{\partial\psi_3}{\partial z} &= \pm\frac{m}{\hbar}\psi_0 \\ -\frac{\partial\psi_2}{\partial t} - \frac{\partial\psi_0}{\partial x} - \frac{\partial\psi_1}{\partial y} - \frac{\partial\psi_2}{\partial z} &= \pm\frac{m}{\hbar}\psi_1 \\ \frac{\partial\psi_1}{\partial t} + \frac{\partial\psi_3}{\partial x} + \frac{\partial\psi_2}{\partial y} - \frac{\partial\psi_1}{\partial z} &= \pm\frac{m}{\hbar}\psi_2 \\ \frac{\partial\psi_0}{\partial t} + \frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_3}{\partial y} - \frac{\partial\psi_0}{\partial z} &= \pm\frac{m}{\hbar}\psi_3 \end{aligned}$$

Implementation #1

	(P1)	(+a)	(+b)	(+c)	(+E)	(+A)	(-B)	(+C)	(-g)	(-h)	(-i)	(+D)	(-f)	(+e)	(-d)	(-F)
(P1)	(P1)	(+a)	(+b)	(+c)	(+E)	(+A)	(-B)	(+C)	(-g)	(-h)	(-i)	(+D)	(-f)	(+e)	(-d)	(-F)
(+a)	(+a)	(P1)	(+A)	(-B)	(-g)	(+b)	(+c)	(+D)	(+E)	(-f)	(+e)	(+C)	(-h)	(-i)	(-F)	(-d)
(+b)	(+b)	(-A)	(P1)	(+C)	(-h)	(-a)	(-D)	(+c)	(+f)	(+E)	(-d)	(+B)	(+g)	(+F)	(-i)	(-e)
(+c)	(+c)	(+B)	(-C)	(P1)	(-i)	(+D)	(-a)	(-b)	(-e)	(+d)	(+E)	(+A)	(-F)	(+g)	(+h)	(-f)
(+E)	(+E)	(+g)	(+h)	(+i)	(M1)	(-f)	(+e)	(-d)	(+a)	(+b)	(+c)	(+F)	(-A)	(+B)	(-C)	(+D)
(+A)	(+A)	(-b)	(+a)	(+D)	(-f)	(M1)	(-C)	(-B)	(+h)	(-g)	(-F)	(-c)	(-E)	(+d)	(+e)	(+i)
(-B)	(-B)	(-c)	(-D)	(+a)	(+e)	(+C)	(M1)	(-A)	(+i)	(+F)	(-g)	(+b)	(-d)	(-E)	(+f)	(-h)
(+C)	(+C)	(+D)	(-c)	(+b)	(-d)	(+B)	(+A)	(M1)	(-F)	(+i)	(-h)	(-a)	(-e)	(-f)	(-E)	(+g)
(-g)	(-g)	(-E)	(+f)	(-e)	(-a)	(-h)	(-i)	(-F)	(P1)	(+A)	(-B)	(+d)	(-b)	(-c)	(-D)	(+C)
(-h)	(-h)	(-f)	(-E)	(+d)	(-b)	(+g)	(+F)	(-i)	(-A)	(P1)	(+C)	(+e)	(+a)	(+D)	(-c)	(+B)
(-i)	(-i)	(+e)	(-d)	(-E)	(-c)	(-F)	(+g)	(+h)	(+B)	(-C)	(P1)	(+f)	(-D)	(+a)	(+b)	(+A)
(+D)	(+D)	(+C)	(+B)	(+A)	(-F)	(-c)	(+b)	(-a)	(-d)	(-e)	(-f)	(M1)	(+i)	(-h)	(+g)	(-E)
(-f)	(-f)	(-h)	(+g)	(+F)	(-A)	(-E)	(+d)	(+e)	(-b)	(+a)	(+D)	(-i)	(P1)	(+C)	(+B)	(-c)
(+e)	(+e)	(-i)	(-F)	(+g)	(+B)	(-d)	(-E)	(+f)	(-c)	(-D)	(+a)	(+h)	(-C)	(P1)	(+A)	(+b)
(-d)	(-d)	(+F)	(-i)	(+h)	(-C)	(-e)	(-f)	(-E)	(+D)	(-c)	(+b)	(-g)	(-B)	(-A)	(P1)	(-a)
(-F)	(-F)	(+d)	(+e)	(+f)	(-D)	(+i)	(-h)	(+g)	(+C)	(+B)	(+A)	(+E)	(+c)	(-b)	(+a)	(M1)

```
    Unity              xyzt
[ 1  0  0  0]       [ 0  1  0  0]
[ 0  1  0  0]       [-1  0  0  0]
[ 0  0  1  0]       [ 0  0  0  1]
[ 0  0  0  1]       [ 0  0 -1  0]

      x             y             z             t
[ 0  1  0  0] [-1  0  0  0] [ 0  0  0  1] [ 0  0  0 -1]
[ 1  0  0  0] [ 0  1  0  0] [ 0  0  1  0] [ 0  0 -1  0]
[ 0  0  0 -1] [ 0  0 -1  0] [ 0  1  0  0] [ 0  1  0  0]
[ 0  0 -1  0] [ 0  0  0  1] [ 1  0  0  0] [ 1  0  0  0]

     xy            xz            yz            xt            yt            zt
[ 0  1  0  0] [ 0  0  1  0] [ 0  0  0 -1] [ 0  0 -1  0] [ 0  0  0  1] [ 1  0  0  0]
[-1  0  0  0] [ 0  0  0  1] [ 0  0  1  0] [ 0  0  0 -1] [ 0  0 -1  0] [ 0  1  0  0]
[ 0  0  0 -1] [-1  0  0  0] [ 0 -1  0  0] [-1  0  0  0] [ 0 -1  0  0] [ 0  0 -1  0]
[ 0  0  1  0] [ 0 -1  0  0] [ 1  0  0  0] [ 0 -1  0  0] [ 1  0  0  0] [ 0  0  0 -1]

    xyz           xyt           xzt           yzt
[ 0  0  1  0] [ 0  0 -1  0] [ 0  1  0  0] [-1  0  0  0]
[ 0  0  0 -1] [ 0  0  0  1] [ 1  0  0  0] [ 0  1  0  0]
[-1  0  0  0] [-1  0  0  0] [ 0  0  0  1] [ 0  0  1  0]
[ 0  1  0  0] [ 0  1  0  0] [ 0  0  1  0] [ 0  0  0 -1]
```

The new set of basis does not have the caprious i terms associated with the conventional Dirac equation. However, I am still annoyed by the zero terms and doubled terms in this implementation. Similar features are in all twelve implementations.

In this real matrix form, in the absense of electromagnetic fields, there is no motivation for ψ to have complex values. We will shortly see that we require fields to get the complex format, and that the fourspace pseudoscalar plays the role of i, changing our fourvector to a four component spinor.

Dirac Equation With Electromagnetic Fields

The previous sections dealt with the Dirac equation in the absense of electromagnetic fields. This is, of course, the simplest case, and a good starting point. Now, we extend our energy and momentum terms to include electromagnetic fields ϕ and $\mathbf{A}$.

Our canonical momentum becomes

$$\begin{aligned}\vec{p} &= q\vec{A} - i\hbar\vec{\nabla} \\ p_x &= qA_x - i\hbar\frac{\partial}{\partial x} \\ p_y &= qA_y - i\hbar\frac{\partial}{\partial y} \\ p_z &= qA_z - i\hbar\frac{\partial}{\partial z}\end{aligned}$$

Our canonical energy becomes

$$E = -q\phi + i\hbar\frac{\partial}{\partial t}$$

Inserting these into the Dirac equation yields

$$\left(\gamma^0 E + \gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z\right)\psi = \pm m\psi$$

$$\left(\gamma^0\left(-q\phi + i\hbar\frac{\partial}{\partial t}\right) + \gamma^1\left(qA_x - i\hbar\frac{\partial}{\partial x}\right) + \gamma^2\left(qA_y - i\hbar\frac{\partial}{\partial y}\right) + \gamma^3\left(qA_z - i\hbar\frac{\partial}{\partial z}\right)\right)\psi = \pm m\psi$$

$$i\hbar\left(+\gamma^0\frac{\partial}{\partial t} - \gamma^1\frac{\partial}{\partial x} - \gamma^2\frac{\partial}{\partial y} - \gamma^3\frac{\partial}{\partial z}\right)\psi = \pm m\psi + \left(\gamma^0 q\phi - \gamma^1 qA_x - \gamma^2 qA_y - \gamma^3 qA_z\right)\psi$$

Shifting to our Γ matrices, we have

$$\hbar\left(+\Gamma^0\frac{\partial}{\partial t}-\Gamma^1\frac{\partial}{\partial x}-\Gamma^2\frac{\partial}{\partial y}-\Gamma^3\frac{\partial}{\partial z}\right)\psi \;=\; \pm m\psi - i\left(\Gamma^0 q\phi - \Gamma^1 qA_x - \Gamma^2 qA_y - \Gamma^3 qA_z\right)\psi$$

In Minkowski geometric algebra, the pseudoscalar corresponds to the product $\Gamma^0\Gamma^1\Gamma^2\Gamma^3$. This term squares to minus one, but anti-commutes with Γ^0, Γ^1, Γ^2, and Γ^3. Replacing i with $\Gamma^0\Gamma^1\Gamma^2\Gamma^3$ poses a concern about the order of multiplication, whether the i is a prefactor or a postfactor. The ambiguity of i as a prefactor or postfactor does give us a sign choice we need to later resolve. Consequently, I place another $\pm$ on the front of the i term, independent of the $\pm$ associated with the mass term. Personal opinion, I think this sign choice will be non-observable, but I need to flag this topic here for later review.

$$\begin{aligned}\hbar\left(+\Gamma^0\frac{\partial}{\partial t}-\Gamma^1\frac{\partial}{\partial x}-\Gamma^2\frac{\partial}{\partial y}-\Gamma^3\frac{\partial}{\partial z}\right)\psi &= \pm m\psi - i\left(\Gamma^0 q\phi - \Gamma^1 qA_x - \Gamma^2 qA_y - \Gamma^3 qA_z\right)\psi \\ &= \pm m\psi \pm \Gamma^0\Gamma^1\Gamma^2\Gamma^3\left(\Gamma^0 q\phi - \Gamma^1 qA_x - \Gamma^2 qA_y - \Gamma^3 qA_z\right)\psi \\ &= \pm m\psi \pm \left(\Gamma^1\Gamma^2\Gamma^3 q\phi - \Gamma^0\Gamma^2\Gamma^3 qA_x - \Gamma^0\Gamma^3\Gamma^1 qA_y - \Gamma^0\Gamma^1\Gamma^2 qA_z\right)\psi\end{aligned}$$

In this interpretation, the mass term will be multiplied by a unit matrix. We now have a purely real, 4x4 multivector format for the Dirac equation with electromagnetic fields. I will revise this shortly to show the explicit matrix and component level equations.

Extending Wavefunction to Full Multivector

As has been remarked earlier, there are only eight components in the standard Dirac wavefunction, while this multivector approach is delighted to support sixteen. Rather than represent the wavefunction as a four component spinor, I am eager to promote the wavefunction to full multivector status using real 4x4 matrices. I believe the even rank multivector components will give us the standard Dirac electromagnetic interactions. I am hopeful that the odd components will cheerfully present the electroweak interactions.

Conclusion

I find the Dirac equation can be implemented as a real multivector equation using a Minkowski (+,+,+,-) metric.

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