The Three Dimensional Dirac Equation

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Three Dimensional Geometric Algebra

Three dimensional geometric algebra has eight multivector components (same number as Dirac wavefunction), and a pseudoscalar which mimics i. I would like to find a mapping of the Dirac equation to a three-space multivector.

In three dimensional Euclidean geometric algebra, we have one scalar e_q, three vector basis e_x, e_y, and e_z three bivector basis e_{xy}, e_{xz}, and e_{yz}, and one trivector e_{xyz}. The basis vectors square to one, and anti-commute among themselves. For example, $e_x e_x = 1$, but $e_x e_y = -e_y e_x = e_{xy}$.

The choice of default orderings for the bivectors and multivectors, in this case, is made by multiplying the vectors by the trivector to get the bivectors. Because the trivector mimics i, we can form complex numbers from (e_q, e_{xyz}), (e_x, e_{yz}), (e_y, e_{zx}), and (e_z, e_{xy}) components. In three-space, we are using e_{zx}, instead of e_{xz}.

In text format, suppressing the letter e in the basis names, we have a multiplication table

q	x	y	z	xy	zx	yz	xyz
x	q	xy	-zx	y	-z	xyz	yz
y	-xy	q	yz	-x	xyz	z	zx
z	zx	-yz	q	xyz	x	-y	xy
xy	-y	x	xyz	-q	yz	-zx	-z
zx	z	xyz	-x	-yz	-q	xy	-y
yz	xyz	-z	y	zx	-xy	-q	-x
xyz	yz	zx	xy	-z	-y	-x	-q

In component form, suitable for programming languages such as C, we can express the product $c = ab$ as

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c.q   =  + a.q*b.q   + a.x*b.x   + a.y*b.y   + a.z*b.z   - a.xy*b.xy  - a.zx*b.zx  - a.yz*b.yz  - a.xyz*b.xyz
c.x   =  + a.q*b.x   + a.x*b.q   - a.y*b.xy  + a.z*b.zx  + a.xy*b.y   - a.zx*b.z   - a.yz*b.xyz - a.xyz*b.yz
c.y   =  + a.q*b.y   + a.x*b.xy  + a.y*b.q   - a.z*b.yz  - a.xy*b.x   - a.zx*b.xyz + a.yz*b.z   - a.xyz*b.zx
c.z   =  + a.q*b.z   - a.x*b.zx  + a.y*b.yz  + a.z*b.q   - a.xy*b.xyz + a.zx*b.x   - a.yz*b.y   - a.xyz*b.xy
c.xy  =  + a.q*b.xy  + a.x*b.y   - a.y*b.x   + a.z*b.xyz + a.xy*b.q   + a.zx*b.yz  - a.yz*b.zx  + a.xyz*b.z
c.zx  =  + a.q*b.zx  - a.x*b.z   + a.y*b.xyz + a.z*b.x   - a.xy*b.yz  + a.zx*b.q   + a.yz*b.xy  + a.xyz*b.y
c.yz  =  + a.q*b.yz  + a.x*b.xyz + a.y*b.z   - a.z*b.y   + a.xy*b.zx  - a.zx*b.xy  + a.yz*b.q   + a.xyz*b.x
c.xyz =  + a.q*b.xyz + a.x*b.yz  + a.y*b.zx  + a.z*b.xy  + a.xy*b.z   + a.zx*b.y   + a.yz*b.x   + a.xyz*b.q
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Trajectories and Streamlines in 3D

In conventional three dimensional geometry, it is possible to describe a curve by means of the Frenet-Serret formulas. We parameterize the curve by path-length s, and at each point on the curve define the curvature κ, torsion τ, and a local orthogonal frame consisting of the unit tangent $\vec{u}$, normal $\vec{n}$ and binormal $\vec{b}$. The curvature measures the deviation of the curve from a straight line, and the torsion measure the deviation of the curve from a plane. Two curves which have the same histories of $\kappa(s)$ and $\tau(s)$ are congruent, regardless of origin or attitude.

We can do the same process in geometric algebra, with slightly different definitions. In the formulas below, I will use a Cartesian orthgonal three space with basis vector e_x, e_y and e_z.

As in the standard approach, we define a differential pathlength along the curve.

$$ds^2 = dx^2 + dy^2 + dz^2$$

Tangent

We define a unit tangent

$$\frac{d\vec{R}}{ds} = \vec{u} = \frac{dx}{ds}e_x + \frac{dy}{ds}e_y + \frac{dz}{ds}e_z$$

We can define a vector differential for ds using this unit tangent.

$$d\vec{s} = \vec{u}ds$$

Because the unit tangent has a constant length of one, the derivative of length squared is zero, and the unit tangent and its derivative are orthogonal.

$$\begin{aligned} \vec{u} \cdot \vec{u} &= 1 \\ \frac{d}{ds}(\vec{u} \cdot \vec{u}) &= 2\vec{u} \cdot \frac{d\vec{u}}{ds} = 0 \\ \vec{u} &\perp \frac{d\vec{u}}{ds} \end{aligned}$$

In geometric algebra, the product of two vectors is the sum of the dot product and wedge product.

$$\vec{a}\vec{b} = \vec{a} \cdot \vec{b} + \vec{a} \wedge \vec{b}$$

For the special case of orthogonal vectors, such as $\vec{u}$ and $d\vec{u}/ds$, the geometric product and wedge product coincide.

Curvature

The wedge product of two vectors is anti-symmetric, $\vec{a} \wedge \vec{b} = -\vec{b} \wedge \vec{a}$. The wedge square $\vec{a} \wedge \vec{a} = 0$ implies that the wedge product measures the deviation of the two vectors. In detail, the wedge product of two vectors is an oriented area for the parallelogram formed by the two vectors. Taking the wedge product of $\vec{u}$ and $d\vec{u}/ds$, we find the curvature bivector

$$\vec{u} \wedge \frac{d\vec{u}}{ds} = \vec{u}\frac{d\vec{u}}{ds} = \kappa(\vec{u} \wedge \vec{n}) = \kappa\vec{u}\vec{n}$$

For a circle, the radius of curvature is the inverse of the curvature, $\rho = 1/\kappa$. The incremental distance along a circular arc is related to angle and radius, $ds = \rho d\theta$. We can invert this equation, to find

$$\begin{aligned} ds &= \rho d\theta \\ d\theta &= \frac{1}{\rho} ds = \kappa ds \end{aligned}$$

When we generalize to geometric algebra, we find that angles become bivectors in the plane of rotation. I will use double overbars to indicate bivectors, here.

$$d\bar{\bar{\theta}} = \bar{\bar{\kappa}} ds = \kappa\vec{u}\vec{n} ds$$

It is important to point out that κ and ρ are locally defined in terms of differential elements, and are not tied to any particular coordinate zero location.

We now look at the interesting unit planar element expression inspired by distance along arc length,

$$\begin{aligned} d\bar{\bar{s}} &= \rho \, d\bar{\bar{\theta}} \\ &= \rho \, \kappa \, \vec{u} \, \vec{n} \, ds \\ &= \vec{u} \, \vec{n} \, ds \end{aligned}$$

We see that this unit bivector associated with curvature is analogous to the unit tangent for linear terms.

Torsion

The triple wedge product

$$\vec{u} \wedge \frac{d\vec{u}}{ds} \wedge \frac{d^2\vec{u}}{ds^2} = \kappa^2 \, \tau \, \vec{u} \, \vec{n} \, \vec{b}$$

defines geometric torsion. In three-space, torsion is a single pseudoscalar. The radius of torsion and the torsion are inverses. $R = 1/\tau$. In a fashion paralleling the unit curvature bivector direction, we have

$$d\bar{\bar{\bar{s}}} = \vec{u} \, \vec{n} \, \vec{b} \, ds$$

Frenet Frame

In standard three dimensional space, the three unit vectors $\vec{u}$, $\vec{n}$ and $\vec{b}$ define a local coordinate system at each space along the curve. When we transistion to geometric algebra, we need a multivector with unit components to achieve the same task. Using $\mathbf{S}$ as a symbol for this multivector frame (state), I find

$$d\mathbf{S} = ds + \vec{u} \, ds + \vec{u} \, \vec{n} \, ds + \vec{u} \, \vec{n} \, \vec{b} \, ds$$

We will find a strong tie-in between this state, and the multivector wavefunction.

$$\psi = \frac{d\mathbf{S}}{ds} = 1 + \vec{u} \; + \vec{u} \, \vec{n} \; + \vec{u} \, \vec{n} \, \vec{b}$$

We will find streamlines of the wavefunction being the unit tangent, and transverse lines similar to equipotentials in field theory associated with the unit bivector. Associating s with t, we will find that the unit tangent implies everything is in constant motion at speed c. When averaging over volumes and time, we will find average positions, and average velocities less than c, with accompanying loss of information due to the averaging process.

Multivector Determinant

Geometric algebra can be implemented using matrices, and we can take matrix tools and use them in geometric algebra. For magnitude of a multivector, I prefer to use the determinant from matrix algebra.

For a generic, three dimensional multivector (note order of e_{zx} term)

$$W = a + be_x + ce_y + de_z + ee_{xy} + fe_{zx} + ge_{xy} + he_{xyz}$$

the determinant can be written in the suggestive form

$$\begin{aligned} \det_{3\mathrm{D}}(W) &= +(a^2 - b^2 - c^2 - d^2 + e^2 + f^2 + g^2 - h^2)^2 \\ &\quad +4 * (ah - bg - cf - de)^2) \end{aligned}$$

For the 3D Euclidean Frenet frame above, we have the determinant.

$$\det\,(1 + \vec{u}\; + \vec{u}\,\vec{n}\; + \vec{u}\,\vec{n}\,\vec{b}\,) = 4$$

The Three Dimensional Dirac Equations

Write the relativistic energy-momentum relationship with $c = 1$.

$$\begin{aligned} E^2 - \vec{p}\cdot\vec{p} &= m^2 \\ E^2 - p_x^2 - p_y^2 - p_z^2 &= m^2 \\ (E + p_x e_x + p_y e_y + p_z e_z)\,(E - p_x e_x - p_y e_y - p_z e_z) &= m^2 \end{aligned}$$

This can be interpreted as the product of two eigen-equations with eigenvalue m,

$$\begin{aligned} (E + p_x e_x + p_y e_y + p_z e_z) &= m \\ (E - p_x e_x - p_y e_y - p_z e_z) &= m \end{aligned}$$

with the assumption that energy and mass are independent of the direction of momentum.

Using the positive sign term, and converting to operators, we have the eigen-equation

$$(E + p_x e_x + p_y e_y + p_z e_z)\,\psi \;=\; m\psi$$

We replace our energy and momentum terms with the quantum operators with electromagnetic source terms.

$$E = i\hbar\frac{\partial}{\partial t} + q\phi \quad p_x = -i\hbar\frac{\partial}{\partial x} + qA_x \quad p_y = -i\hbar\frac{\partial}{\partial y} + qA_y \quad p_z = -i\hbar\frac{\partial}{\partial z} + qA_z$$

$$(E + p_x e_x + p_y e_y + p_z e_z)\,\psi \;=\; m\psi$$

$$\left((i\hbar\frac{\partial}{\partial t} + q\phi) + (-i\hbar\frac{\partial}{\partial x} + qA_x)e_x + (-i\hbar\frac{\partial}{\partial y} + qA_y)e_y + (-i\hbar\frac{\partial}{\partial z} + qA_z)e_z\right)\psi \;=\; m\psi$$

Expand our parenthesis.

$$-m\psi + i\hbar\frac{\partial\psi}{\partial t} + q\phi\psi - ie_x\hbar\frac{\partial\psi}{\partial x} + e_x qA_x\psi - ie_y\hbar\frac{\partial\psi}{\partial y} + e_y qA_y\psi - ie_z\hbar\frac{\partial\psi}{\partial z} + e_z qA_z\psi \;=\; 0$$

Re-order our terms.

$$-m\psi + q\phi\psi + e_x qA_x\psi + e_y qA_y\psi + e_z qA_z\psi + i\hbar\frac{\partial\psi}{\partial t} - ie_x\hbar\frac{\partial\psi}{\partial x} - ie_y\hbar\frac{\partial\psi}{\partial y} - ie_z\hbar\frac{\partial\psi}{\partial z} \;=\; 0$$

We now replace i with the trivector e_{xyz}, and simplify products

$$\begin{aligned}&-m\psi + q\phi\psi + e_x qA_x\psi + e_y qA_y\psi + e_z qA_z\psi\\ &+e_{xyz}\hbar\frac{\partial\psi}{\partial t} - e_{xyz}e_x\hbar\frac{\partial\psi}{\partial x} - e_{xyz}e_y\hbar\frac{\partial\psi}{\partial y} - e_{xyz}e_z\hbar\frac{\partial\psi}{\partial z} \;=\; 0\end{aligned}$$

with the result

$$-m\psi + q\phi\psi + e_x qA_x\psi + e_y qA_y\psi + e_z qA_z\psi + e_{xyz}\hbar\frac{\partial\psi}{\partial t} - e_{yz}\hbar\frac{\partial\psi}{\partial x} - e_{zx}\hbar\frac{\partial\psi}{\partial y} - e_{xy}\hbar\frac{\partial\psi}{\partial z} \;=\; 0$$

We look at this equation, and note the superficial resemblance to the classic Dirac equation. Our next step is to express ψ in terms of multivector components, substitute into the equation above, and separate components to obtain eight real (as opposed to complex) equations.

As a three dimensional multivector, ψ has eight components.

$$\psi = \psi_0 + \psi_x e_x + \psi_y e_y + \psi_z e_z + \psi_{xy} e_{ey} + \psi_{zx} e_{zx} + \psi_{yz} e_{yz} + \psi_{xyz} e_{xyz}$$

We substitute into our equation above.

$$\begin{aligned}
&-m\psi_0 + q\phi\psi_0 + e_x q A_x \psi_0 + e_y q A_y \psi_0 + e_z q A_z \psi_0 \\
&+e_{xyz}\hbar\frac{\partial\psi_0}{\partial t} - e_{yz}\hbar\frac{\partial\psi_0}{\partial x} - e_{zx}\hbar\frac{\partial\psi_0}{\partial y} - e_{xy}\hbar\frac{\partial\psi_0}{\partial z} + \\
&-m\psi_x e_x + q\phi\psi_x e_x + e_x q A_x \psi_x e_x + e_y q A_y \psi_x e_x + e_z q A_z \psi_x e_x \\
&+e_{xyz}\hbar\frac{\partial\psi_x e_x}{\partial t} - e_{yz}\hbar\frac{\partial\psi_x e_x}{\partial x} - e_{zx}\hbar\frac{\partial\psi_x e_x}{\partial y} - e_{xy}\hbar\frac{\partial\psi_x e_x}{\partial z} + \\
&-m\psi_y e_y + q\phi\psi_y e_y + e_x q A_x \psi_y e_y + e_y q A_y \psi_y e_y + e_z q A_z \psi_y e_y \\
&+e_{xyz}\hbar\frac{\partial\psi_y e_y}{\partial t} - e_{yz}\hbar\frac{\partial\psi_y e_y}{\partial x} - e_{zx}\hbar\frac{\partial\psi_y e_y}{\partial y} - e_{xy}\hbar\frac{\partial\psi_y e_y}{\partial z} + \\
&-m\psi_z e_z + q\phi\psi_z e_z + e_x q A_x \psi_z e_z + e_y q A_y \psi_z e_z + e_z q A_z \psi_z e_z \\
&+e_{xyz}\hbar\frac{\partial\psi_z e_z}{\partial t} - e_{yz}\hbar\frac{\partial\psi_z e_z}{\partial x} - e_{zx}\hbar\frac{\partial\psi_z e_z}{\partial y} - e_{xy}\hbar\frac{\partial\psi_z e_z}{\partial z} + \\
&-m\psi_{xy} e_{xy} + q\phi\psi_{xy} e_{xy} + e_x q A_x \psi_{xy} e_{xy} + e_y q A_y \psi_{xy} e_{xy} + e_z q A_z \psi_{xy} e_{xy} \\
&+e_{xyz}\hbar\frac{\partial\psi_{xy} e_{xy}}{\partial t} - e_{yz}\hbar\frac{\partial\psi_{xy} e_{xy}}{\partial x} - e_{zx}\hbar\frac{\partial\psi_{xy} e_{xy}}{\partial y} - e_{xy}\hbar\frac{\partial\psi_{xy} e_{xy}}{\partial z} + \\
&-m\psi_{zx} e_{zx} + q\phi\psi_{zx} e_{zx} + e_x q A_x \psi_{zx} e_{zx} + e_y q A_y \psi_{zx} e_{zx} + e_z q A_z \psi_{zx} e_{zx} \\
&+e_{xyz}\hbar\frac{\partial\psi_{zx} e_{zx}}{\partial t} - e_{yz}\hbar\frac{\partial\psi_{zx} e_{zx}}{\partial x} - e_{zx}\hbar\frac{\partial\psi_{zx} e_{zx}}{\partial y} - e_{xy}\hbar\frac{\partial\psi_{zx} e_{zx}}{\partial z} + \\
&-m\psi_{yz} e_{yz} + q\phi\psi_{yz} e_{yz} + e_x q A_x \psi_{yz} e_{yz} + e_y q A_y \psi_{yz} e_{yz} + e_z q A_z \psi_{yz} e_{yz} \\
&+e_{xyz}\hbar\frac{\partial\psi_{yz} e_{yz}}{\partial t} - e_{yz}\hbar\frac{\partial\psi_{yz} e_{yz}}{\partial x} - e_{zx}\hbar\frac{\partial\psi_{yz} e_{yz}}{\partial y} - e_{xy}\hbar\frac{\partial\psi_{yz} e_{yz}}{\partial z} + \\
&-m\psi_{xyz} e_{xyz} + q\phi\psi_{xyz} e_{xyz} + e_x q A_x \psi_{xyz} e_{xyz} + e_y q A_y \psi_{xyz} e_{xyz} + e_z q A_z \psi_{xyz} e_{xyz} \\
&+e_{xyz}\hbar\frac{\partial\psi_{xyz} e_{xyz}}{\partial t} - e_{yz}\hbar\frac{\partial\psi_{xyz} e_{xyz}}{\partial x} - e_{zx}\hbar\frac{\partial\psi_{xyz} e_{xyz}}{\partial y} - e_{xy}\hbar\frac{\partial\psi_{xyz} e_{xyz}}{\partial z} = 0
\end{aligned}$$

We now group our terms by component. Each component separately sums to zero, yielding eight equations.

$$-m\psi_0 + q\phi\psi_0 + qA_x\psi_x + qA_y\psi_y + qA_z\psi_z$$
$$-\hbar\frac{\partial\psi_{xyz}}{\partial t} + \hbar\frac{\partial\psi_{yz}}{\partial x} + \hbar\frac{\partial\psi_{zx}}{\partial y} + \hbar\frac{\partial\psi_{xy}}{\partial z} = 0$$

$$-m\psi_x e_x + q\phi\psi_x e_x + e_x qA_x\psi_0 - e_x qA_y\psi_{xy} + e_x qA_z\psi_{zx}$$
$$-e_x\hbar\frac{\partial\psi_{yz}}{\partial t} + e_x\hbar\frac{\partial\psi_{xyz}}{\partial x} + e_x\hbar\frac{\partial\psi_z}{\partial y} - e_x\hbar\frac{\partial\psi_y}{\partial z} = 0$$

$$-m\psi_y e_y + q\phi\psi_y e_y + e_y qA_x\psi_{xy} + e_y qA_y\psi_0 - e_y qA_z\psi_{yz}$$
$$-e_y\hbar\frac{\partial\psi_{zx}}{\partial t} - e_y\hbar\frac{\partial\psi_z}{\partial x} + e_y\hbar\frac{\partial\psi_{xyz}}{\partial y} + e_y\hbar\frac{\partial\psi_x}{\partial z} = 0$$

$$-m\psi_z e_z + q\phi\psi_z e_z - e_z qA_x\psi_{zx} + e_z qA_y\psi_{yz} + e_z qA_z\psi_0$$
$$-e_z\hbar\frac{\partial\psi_{xy}}{\partial t} + e_z\hbar\frac{\partial\psi_y}{\partial x} - e_z\hbar\frac{\partial\psi_x}{\partial y} + e_z\hbar\frac{\partial\psi_{xyz}}{\partial z} = 0$$

$$-m\psi_{xy}e_{xy} + q\phi\psi_{xy}e_{xy} + e_{xy}qA_x\psi_y - e_{xy}qA_y\psi_x + e_{xy}qA_z\psi_{xyz}$$
$$+e_{xy}\hbar\frac{\partial\psi_z}{\partial t} + e_{xy}\hbar\frac{\partial\psi_{zx}}{\partial x} - e_{xy}\hbar\frac{\partial\psi_{yz}}{\partial y} - e_{xy}\hbar\frac{\partial\psi_0}{\partial z} = 0$$

$$-m\psi_{zx}e_{zx} + q\phi\psi_{zx}e_{zx} - e_{zx}qA_x\psi_z + e_{zx}qA_y\psi_{xyz} + e_{zx}qA_z\psi_x$$
$$+e_{zx}\hbar\frac{\partial\psi_y}{\partial t} - e_{zx}\hbar\frac{\partial\psi_{xy}}{\partial x} - e_{zx}\hbar\frac{\partial\psi_0}{\partial y} + e_{zx}\hbar\frac{\partial\psi_{yz}}{\partial z} = 0$$

$$-m\psi_{yz}e_{yz} + q\phi\psi_{yz}e_{yz} + e_{yz}qA_x\psi_{xyz} + e_{yz}qA_y\psi_z - e_{yz}qA_z\psi_y$$
$$+e_{yz}\hbar\frac{\partial\psi_x}{\partial t} - e_{yz}\hbar\frac{\partial\psi_0}{\partial x} + e_{yz}\hbar\frac{\partial\psi_{xy}}{\partial y} - e_{yz}\hbar\frac{\partial\psi_{zx}}{\partial z} = 0$$

$$-m\psi_{xyz}e_{xyz} + q\phi\psi_{xyz}e_{xyz} + e_{xyz}qA_x\psi_{yz} + e_{xyz}qA_y\psi_{zx} + e_{xyz}qA_z\psi_{xy}$$
$$+e_{xyz}\hbar\frac{\partial\psi_0}{\partial t} - e_{xyz}\hbar\frac{\partial\psi_x}{\partial x} - e_{xyz}\hbar\frac{\partial\psi_y}{\partial y} - e_{xyz}\hbar\frac{\partial\psi_z}{\partial z} = 0$$

We now remove the common multivector factor, resulting in eight scalar equations.

$$\begin{aligned}
-m\psi_0 + q\phi\psi_0 + qA_x\psi_x + qA_y\psi_y + qA_z\psi_z - \hbar\frac{\partial\psi_{xyz}}{\partial t} + \hbar\frac{\partial\psi_{yz}}{\partial x} + \hbar\frac{\partial\psi_{zx}}{\partial y} + \hbar\frac{\partial\psi_{xy}}{\partial z} &= 0 \\
-m\psi_x + q\phi\psi_x + qA_x\psi_0 - qA_y\psi_{xy} + qA_z\psi_{zx} - \hbar\frac{\partial\psi_{yz}}{\partial t} + \hbar\frac{\partial\psi_{xyz}}{\partial x} + \hbar\frac{\partial\psi_z}{\partial y} - \hbar\frac{\partial\psi_y}{\partial z} &= 0 \\
-m\psi_y + q\phi\psi_y + qA_x\psi_{xy} + qA_y\psi_0 - qA_z\psi_{yz} - \hbar\frac{\partial\psi_{zx}}{\partial t} - \hbar\frac{\partial\psi_z}{\partial x} + \hbar\frac{\partial\psi_{xyz}}{\partial y} + \hbar\frac{\partial\psi_x}{\partial z} &= 0 \\
-m\psi_z + q\phi\psi_z - qA_x\psi_{zx} + qA_y\psi_{yz} + qA_z\psi_0 - \hbar\frac{\partial\psi_{xy}}{\partial t} + \hbar\frac{\partial\psi_y}{\partial x} - \hbar\frac{\partial\psi_x}{\partial y} + \hbar\frac{\partial\psi_{xyz}}{\partial z} &= 0 \\
-m\psi_{xy} + q\phi\psi_{xy} + qA_x\psi_y - qA_y\psi_x + qA_z\psi_{xyz} + \hbar\frac{\partial\psi_z}{\partial t} + \hbar\frac{\partial\psi_{zx}}{\partial x} - \hbar\frac{\partial\psi_{yz}}{\partial y} - \hbar\frac{\partial\psi_0}{\partial z} &= 0 \\
-m\psi_{zx} + q\phi\psi_{zx} - qA_x\psi_z + qA_y\psi_{xyz} + qA_z\psi_x + \hbar\frac{\partial\psi_y}{\partial t} - \hbar\frac{\partial\psi_{xy}}{\partial x} - \hbar\frac{\partial\psi_0}{\partial y} + \hbar\frac{\partial\psi_{yz}}{\partial z} &= 0 \\
-m\psi_{yz} + q\phi\psi_{yz} + qA_x\psi_{xyz} + qA_y\psi_z - qA_z\psi_y + \hbar\frac{\partial\psi_x}{\partial t} - \hbar\frac{\partial\psi_0}{\partial x} + \hbar\frac{\partial\psi_{xy}}{\partial y} - \hbar\frac{\partial\psi_{zx}}{\partial z} &= 0 \\
-m\psi_{xyz} + q\phi\psi_{xyz} + qA_x\psi_{yz} + qA_y\psi_{zx} + qA_z\psi_{xy} + \hbar\frac{\partial\psi_0}{\partial t} - \hbar\frac{\partial\psi_x}{\partial x} - \hbar\frac{\partial\psi_y}{\partial y} - \hbar\frac{\partial\psi_z}{\partial z} &= 0
\end{aligned}$$

We see similarity to the standard Dirac equation, but differences as well.

Grouping pairs of equations which mimic complex numbers, we have

$$\begin{aligned}
-m\psi_0 + q\phi\psi_0 + qA_x\psi_x + qA_y\psi_y + qA_z\psi_z - \hbar\frac{\partial\psi_{xyz}}{\partial t} + \hbar\frac{\partial\psi_{yz}}{\partial x} + \hbar\frac{\partial\psi_{zx}}{\partial y} + \hbar\frac{\partial\psi_{xy}}{\partial z} &= 0 \\
-m\psi_{xyz} + q\phi\psi_{xyz} + qA_x\psi_{yz} + qA_y\psi_{zx} + qA_z\psi_{xy} + \hbar\frac{\partial\psi_0}{\partial t} - \hbar\frac{\partial\psi_x}{\partial x} - \hbar\frac{\partial\psi_y}{\partial y} - \hbar\frac{\partial\psi_z}{\partial z} &= 0
\end{aligned}$$

$$
\begin{aligned}
-m\psi_x + q\phi\psi_x + qA_x\psi_0 - qA_y\psi_{xy} + qA_z\psi_{zx} - \hbar\frac{\partial\psi_{yz}}{\partial t} + \hbar\frac{\partial\psi_{xyz}}{\partial x} + \hbar\frac{\partial\psi_z}{\partial y} - \hbar\frac{\partial\psi_y}{\partial z} &= 0 \\
-m\psi_{yz} + q\phi\psi_{yz} + qA_x\psi_{xyz} + qA_y\psi_z - qA_z\psi_y + \hbar\frac{\partial\psi_x}{\partial t} - \hbar\frac{\partial\psi_0}{\partial x} + \hbar\frac{\partial\psi_{xy}}{\partial y} - \hbar\frac{\partial\psi_{zx}}{\partial z} &= 0
\end{aligned}
$$

$$
\begin{aligned}
-m\psi_y + q\phi\psi_y + qA_x\psi_{xy} + qA_y\psi_0 - qA_z\psi_{yz} - \hbar\frac{\partial\psi_{zx}}{\partial t} - \hbar\frac{\partial\psi_z}{\partial x} + \hbar\frac{\partial\psi_{xyz}}{\partial y} + \hbar\frac{\partial\psi_x}{\partial z} &= 0 \\
-m\psi_{zx} + q\phi\psi_{zx} - qA_x\psi_z + qA_y\psi_{xyz} + qA_z\psi_x + \hbar\frac{\partial\psi_y}{\partial t} - \hbar\frac{\partial\psi_{xy}}{\partial x} - \hbar\frac{\partial\psi_0}{\partial y} + \hbar\frac{\partial\psi_{yz}}{\partial z} &= 0
\end{aligned}
$$

$$
\begin{aligned}
-m\psi_z + q\phi\psi_z - qA_x\psi_{zx} + qA_y\psi_{yz} + qA_z\psi_0 - \hbar\frac{\partial\psi_{xy}}{\partial t} + \hbar\frac{\partial\psi_y}{\partial x} - \hbar\frac{\partial\psi_x}{\partial y} + \hbar\frac{\partial\psi_{xyz}}{\partial z} &= 0 \\
-m\psi_{xy} + q\phi\psi_{xy} + qA_x\psi_y - qA_y\psi_x + qA_z\psi_{xyz} + \hbar\frac{\partial\psi_z}{\partial t} + \hbar\frac{\partial\psi_{zx}}{\partial x} - \hbar\frac{\partial\psi_{yz}}{\partial y} - \hbar\frac{\partial\psi_0}{\partial z} &= 0
\end{aligned}
$$

In a sense, this arrangement is more satisfactory to me than is the standard Bjorken-Drell or Dirac alpha implementations. In this three dimensional implementation, i is e_{xyz}, as opposed to e_{wxyzt} in five-space. We see no 'gaposis' with respect to each multivector element used in each equation. I like the way this looks.

Three Dimensional Dirac Equation Interpreted

$$
\begin{aligned}
-m\psi_0 + q\phi\psi_0 + qA_x\psi_x + qA_y\psi_y + qA_z\psi_z - \hbar\frac{\partial\psi_{xyz}}{\partial t} + \hbar\frac{\partial\psi_{yz}}{\partial x} + \hbar\frac{\partial\psi_{zx}}{\partial y} + \hbar\frac{\partial\psi_{xy}}{\partial z} &= 0 \\
-m\psi_x + q\phi\psi_x + qA_x\psi_0 - qA_y\psi_{xy} + qA_z\psi_{zx} - \hbar\frac{\partial\psi_{yz}}{\partial t} + \hbar\frac{\partial\psi_{xyz}}{\partial x} + \hbar\frac{\partial\psi_z}{\partial y} - \hbar\frac{\partial\psi_y}{\partial z} &= 0 \\
-m\psi_y + q\phi\psi_y + qA_x\psi_{xy} + qA_y\psi_0 - qA_z\psi_{yz} - \hbar\frac{\partial\psi_{zx}}{\partial t} - \hbar\frac{\partial\psi_z}{\partial x} + \hbar\frac{\partial\psi_{xyz}}{\partial y} + \hbar\frac{\partial\psi_x}{\partial z} &= 0 \\
-m\psi_z + q\phi\psi_z - qA_x\psi_{zx} + qA_y\psi_{yz} + qA_z\psi_0 - \hbar\frac{\partial\psi_{xy}}{\partial t} + \hbar\frac{\partial\psi_y}{\partial x} - \hbar\frac{\partial\psi_x}{\partial y} + \hbar\frac{\partial\psi_{xyz}}{\partial z} &= 0 \\
-m\psi_{xy} + q\phi\psi_{xy} + qA_x\psi_y - qA_y\psi_x + qA_z\psi_{xyz} + \hbar\frac{\partial\psi_z}{\partial t} + \hbar\frac{\partial\psi_{zx}}{\partial x} - \hbar\frac{\partial\psi_{yz}}{\partial y} - \hbar\frac{\partial\psi_0}{\partial z} &= 0 \\
-m\psi_{zx} + q\phi\psi_{zx} - qA_x\psi_z + qA_y\psi_{xyz} + qA_z\psi_x + \hbar\frac{\partial\psi_y}{\partial t} - \hbar\frac{\partial\psi_{xy}}{\partial x} - \hbar\frac{\partial\psi_0}{\partial y} + \hbar\frac{\partial\psi_{yz}}{\partial z} &= 0 \\
-m\psi_{yz} + q\phi\psi_{yz} + qA_x\psi_{xyz} + qA_y\psi_z - qA_z\psi_y + \hbar\frac{\partial\psi_x}{\partial t} - \hbar\frac{\partial\psi_0}{\partial x} + \hbar\frac{\partial\psi_{xy}}{\partial y} - \hbar\frac{\partial\psi_{zx}}{\partial z} &= 0 \\
-m\psi_{xyz} + q\phi\psi_{xyz} + qA_x\psi_{yz} + qA_y\psi_{zx} + qA_z\psi_{xy} + \hbar\frac{\partial\psi_0}{\partial t} - \hbar\frac{\partial\psi_x}{\partial x} - \hbar\frac{\partial\psi_y}{\partial y} - \hbar\frac{\partial\psi_z}{\partial z} &= 0
\end{aligned}
$$

I suspect the first equation is an energy equation, followed by three components of linear momentum (corresponding to unit tangent), three components of angular momentum (corresponding to curvature), and a single equation related to torsion. Let us see if we can interpret ψ in a fashion to achieve this interpretation.

Re-arrange terms slightly

$$
\begin{aligned}
-m\psi_0 + q\phi\psi_0 + qA_x\psi_x + qA_y\psi_y + qA_z\psi_z - \hbar\frac{\partial\psi_{xyz}}{\partial t} + \hbar\frac{\partial\psi_{yz}}{\partial x} + \hbar\frac{\partial\psi_{zx}}{\partial y} + \hbar\frac{\partial\psi_{xy}}{\partial z} &= 0 \\
-m\psi_x + q\phi\psi_x + qA_x\psi_0 - qA_y\psi_{xy} + qA_z\psi_{zx} - \hbar\frac{\partial\psi_{yz}}{\partial t} + \hbar\frac{\partial\psi_{xyz}}{\partial x} + \hbar\frac{\partial\psi_z}{\partial y} - \hbar\frac{\partial\psi_y}{\partial z} &= 0 \\
-m\psi_y + q\phi\psi_y + qA_y\psi_0 - qA_z\psi_{yz} + qA_x\psi_{xy} - \hbar\frac{\partial\psi_{zx}}{\partial t} - \hbar\frac{\partial\psi_z}{\partial x} + \hbar\frac{\partial\psi_{xyz}}{\partial y} + \hbar\frac{\partial\psi_x}{\partial z} &= 0 \\
-m\psi_z + q\phi\psi_z + qA_z\psi_0 - qA_x\psi_{zx} + qA_y\psi_{yz} - \hbar\frac{\partial\psi_{xy}}{\partial t} + \hbar\frac{\partial\psi_y}{\partial x} - \hbar\frac{\partial\psi_x}{\partial y} + \hbar\frac{\partial\psi_{xyz}}{\partial z} &= 0 \\
-m\psi_{xy} + q\phi\psi_{xy} + qA_z\psi_{xyz} + qA_x\psi_y - qA_y\psi_x + \hbar\frac{\partial\psi_z}{\partial t} + \hbar\frac{\partial\psi_{zx}}{\partial x} - \hbar\frac{\partial\psi_{yz}}{\partial y} - \hbar\frac{\partial\psi_0}{\partial z} &= 0 \\
-m\psi_{zx} + q\phi\psi_{zx} + qA_y\psi_{xyz} + qA_z\psi_x - qA_x\psi_z + \hbar\frac{\partial\psi_y}{\partial t} - \hbar\frac{\partial\psi_{xy}}{\partial x} - \hbar\frac{\partial\psi_0}{\partial y} + \hbar\frac{\partial\psi_{yz}}{\partial z} &= 0 \\
-m\psi_{yz} + q\phi\psi_{yz} + qA_x\psi_{xyz} + qA_y\psi_z - qA_z\psi_y + \hbar\frac{\partial\psi_x}{\partial t} - \hbar\frac{\partial\psi_0}{\partial x} + \hbar\frac{\partial\psi_{xy}}{\partial y} - \hbar\frac{\partial\psi_{zx}}{\partial z} &= 0 \\
-m\psi_{xyz} + q\phi\psi_{xyz} + qA_x\psi_{yz} + qA_y\psi_{zx} + qA_z\psi_{xy} + \hbar\frac{\partial\psi_0}{\partial t} - \hbar\frac{\partial\psi_x}{\partial x} - \hbar\frac{\partial\psi_y}{\partial y} - \hbar\frac{\partial\psi_z}{\partial z} &= 0
\end{aligned}
$$

I assume that ψ carries geometric information, akin to the deBroglie pilot wave of 1927.

I assume that in the low speed limit, $\psi_0 \approx 1$, $\psi_x \approx v_x$, $\psi_y \approx v_y$, and $\psi_z \approx v_z$, The red terms are macroscopic energy with homopolar DC contribution. The green terms are macroscopic linear momentum. The blue terms are akin to macroscopic precession, or gyroscopic effects. The magenta terms deal with curvature (away from tangent line) effects, while the cyan line deals with torsion (out of plane) effects.

The black terms are the microscopic quantum effects, and are basically continuity equations among the multivector components.

To gain a feel for the meaning of the ψ terms, lets look at the free space case where there is no charge or mass. The eight black equation components become

$$
\begin{aligned}
-\hbar\frac{\partial\psi_{xyz}}{\partial t}+\hbar\frac{\partial\psi_{yz}}{\partial x}+\hbar\frac{\partial\psi_{zx}}{\partial y}+\hbar\frac{\partial\psi_{xy}}{\partial z} &= 0\\
-\hbar\frac{\partial\psi_{yz}}{\partial t}+\hbar\frac{\partial\psi_{xyz}}{\partial x}+\hbar\frac{\partial\psi_{z}}{\partial y}-\hbar\frac{\partial\psi_{y}}{\partial z} &= 0\\
-\hbar\frac{\partial\psi_{zx}}{\partial t}-\hbar\frac{\partial\psi_{z}}{\partial x}+\hbar\frac{\partial\psi_{xyz}}{\partial y}+\hbar\frac{\partial\psi_{x}}{\partial z} &= 0\\
-\hbar\frac{\partial\psi_{xy}}{\partial t}+\hbar\frac{\partial\psi_{y}}{\partial x}-\hbar\frac{\partial\psi_{x}}{\partial y}+\hbar\frac{\partial\psi_{xyz}}{\partial z} &= 0\\
+\hbar\frac{\partial\psi_{z}}{\partial t}+\hbar\frac{\partial\psi_{zx}}{\partial x}-\hbar\frac{\partial\psi_{yz}}{\partial y}-\hbar\frac{\partial\psi_{0}}{\partial z} &= 0\\
+\hbar\frac{\partial\psi_{y}}{\partial t}-\hbar\frac{\partial\psi_{xy}}{\partial x}-\hbar\frac{\partial\psi_{0}}{\partial y}+\hbar\frac{\partial\psi_{yz}}{\partial z} &= 0\\
+\hbar\frac{\partial\psi_{x}}{\partial t}-\hbar\frac{\partial\psi_{0}}{\partial x}+\hbar\frac{\partial\psi_{xy}}{\partial y}-\hbar\frac{\partial\psi_{zx}}{\partial z} &= 0\\
+\hbar\frac{\partial\psi_{0}}{\partial t}-\hbar\frac{\partial\psi_{x}}{\partial x}-\hbar\frac{\partial\psi_{y}}{\partial y}-\hbar\frac{\partial\psi_{z}}{\partial z} &= 0
\end{aligned}
$$

The first equation, for example, states that the rate of change of torsion is the divergence of the curvatures.

The eight equation is an interesting conservation law for spacetime, showing that changes in ψ_0 in time are driven by the divergence of ψ_x, ψ_y, and ψ_z.

Let's look at this eighth equation, and see how virtual particles can be described as such. Start with the eight equation in free space, with no mass and no charges.

$$
+\hbar\frac{\partial\psi_{0}}{\partial t}-\hbar\frac{\partial\psi_{x}}{\partial x}-\hbar\frac{\partial\psi_{y}}{\partial y}-\hbar\frac{\partial\psi_{z}}{\partial z} = 0
$$

One specific solution is a tilted ellipse in three space, and associated vibration in time.

$$
\begin{aligned}
\psi_0 &= r\cos\omega t\\
\psi_x &= r\cos\omega x\\
\psi_y &= \rho\sin\gamma y\\
\psi_z &= -\rho\sin\gamma z
\end{aligned}
$$

This is not meant as the most general solution, just an illustration of a simple solution showing independence of location in space time, yet limit extented around an event. We would see this feature only for a short time around $t = 0$, around the spatial origin.