Determinants, Idempotents and Nilpotents in Geometric Algebra

Kurt Nalty

July 6, 2018

Abstract

I present formulas for determinants, nilpotents (non-zero expressions which square to zero) and idempotents (expressions which square to themselves) in two through five dimensional Geometric Algebras. Formulas implemented using nilpotents and idempotents are simple, and aggressively linear.

Determinants in Geometric Algebra

Geometric algebra is a non-commutative associative algebra which can always be implemented in a (large enough) matrix formulation. As such, matrix tools such as determinants, inverses, and the like can be defined for specific geometric algebra implementations. For a matrix to be invertible, the determinant must be non-zero. Likewise, for a multivector to be invertible, the determinant must be non-zero. Algebraically, the determinant is a measure of the hypervolume spanned by the matrix rows or columns. For matrices, a zero determinant indicates a condition where the rows or columns do not fully span the space available. Likewise, for multivectors, a zero determinant indicates that the multivector inherently spans only a subset of the space available. An example is a photon. Travelling at light speed, the photon does not experience proper time or linear distance in the axial direction. From emission to absorption, the photon only has a contact experience. In the lab frame, we see the photon in a four dimensional Minkowski spacetime, but from the photon's point of view, it has only a two dimensional experience. The zero determinant reflects this reduced spatial condition, embedded in a higher spacetime. In my opinion, all particle physics will involve zero multivector determinant descriptions. Potents, nilpotents and idempotents will be very useful tools, and their most useful forms all have zero determinants.

Potents and Eigenvalue Problems

Assume that a multivector square is proportional to the original multivector. Also note that the determinant of a product is also the product of the individual determinants.

$$\begin{aligned} M^2 &= \lambda M \\ M^2 - \lambda M &= 0 \\ M(M-\lambda) &= 0 \\ \det\left(M\left(M-\lambda\right)\right) &= \det(0) = 0 \\ \det(M)\det(M-\lambda) &= 0 \end{aligned}$$

We see that a consequence of a multivector square being non-singular (meaning non-zero determinant) and similar to the original is that it satisfies the eigenvalue equation of matrix algebra. As a scalar, λ works like a DC offset against the scalar component q. For the 4x4 matrices which describe Minkowski and five dimensional space, we have a quartic equation for λ with potentially four solutions. I have hopes that the masses of leptons, including a fourth generation not currently part of the standard model, will arise from a geometric algebra description, and that the zero determinant feature will be the key for finite or zero self-interaction energies. I also note that if the multivector M above is singular, then λ supports a continuous spectrum, like photons.

Two specialized potents are of great interest. Idempotents are potents where $\lambda = 1$, and nilpotents are potents where $\lambda = 0$.

Idempotents, Null Factors and Nilpotents

Idempotents are expressions which square to themselves, under some form of multiplication. For regular numbers, this relationship is $P^2 = P$, with solutions $P_+ = 1$ and $P_- = 0$. Looking at this relationship as a quadratic equation with roots P_+ and and paying attention to non-commutative order of multiplication, we have

$$(P - P_+)(P - P_-) = P^2 - (PP_- + P_+P) + P_+P_- = P^2 - P = 0$$

We see that the product of roots is zero ($P_+P_- = 0$). From the linear terms, we see $(PP_- + P_+P) = P$. Because multiplication is usually non-commutative, we cannot simply factor out P in the equation above. However, because we have $P^2 = P$ as part of the problem statement, we can do the following.

$$\begin{aligned}
PP_- + P_+P &= P \\
PP_-P + P_+P^2 &= P^2 && \text{postmultiply by } P \\
P^2P_-P + PP_+P^2 &= P^3 && \text{premultiply by } P \\
PP_-P + PP_+P &= P^2 && \text{selectively downshift exponents} \\
P(P_-P + P_+P) &= P^2 && \text{factor out forefactor} \\
P(P_- + P_+)P &= P^2 && \text{factor out postfactor} \\
(P_- + P_+) &= 1 && \text{is one solution of the above}
\end{aligned}$$

We have some clues about the form of our idempotents from the quadraric expressions. Define symmetrical and antisymmetrical combinations of our roots.

$$\begin{aligned}
S &= \frac{P_+ + P_-}{2} \\
A &= \frac{P_+ - P_-}{2} \\
P_+ &= S + A \\
P_- &= S - A
\end{aligned}$$

Form the square of these terms, and remember $P_+P_- = 0$

$$\begin{aligned}
S^2 &= \frac{P_+^2 + P_+P_- + P_-P_+ + P_-^2}{4} = \frac{P_+^2 + P_-^2}{4} = \frac{P_+ + P_-}{4} = \frac{S}{2} \\
A^2 &= \frac{P_+^2 - P_+P_- - P_-P_+ + P_-^2}{4} = \frac{P_+^2 + P_-^2}{4} = \frac{P_+ + P_-}{4} = \frac{S}{2}
\end{aligned}$$

We have the nice result that

$$S^2 = A^2 = \frac{S}{2}$$

Expand our null product,

$$\begin{aligned}
P_+P_- &= 0 \\
(S + A)(S - A) &= S^2 - SA + AS - A^2 = 0 \\
&= -SA + AS = 0 \quad \text{(because } S^2 = A^2\text{)} \\
AS &= SA
\end{aligned}$$

We now know that $AS = SA$. Now, expand our squares of the roots.

$$\begin{aligned} P_+^2 &= P_+ \\ S^2 + SA + AS + A^2 &= S + A \\ \frac{S}{2} + 2SA + \frac{S}{2} &= S + A \\ SA &= \frac{A}{2} \end{aligned}$$

We now know more specifically that $AS = SA = A/2$. Summarizing,

$$\begin{aligned} S^2 = A^2 &= \frac{S}{2} \\ AS = SA &= \frac{A}{2} \\ P_+ &= S + A \\ P_- &= S - A \end{aligned}$$

If we restrict S to be a real number, then $S = 0$ or $S = 1/2$. More exciting is to let S be a scaled multivector idempotent, with A being S scaled by a unitary, commutating factor. (We may find an example in planar plus time subsets of Minkowki spacetime.)

Null factors are terms which have a zero product, such as P_+ and P_- above. Obviously, zero is a null factor, but vector products and the geometric product also have non-zero null factors. If we can express functions in terms of a spectrum of null factors, calculations are enormously simplified. Consequently, we seek out and get excited when we find useful null factors.

Solving $P^2 = P$ yielded two solutions, P_+ and P_-. Between themselves, these solutions commute, with a product of zero. These solutions are our primal null factors.

More general null factors are made up of the geometric product of arbitrary multivectors with our idempotents terms. For example, let $A = MP_+$ and $B = P_-N$. The product $AB = MP_+P_-N = 0$. However, revealing our order of multiplication dependancy, $BA = P_-NMP_+$ is generally *not* zero.

Sandwich products are a chain of three multiplications, typically involving an operation such as reflection, translation or the like. A sandwich product such as $A = P_+MP_-$, where the three factors don't commute, creates a nilpotent, a non-zero item which squares to zero.

$$\begin{aligned} A &= P_+ M P_- \\ A^2 &= (P_+ M P_-)(P_+ M P_-) \\ &= (P_+ M)(P_- P_+)(M P_-) \\ &= (P_+ M)0(M P_-) = 0 \end{aligned}$$

As an aside, the special case of a sandwich product of idempotents where the middle term commutes with the idempotents is automatically zero. $A = P_+ C P_- = C(P_+ P_-) = C * 0 = 0$.

Nilpotents and Idempotents in Calculations

Math and physics use polynomials to represent non-linear functions, as well as solutions to differential equations. When nilpotents and idempotents enter into polynomials, some amazing simplifications occur.

Start with a nilpotent, such as $z = e_x$ which squares to zero. An example is a basis vector, such as e_x, under the wedge product, where $e_x \wedge e_x = 0$. A multivector made from a real number r combined with z has the following powers.

$$\begin{aligned} (r+z)^2 &= r^2 + 2rz + z^2 \\ &= r^2 + 2rz \\ (r+z)^3 &= (r+z) * (r^2 + 2rz) \\ &= r^3 + 3r^2 z \\ (r+z)^4 &= r^4 + 4r^3 z \end{aligned}$$

What is happening here, is that in our binomial expansion, all terms with the power of z greater than one, drop out. We have a structure which is stubbornly linear in z, no matter how we try to non-linearly manipulate it. This is a defining characteristic of wavefunctions in quantum mechanics.

How about taking the exponential of this combination?

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \\
e^{(r+z)} &= 1 + (r+z) + \frac{(r+z)^2}{2!} + \frac{(r+z)^3}{3!} + \frac{(r+z)^4}{4!} + \ldots \\
&= 1 + (r+z) + \frac{r^2 + 2rz}{2!} + \frac{r^3 + 3r^2 z}{3!} + \frac{r^4 + 4r^3 z}{4!} + \ldots \\
e^{(r+z)} &= \left[1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \frac{r^4}{4!} + \ldots\right] + \\
&\quad z\left[1 + \frac{2r}{2!} + \frac{3r^2}{3!} + \frac{4r^3}{4!} + \ldots\right] \\
e^{(r+z)} &= \left[1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \frac{r^4}{4!} + \ldots\right] + \\
&\quad z\left[1 + \frac{r}{1} + \frac{r^2}{2!} + \frac{r^3}{3!} + \ldots\right] \\
e^{(r+z)} &= e^r + ze^r = (1+z)e^r
\end{aligned}$$

This same result is consistent with the series expansion, $e^z = (1+z)$, with all squared and higher powers of z equal to zero.

Now we look at powers of real and idempotent combinations. I will use the letter P for my idempotent, with $P^2 = P$ the defining condition. The common example of an idempotent are the numbers $0^2 = 0$ and $1^2 = 1$ under simple multiplication, and the multivector $P = (1 + e_x)/2$ under the geometric product.

$$\begin{aligned}
(r+P)^2 &= r^2 + 2rP + P^2 \\
&= r^2 + (2r+1)P \\
(r+P)^3 &= r^3 + 3r^2 P + 3rP^2 + P^3 \\
&= r^3 + (3r^2 + 3r + 1)P \\
(r+P)^4 &= r^4 + 4r^3 P + 6r^2 P^2 + 4rP^3 + P^4 \\
&= r^4 + (4r^3 + 6r^2 + 4r + 1)P
\end{aligned}$$

We see again, stubbornly linear behavior, this time in P.

Let's take the exponential of P.

$$\begin{aligned}
e^P &= 1 + P + \frac{P^2}{2!} + \frac{P^3}{3!} + \frac{P^4}{4!} + \dots \\
&= 1 + P + \frac{P}{2!} + \frac{P}{3!} + \frac{P}{4!} + \dots \\
&= 1 + P\left(1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots\right) \\
&= 1 + P(e-1)
\end{aligned}$$

With this as guidance, we expect

$$e^{(r+P)} = e^r + Pe^r(e-1)$$

Next, examine the case of two orthogonal idempotents. In geometric algebra, these terms arise in the simplest form as expressions involving unit directions, such as a_x, in the form

$$\begin{aligned}
P_+ &= \frac{1}{2} + \frac{a_x}{2} \\
P_- &= \frac{1}{2} - \frac{a_x}{2} \qquad \text{where} \\
a_x^2 &= 1
\end{aligned}$$

Verifying,

$$\begin{aligned}
(P_+)^2 &= \left(\frac{1}{2} + \frac{a_x}{2}\right)^2 = \frac{1}{4} + 2\frac{1}{2}\frac{a_x}{2} + \frac{a_x^2}{4} = \frac{1}{2} + \frac{a_x}{2} = P_+ \\
(P_-)^2 &= \left(\frac{1}{2} - \frac{a_x}{2}\right)^2 = \frac{1}{4} - 2\frac{1}{2}\frac{a_x}{2} + \frac{a_x^2}{4} = \frac{1}{2} - \frac{a_x}{2} = P_- \\
(P_+)(P_-) &= \left(\frac{1}{2} + \frac{a_x}{2}\right)\left(\frac{1}{2} - \frac{a_x}{2}\right) = \frac{1}{4} - \frac{1}{4} = 0
\end{aligned}$$

Now look at a general combination of two orthogonal idempotents.

$$\begin{aligned}
(aP_+ + bP_-)^2 &= a^2P_+^2 + 2abP_+P_- + b^2P_-^2 \\
&= a^2P_+ + b^2P_- \\
(aP_+ + bP_-)^3 &= a^3P_+^3 + 3a^2P_+^2bP_- + 3aP_+b^2P_-^2 + b^3P_-^3 \\
&= a^3P_+ + b^3P_- \\
(aP_+ + bP_-)^n &= a^nP_+ + b^nP_-
\end{aligned}$$

Let's take the exponential of this combination.

$$\begin{aligned}
e^{(aP_+ + bP_-)} &= 1 + (aP_+ + bP_-) + \frac{(aP_+ + bP_-)^2}{2!} + \frac{(aP_+ + bP_-)^3}{3!} + \ldots \\
&= 1 + (aP_+ + bP_-) + \frac{(a^2P_+ + b^2P_-)}{2!} + \frac{(a^3P_+ + b^3P_-)}{3!} + \ldots \\
&= 1 + P_+(-1 + e^a) + P_-(-1 + e^b) \\
&= 1 - (P_+ + P_-) + e^aP_+ + e^bP_- \\
&= e^aP_+ + e^bP_-
\end{aligned}$$

We see again, that idempotents and nilpotents keep the system linear with regard to these potents, regardless of our nonlinear polynomial abuse.

Two Dimensional Euclidean Algebra

I assume basic familiarity with geometric algebra. For two and three dimensions, I use Euclidean algebra with positive metric such that the basis vectors square to positive unity ($e_xe_x = e_ye_y = 1$).

In two dimensions, the generic multivector is

$$g = a + be_x + ce_y + de_xe_y$$

Two Dimensional Square

The square of this generic 2D multivector is

$$\begin{aligned}
g^2 = gg &= (a^2 + b^2 + c^2 - d^2) \\
&+(2ab)e_x \\
&+(2ac)e_y \\
&+(2ad)e_xe_y
\end{aligned}$$

Two Dimensional Determinant

For the multivector $g = a + be_x + ce_y + de_xe_y$, the determinant is

$$\det = (a^2 - b^2) + (d^2 - c^2)$$

.

For zero determinant, we require $b^2+c^2 = a^2+d^2$. Using polar coordinates for the xy plane, I have a generic null determinant multivector of

$$\begin{aligned} r^2 &= b^2 + c^2 \\ x &= b = r\cos\theta \\ y &= c = r\sin\theta \\ a &= r\cos\phi \\ d &= r\sin\phi \\ N &= r\cos\phi + (r\cos\theta)a_x + (r\sin\theta)a_y + (r\sin\phi)a_{xy} \end{aligned}$$

Notice that $\det(g) + g^2 = 2ag$. When $\det(g) = 0$, $g^2 = 2ag$. This self-similarity is a generic potent and eigenvalue expression. Looking in more detail at the determinant,

$$\begin{aligned} \det(g) + g^2 &= 2ag \\ \det(g) &= -g^2 + 2ag \\ &= g(-g + 2a) \\ &= (a + be_x + ce_y + de_xe_y)(a - be_x - ce_y - de_xe_y) \end{aligned}$$

This is the product of the multivector and its Clifford conjugation.

Two Dimensional Idempotent

To find our generic idempotent, where $P^2 = P$, we equate components of the square and original multivector. $P = a + be_x + ce_y + de_xe_y$.

$$\begin{aligned} a &= (a^2 + b^2 + c^2 - d^2) \\ b &= +(2ab)e_x \\ c &= +(2ac)e_y \\ d &= +(2ad)e_xe_y \end{aligned}$$

For the case where $b = c = d = 0$, we have $a = 0$ and $a = 1$ as our standard numerical idempotents. However, when b, c and d are not zero, we must have $a = 1/2$, and $b^2 + c^2 - d^2 = 1/4$. This later form I will call our generic idempotent P. Idempotents require a minimum radius of 1/2 in the spatial plane to counterbalance the scalar term of $a = 1/2$.

$$P = \frac{1}{2} + be_x + ce_y \pm \left(\sqrt{b^2 + c^2 - \frac{1}{4}}\right) e_{xy}$$

When implemented using 2x2 matrices, the two dimensional determinant of a multivector is $\det = a^2 - b^2 - c^2 + d^2$. For the generic idempotent, this determinant is zero.

Given an idempotent P, we find $1 - P$ is also an idempotent. To verify, $(1-P)^2 = 1 + P^2 - 2P = 1 + P - 2P = 1 - P$. Further, this idempotent times the original idempotent gives a zero product. To verify, $P(1-P) = P - P^2 = P - P = 0$. Consequently, we commonly use these pairs, and have a notation

$$\begin{aligned} P_+ &= \frac{1}{2} + be_x + ce_y \pm \left(\sqrt{b^2 + c^2 - \frac{1}{4}}\right) e_{xy} \\ P_- &= (1 - P_+) = \frac{1}{2} - be_x - ce_y \mp \left(\sqrt{b^2 + c^2 - \frac{1}{4}}\right) e_{xy} \end{aligned}$$

Two Dimensional Nilpotent

To find our generic nilpotent, we equate components to zero

$$\begin{aligned} 0 &= (a^2 + b^2 + c^2 - d^2) \\ 0 &= +e_x(2ab) \\ 0 &= +e_y(2ac) \\ 0 &= +e_x e_y(2ad) \end{aligned}$$

We see that $a = 0$, and $b^2 + c^2 = d^2$, with no minimum radius, unlike idempotents. Our generic nilpotent Z is

$$Z = be_x + ce_y \pm \left(\sqrt{b^2 + c^2}\right) e_{xy}$$

Like idempotents, our determinant is zero.

As an example nilpotent, take the case of $b = 1$, $c = 0$, $Z = e_x + e_{xy}$. By direct calculation we see

$$Z^2 = e_x e_x + e_x e_{xy} + e_{xy} e_x + e_{xy} e_{xy} = 1 + e_y - e_y - 1 = 0$$

Remembering zerofactors from the idempotents above, we look at this nilpotent for such a pattern.

$$\begin{aligned}
e_x + e_{xy} &= e_x\left(1+e_y\right) \\
&= 2e_x\left(\frac{1}{2}+\frac{e_y}{2}\right) \\
(e_x + e_{xy})^2 &= \left[2e_x\left(\frac{1}{2}+\frac{e_y}{2}\right)\right]\left[2e_x\left(\frac{1}{2}+\frac{e_y}{2}\right)\right] \\
&= 2e_x\left(\frac{1}{2}+\frac{e_y}{2}\right)2e_x\left(\frac{1}{2}+\frac{e_y}{2}\right) \\
&= 2e_x 2e_x\left(\frac{1}{2}-\frac{e_y}{2}\right)\left(\frac{1}{2}+\frac{e_y}{2}\right) \\
&= 4\left(\frac{1}{2}-\frac{e_y}{2}\right)\left(\frac{1}{2}+\frac{e_y}{2}\right)
\end{aligned}$$

I suspect all nilpotents will have idempotent pairs revealed by product order transposition.

Comparison of 2D Potent, Idempotent and Nilpotents

Our generic potent is

$$\begin{aligned}
N &= r\cos\phi + (r\cos\theta)a_x + (r\sin\theta)a_y + (r\sin\phi)a_{xy} \\
N^2 &= (2r\cos\phi)N
\end{aligned}$$

Our generic idempotent is

$$\begin{aligned}
P_+ &= \frac{1}{2} + be_x + ce_y \pm \left(\sqrt{b^2+c^2-\frac{1}{4}}\right)e_{xy} \\
P_- &= \frac{1}{2} - be_x - ce_y \mp \left(\sqrt{b^2+c^2-\frac{1}{4}}\right)e_{xy} \\
P^2 &= P
\end{aligned}$$

Our generic nilpotent is

$$\begin{aligned}
Z &= be_x + ce_y \pm \left(\sqrt{b^2+c^2}\right)e_{xy} \\
Z^2 &= 0
\end{aligned}$$

Three Dimensional Euclidean Algebra

The generic three dimensional multivector is $M = a + be_x + ce_y + de_z + ee_{xy} + fe_{xz} + ge_{yz} + he_{xyz}$. The trivector e_{xyz} squares to -1, and commutes with all other multivector components, and thus matches the behavior of $i = \sqrt{-1}$.

We sometimes re-write the three dimensional multivector as a complexified four-vector, where the four-vector is made from the scalar and three vector.

$$\begin{aligned} M &= a + be_x + ce_y + de_z + ee_{xy} + fe_{xz} + ge_{yz} + he_{xyz} \\ &= (a + be_x + ce_y + de_z) + e_{xyz}\,(h + ge_x - fe_y + ee_z) \end{aligned}$$

A different point of view separates the terms into even and odd grades. This becomes the biquaternions of the 1880's.

$$\begin{aligned} M &= a + be_x + ce_y + de_z + ee_{xy} + fe_{xz} + ge_{yz} + he_{xyz} \\ &= (a + ee_{xy} + fe_{xz} + ge_{yz}) + e_{xyz}\,(h - de_{xy} + ce_{xz} - be_{yz}) \end{aligned}$$

This biquaternion approach is a better fit for the determinant and potent formulas which follow.

Three Dimension Determinant

The determinant, in a complexified two dimensional implementation, is

$$\det = \left(a^2 - b^2 - c^2 - d^2 + e^2 + f^2 + g^2 - h^2\right) + e_{xyz}\,(2ah - 2bg + 2cf - 2de)$$

Grouping terms in a suggestive fashion, the scalar portion of the complexified determinant becomes

$$(a^2 + e^2 + f^2 + g^2) - (h^2 + d^2 + c^2 + b^2)$$

For a zero determinant, this requires that quaternions $(a + ee_{xy} + fe_{xz} + ge_{yz})$ and $(h - de_{xy} + ce_{xz} - be_{yz})$ have equal magnitude.

The imaginary component of the complexified determinant is

$$2(ah - bg + cf - de)$$

For a zero determinant, this requires two quaternions $(a + ee_{xy} + fe_{xz} + ge_{yz})$ and $(h - de_{xy} + ce_{xz} - be_{yz})$ be orthogonal.

Three Dimensional Square

Squaring the multivector, we have

$$
\begin{aligned}
M &= a + be_x + ce_y + de_z + ee_{xy} + fe_{xz} + ge_{yz} + he_{xyz} \\
s &= M^2 \\
s.q &= = (a^2 + b^2 + c^2 + d^2) - (h^2 + g^2 + f^2 + e^2) \\
s.x &= = 2(ab - gh) \\
s.y &= = 2(ac + fh) \\
s.z &= = 2(ad - eh) \\
s.xy &= = 2(ae + dh) \\
s.xz &= = 2(af - ch) \\
s.yz &= = 2(ag + bh) \\
s.xyz &= = 2(ah + bg - cf + de)
\end{aligned}
$$

Note that $\det(M) + M^2 = 2aM + 2h(M * e_{xyz}) = M(2a + 2he_{xyz})$. When the determinant is zero, $M^2 = M(2a + 2he_{xyz})$. The square is proportional to the linear term scaled by a commutating multivector.

Three Dimensional Idempotent

Squaring the multivector, and equating components, we have

$$
\begin{aligned}
M &= a + be_x + ce_y + de_z + ee_{xy} + fe_{xz} + ge_{yz} + he_{xyz} \\
s &= M^2 \\
s.q &= a = (a^2 + b^2 + c^2 + d^2) - (h^2 + g^2 + f^2 + e^2) \\
s.x &= b = 2(ab - gh) \\
s.y &= c = 2(ac + fh) \\
s.z &= d = 2(ad - eh) \\
s.xy &= e = 2(ae + dh) \\
s.xz &= f = 2(af - ch) \\
s.yz &= g = 2(ag + bh) \\
s.xyz &= h = 2(ah + bg - cf + de)
\end{aligned}
$$

From the vector and bivector components, we find by substitution

$$\begin{aligned} b = 2(ab - gh) &\rightarrow b(1-2a) = -2gh \\ g = 2(ag + bh) &\rightarrow g(1-2a) = 2bh \\ b(1-2a)^2 = -2g(1-2a)h = -4bh^2 &\rightarrow (1-2a)^2 = -4h^2 \end{aligned}$$

Keeping our numbers real forces $h = 0$, and in turn, $a = 1/2$. Our vector and bivector components are now trivially satisfied. From the trivector component, we have the vector and bivector perpendicular (using my e_{xz} convention). From the scalar component, we have $(b^2+c^2+d^2) = 1/4+(g^2+f^2+e^2)$. The minimum radius for the vector component indicates that the bivector is more fundamental than the vector for this expression. The bivector components can be any value whatsoever. However, the radius must greater than 1/2.

My recipe for generating idempotents begins with three independent numbers for the bivector component W. A unit vector sets an arbitrary direction for a vector u. This is equivalent to two more degrees of freedom, such as angles for azimuth and longitude. The product $V = uW - Wu$ is a pure vector, mutually perpendicular to $\vec{u}$ and bivector W, with the trivector component identically zero. We set the magnitude of this vector to $(1/4 + |W|)$. Setting the scalar component to 1/2, and the trivector component to zero completes this process.

The fixed difference in magnitude between bivector and vector components is a natural fit for hyperbolic functions. Using the hyperbolic functions, our generic projection idempotent with unit vector $\vec{u}$ and orthogonal unit bivector U becomes

$$\begin{aligned} p &= \frac{1}{2} \pm \frac{1}{2}\left(\vec{u}\cosh(t) + U\sinh(t)\right) \\ U \perp \vec{u} &\rightarrow U\vec{u} + \vec{u}U = 0 \end{aligned}$$

where

$$\begin{aligned} W &= \text{Arbitrary three component bivector} \\ U &= \frac{W}{|W|} \\ t &= \sinh^{-1}(2|W|) \\ w &= \vec{r}W - W\vec{r} \qquad \text{arbitrary vector } \vec{r} \\ u &= \frac{\vec{w}}{|w|} \end{aligned}$$

Like the 2D case, we have annihilating idempotents.

$$\begin{aligned} p_+ &= \frac{1}{2} + \frac{1}{2}\left(\vec{u}\cosh(t) + U\sinh(t)\right) \\ p_- &= \frac{1}{2} - \frac{1}{2}\left(\vec{u}\cosh(t) + U\sinh(t)\right) \end{aligned}$$

Three Dimensional Nilpotent

Squaring the multivector, and equating components to zero, we have

$$\begin{aligned} M &= a + be_x + ce_y + de_z + ee_{xy} + fe_{xz} + ge_{yz} + he_{xyz} \\ s &= M^2 \\ s.q &= 0 = (a^2 + b^2 + c^2 + d^2) - (h^2 + g^2 + f^2 + e^2) \\ s.x &= 0 = 2(ab - gh) \\ s.y &= 0 = 2(ac + fh) \\ s.z &= 0 = 2(ad - eh) \\ s.xy &= 0 = 2(ae + dh) \\ s.xz &= 0 = 2(af - ch) \\ s.yz &= 0 = 2(ag + bh) \\ s.xyz &= 0 = 2(ah + bg - cf + de) \end{aligned}$$

The easy solution is to set $a = h = 0$, with $(b^2 + c^2 + d^2) = (g^2 + f^2 + e^2)$ and $(bg - cf + de) = 0$. This corresponds to mutually perpendicular vector and bivector components, with equal magnitudes.

Attempting to use (a, b, c, d) and (h, g, f, e) as equal magnitude, orthogonal fourvectors leads to the requirement that $a^2 = -h^2$ $(= 0)$ from the middle vector and bivector terms.

Determinants and Clifford Conjugation in 3D

Look at a generic 3D multivector and the square.

$$
\begin{aligned}
M &= a + be_x + ce_y + de_z + ee_{xy} + fe_{xz} + ge_{yz} + he_{xyz} \\
s &= M^2 \\
s.q &= (a^2 + b^2 + c^2 + d^2) - (h^2 + g^2 + f^2 + e^2) \\
s.x &= 2(ab - gh) \\
s.y &= 2(ac + fh) \\
s.z &= 2(ad - eh) \\
s.xy &= 2(ae + dh) \\
s.xz &= 2(af - ch) \\
s.yz &= 2(ag + bh) \\
s.xyz &= 2(ah + bg - cf + de)
\end{aligned}
$$

Inside this square, we easily see a contribution of $a * M = M.q * M$. We also detect a component $h * e_{xyz} * M$. Looking at the square minus these two terms, we find

$$
\begin{aligned}
C &= M.q + M.xyze_{xyz} \\
W &= M * M - C * M \\
&= (-a^2 + b^2 + c^2 + d^2 - e^2 - f^2 - g^2 + h^2) + 2 * (-ah + bg - cf + de)e_{xyz}
\end{aligned}
$$

This last expression is the determinant in complex number format.

Look at this a bit more, now

$$
\begin{aligned}
\det(M) &= M^2 - CM \\
&= M(M - C) \\
&= (a,\ b, c, d,\ e, f, g,\ h)\,[(a,\ b, c, d,\ e, f, g,\ h) - 2(a,\ 0, 0, 0,\ 0, 0, 0, h)] \\
&= (a,\ b, c, d,\ e, f, g,\ h)(-a,\ b, c, d,\ e, f, g,\ -h) \\
&= -(a,\ b, c, d,\ e, f, g,\ h)(a,\ -b, -c, -d,\ -e, -f, -g,\ h) \\
\det(M) &= -M * \text{CliffordConjugation}(M)
\end{aligned}
$$

Four Dimensional Euclidean Algebra

The four dimensional Euclidean geometric algebra has four anti-commuting basis vectors e_x, e_y, e_z, and e_w which square to one. We have six bivectors, four trivectors, and one quadvector, which squares to positive one, and anti-commutes with vectors and trivectors while commuting with scalars and bivectors. My generic multivector is

$$\begin{aligned} M &= a + (be_x + ce_y + de_z + ee_w) + (fe_{xy} + ge_{xz} + he_{yz} + ie_{xw} + je_{yw} + ke_{zw}) \\ &\quad +(le_{xyz} + me_{xyw} + ne_{xzw} + oe_{yzw}) + pe_{xyzw} \end{aligned}$$

The parenthesis above are used to group vector, bivector and trivector terms. To save space, I often write this multivector in function form as used in C, such as `M = GA4E(a, b,c,d,e, f,g,h,i,j,k, l,m,n,o, p)`. (GA4E stands for Geometric Algebra, 4 dimensional, Euclidean.)

Four Dimensional Determinant

The expanded formula for the four dimensional determinant occupies a half page, and is not very informative. Instead, I use formulas based upon the product of the multivector and either the reverse, or the Clifford conjugation.

I have four variations, all of which lead to the same determinant.

```
r = ( a, b,c,d,e, f,g,h,i,j,k, l,m,n,o, p)
Reverse(r) = ( a, b,c,d,e, -f,-g,-h,-i,-j,-k, -l,-m,-n,-o, p)
(t = r*Reverse(r) )
t.q    = +a^2+(b^2+c^2+d^2+e^2)+(f^2+g^2+h^2+i^2+j^2+k^2)+(l^2+m^2+n^2+o^2)+p^2
t.x    = +2*a*b+2*c*f+2*d*g+2*e*i+2*h*l+2*j*m+2*k*n+2*o*p
t.y    = +2*a*c-2*b*f+2*d*h+2*e*j-2*g*l-2*i*m+2*k*o-2*n*p
t.z    = +2*a*d-2*b*g-2*c*h+2*e*k+2*f*l-2*i*n-2*j*o+2*m*p
t.w    = +2*a*e-2*b*i-2*c*j-2*d*k+2*f*m+2*g*n+2*h*o-2*l*p
t.xyzw = +2*a*p-2*b*o+2*c*n-2*d*m+2*e*l-2*f*k+2*g*j-2*h*i
Determinant = t.q^2 - t.x^2 - t.y^2 - t.z^2 - t.w^2 - t.xyzw^2

(u = Reverse(r)*r )
u.q    = +a^2+(b^2+c^2+d^2+e^2)+(f^2+g^2+h^2+i^2+j^2+k^2)+(l^2+m^2+n^2+o^2)+p^2
u.x    = +2*a*b-2*c*f-2*d*g-2*e*i+2*h*l+2*j*m+2*k*n-2*o*p
u.y    = +2*a*c+2*b*f-2*d*h-2*e*j-2*g*l-2*i*m+2*k*o+2*n*p
u.z    = +2*a*d+2*b*g+2*c*h-2*e*k+2*f*l-2*i*n-2*j*o-2*m*p
u.w    = +2*a*e+2*b*i+2*c*j+2*d*k+2*f*m+2*g*n+2*h*o+2*l*p
u.xyzw = +2*a*p+2*b*o-2*c*n+2*d*m-2*e*l-2*f*k+2*g*j-2*h*i
Determinant = u.q^2 - u.x^2 - u.y^2 - u.z^2 - u.w^2 - u.xyzw^2
```

```
CliffordConjugation(r) = ( a, -b,-c,-d,-e, -f,-g,-h,-i,-j,-k, l,m,n,o, p)
(v = r*CliffordConjugation(r) )
v.q    = +a^2-(b^2+c^2+d^2+e^2)+(f^2+g^2+h^2+i^2+j^2+k^2)-(l^2+m^2+n^2+o^2)+p^2
v.xyz  = +2*a*l-2*b*h+2*c*g-2*d*f-2*e*p+2*i*o-2*j*n+2*k*m
v.xyw  = +2*a*m-2*b*j+2*c*i+2*d*p-2*e*f+2*h*n-2*g*o-2*k*l
v.xzw  = +2*a*n-2*b*k-2*c*p+2*d*i-2*e*g+2*f*o-2*h*m+2*j*l
v.yzw  = +2*a*o+2*b*p-2*c*k+2*d*j-2*e*h-2*f*n+2*g*m-2*i*l
v.xyzw = +2*a*p+2*b*o-2*c*n+2*d*m-2*e*l-2*f*k+2*g*j-2*h*i
Determinant = v.q^2 + v.xyz^2 + v.xyw^2 + v.xzw^2 + v.yzw^2 - v.xyzw^2

(W = CliffordConjugation(r)*r )
W.q    = +a^2-(b^2+c^2+d^2+e^2)+(f^2+g^2+h^2+i^2+j^2+k^2)-(l^2+m^2+n^2+o^2)+p^2
W.xyz  = +2*a*l-2*b*h+2*c*g-2*d*f+2*e*p-2*i*o+2*j*n-2*k*m
W.xyw  = +2*a*m-2*b*j+2*c*i-2*d*p-2*e*f-2*h*n+2*g*o+2*k*l
W.xzw  = +2*a*n-2*b*k+2*c*p+2*d*i-2*e*g-2*f*o+2*h*m-2*j*l
W.yzw  = +2*a*o-2*b*p-2*c*k+2*d*j-2*e*h+2*f*n-2*g*m+2*i*l
W.xyzw = +2*a*p-2*b*o+2*c*n-2*d*m+2*e*l-2*f*k+2*g*j-2*h*i
Determinant = W.q^2 + W.xyz^2 + W.xyw^2 + W.xzw^2 + W.yzw^2 - W.xyzw^2
```

With the three dimensional determinant, we had a positive sum of two squares formula which had each term independently zero for a zero determinant. We do not have this luxury in the formulas above. Instead, the determinant looks like the difference between a pseudo five dimensional vector magnitude and a scalar.

For the first two forms, the scalar `t.q` and `u.q` is the simple magnitude of the multivector. By contrast, the latter two forms have a scalar component being the difference between even and odd grade components of the multivector, which is interesting in its own right. The pseudo-scalar component `t.xyzw` and `W.xyzw` of the first and fourth forms match, as well as the second and third components `u.xyzw` and `v.xyzw`.

Interesting Specialized Forms

As the idempotents and nilpotents have typically zero determinant, and as we commonly work with simpler multivectors, I present a list of my favorite interesting multivectors and determinants.

We can embed a generic three dimensional multivector in four dimensional space at least four ways. The zero determinant clearly indicates the subspace nature of this embedded object.

$$\texttt{MV3D} = a + (be_x + ce_y + de_z) + (fe_{xy} + ge_{xz} + he_{yz}) + ee_{xyz}$$

$$
\begin{aligned}
\texttt{MV3D\_In\_4} &= +a + (be_x + ce_y + de_z + ee_w) \\
&\quad +(fe_{xy} + ge_{xz} + he_{yz} + he_{xw} - ge_{yw} + fe_{zw}) \\
&\quad +(ee_{xyz} - de_{xyw} + ce_{xzw} - be_{yzw}) - ae_{xyzw} \\
\det(\texttt{MV3D\_In\_4}) &= 0
\end{aligned}
$$

In function format, the four embeddings are

```
r1 = GA4E( a, b,c,d,e, f,g,h, h,-g, f,  e,-d, c,-b, -a)
r2 = GA4E( a, b,c,d,e, f,g,h,-h, g,-f,  e,-d, c,-b,  a)
r3 = GA4E( a, b,c,d,e, f,g,h, h,-g, f, -e, d,-c, b, -a)
r4 = GA4E( a, b,c,d,e, f,g,h,-h, g,-f, -e, d,-c, b,  a)
```

Now we present a list of simpler forms and their determinants.

$$
\begin{aligned}
\texttt{MV1} &= a \\
\det(\texttt{MV1}) &= a^4 \\
\texttt{MV2} &= (be_x + ce_y + de_z + ee_w) \\
\det(\texttt{MV2}) &= (b^2 + c^2 + d^2 + e^2)^2 \\
\texttt{MV3} &= (fe_{xy} + ge_{xz} + he_{yz} + ie_{xw} + je_{yw} + ke_{zw}) \\
\det(\texttt{MV3}) &= (f^2 + g^2 + h^2 + i^2 + j^2 + k^2)^2 - (2fk - 2gj + 2hi)^2 \\
\texttt{MV4} &= (le_{xyz} + me_{xyw} + ne_{xzw} + oe_{yzw}) \\
\det(\texttt{MV4}) &= (l^2 + m^2 + n^2 + o^2)^2 \\
\texttt{MV5} &= pe_{xyzw} \\
\det(\texttt{MV5}) &= p^4
\end{aligned}
$$

Now we list some useful combinations.

$$
\begin{aligned}
\texttt{MV6} &= a + (be_x + ce_y + de_z + ee_w) + pe_{xyzw} \\
\det(\texttt{MV6}) &= (+a^2 - b^2 - c^2 - d^2 - e^2 - p^2)^2 \\
\texttt{MV7} &= a + (le_{xyz} + me_{xyw} + ne_{xzw} + oe_{yzw}) + pe_{xyzw} \\
\det(\texttt{MV7}) &= (+a^2 + l^2 + m^2 + n^2 + o^2 - p^2)^2 \\
\texttt{MV8} &= a + (fe_{xy} + ge_{xz} + he_{yz} + ie_{xw} + je_{yw} + ke_{zw}) + pe_{xyzw} \\
\det(\texttt{MV8}) &= (a^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + p^2)^2 - (-2ap + 2fk - 2gj + 2hi)^2 \\
&= \big((a^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + p^2) + (-2ap + 2fk - 2gj + 2hi)\big) * \\
&\quad \big((a^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + p^2) - (-2ap + 2fk - 2gj + 2hi)\big) \\
&= \big((a-p)^2 + (f+k)^2 + (g-j)^2 + (h+i)^2\big) * \\
&\quad \big((a+p)^2 + (f-k)^2 + (g+j)^2 + (h-i)^2\big) \\
\texttt{MV9} &= (be_x + ce_y + de_z + ee_w) + (le_{xyz} + me_{xyw} + ne_{xzw} + oe_{yzw}) \\
\det(\texttt{MV9}) &= (b^2 + c^2 + d^2 + e^2 + l^2 + m^2 + n^2 + o^2)^2 - (2bo - 2cn + 2dm - 2el)^2 \\
&= \big((b-o)^2 + (c+n)^2 + (d-m)^2 + (e+l)^2\big) * \\
&\quad \big((b+o)^2 + (c-n)^2 + (d+m)^2 + (e-l)^2\big)
\end{aligned}
$$

Four Dimensional Square

The four dimensional square, in multiline functional format, is

```
r = ( a, b,c,d,e, f,g,h,i,j,k, l,m,n,o, p)
s = r^2 = (
+a^2+(b^2+c^2+d^2+e^2)-(f^2+g^2+h^2+i^2+j^2+k^2)-(l^2+m^2+n^2+o^2)+p^2,

+2*a*b-2*h*l-2*j*m-2*k*n,  +2*a*c+2*g*l+2*i*m-2*k*o,
+2*a*d-2*f*l+2*i*n+2*j*o,  +2*a*e-2*f*m-2*g*n-2*h*o,

+2*a*f+2*d*l+2*e*m-2*k*p,  +2*a*g-2*c*l+2*e*n+2*j*p,  +2*a*h+2*b*l+2*e*o-2*i*p,
+2*a*i-2*c*m-2*d*n-2*h*p,  +2*a*j+2*b*m-2*d*o+2*g*p,  +2*a*k+2*b*n+2*c*o-2*f*p,

+2*a*l+2*b*h-2*c*g+2*d*f,  +2*a*m+2*b*j-2*c*i+2*e*f,
+2*a*n+2*b*k-2*d*i+2*e*g,  +2*a*o+2*c*k-2*d*j+2*e*h,

+2*a*p+2*f*k-2*g*j+2*h*i)
```

Four Dimensional Idempotent

For the idempotent, we set $r^2 = r$. Equating components, we have

```
r = ( a, b,c,d,e, f,g,h,i,j,k, l,m,n,o, p)
s = r^2 = r =>

s.q = a = +a^2+(b^2+c^2+d^2+e^2)-(f^2+g^2+h^2+i^2+j^2+k^2)-(l^2+m^2+n^2+o^2)+p^2,

s.x = b = +2*a*b-2*h*l-2*j*m-2*k*n,
s.y = c = +2*a*c+2*g*l+2*i*m-2*k*o,
s.z = d = +2*a*d-2*f*l+2*i*n+2*j*o,
s.w = e = +2*a*e-2*f*m-2*g*n-2*h*o,

s.xy = f = +2*a*f+2*d*l+2*e*m-2*k*p,
s.xz = g = +2*a*g-2*c*l+2*e*n+2*j*p,
s.yz = h = +2*a*h+2*b*l+2*e*o-2*i*p,
s.xw = i = +2*a*i-2*c*m-2*d*n-2*h*p,
s.yw = j = +2*a*j+2*b*m-2*d*o+2*g*p,
s.zw = k = +2*a*k+2*b*n+2*c*o-2*f*p,

s.xyz = l = +2*a*l+2*b*h-2*c*g+2*d*f,
s.xyw = m = +2*a*m+2*b*j-2*c*i+2*e*f,
s.xzw = n = +2*a*n+2*b*k-2*d*i+2*e*g,
s.yzw = o = +2*a*o+2*c*k-2*d*j+2*e*h,

s.xyzw = p = +2*a*p+2*f*k-2*g*j+2*h*i
```

In three dimensions, we examined the `s.x` and `s.yz` equations, and deduced that the coefficient of e_{xyz} had to be zero to prevent imaginary components, and in turn, this set the scalar $a = 1/2$. In present notation, we required $l = 0$ and $a = 1/2$. In four dimensions, I have not yet found such restrictions, but this is a great place to start.

Seeing that the quadvector squares to one, we use it along with the vectors, and find our first uniquely four dimensional idempotent.

$$\begin{aligned} P_1 &= (1/2) + be_x + ce_y + de_z + ee_w + pe_{xyzw} \\ (1/4) &= b^2 + c^2 + d^2 + e^2 + p^2 \end{aligned}$$

Combining trivectors and quadvector, we have

$$\begin{aligned} P_2 &= (1/2) + le_{xyz} + me_{xyw} + ne_{xzw} + oe_{yzw} + pe_{xyzw} \\ (1/4) &= p^2 - l^2 - m^2 - n^2 - o^2 \end{aligned}$$

I will come back to find more general forms later. This is enough to demonstrate circular and hyperbolic uniquely four dimensional idempotent forms exist.

Four Dimensional Nilpotent

For the nilpotent, we set $r^2 = 0$. Equating components, we have

```
r = ( a, b,c,d,e, f,g,h,i,j,k, l,m,n,o, p)
s = r^2 = 0 =>

s.q = 0 = +a^2+(b^2+c^2+d^2+e^2)-(f^2+g^2+h^2+i^2+j^2+k^2)-(l^2+m^2+n^2+o^2)+p^2,

s.x = 0 = +2*a*b-2*h*l-2*j*m-2*k*n,
s.y = 0 = +2*a*c+2*g*l+2*i*m-2*k*o,
s.z = 0 = +2*a*d-2*f*l+2*i*n+2*j*o,
s.w = 0 = +2*a*e-2*f*m-2*g*n-2*h*o,

s.xy = 0 = +2*a*f+2*d*l+2*e*m-2*k*p,
s.xz = 0 = +2*a*g-2*c*l+2*e*n+2*j*p,
s.yz = 0 = +2*a*h+2*b*l+2*e*o-2*i*p,
s.xw = 0 = +2*a*i-2*c*m-2*d*n-2*h*p,
s.yw = 0 = +2*a*j+2*b*m-2*d*o+2*g*p,
s.zw = 0 = +2*a*k+2*b*n+2*c*o-2*f*p,

s.xyz = 0 = +2*a*l+2*b*h-2*c*g+2*d*f,
s.xyw = 0 = +2*a*m+2*b*j-2*c*i+2*e*f,
s.xzw = 0 = +2*a*n+2*b*k-2*d*i+2*e*g,
s.yzw = 0 = +2*a*o+2*c*k-2*d*j+2*e*h,

s.xyzw = 0 = +2*a*p+2*f*k-2*g*j+2*h*i
```

From the scalar equation, we have

$$a^2+(b^2+c^2+d^2+e^2)+p^2 = (f^2+g^2+h^2+i^2+j^2+k^2)+(l^2+m^2+n^2+o^2)$$

The remaining equations look like a collection of four dimensional orthogonality requirements. In three dimensions, we set the scalar and trivector components to zero.

In these higher dimensions, I find synthesis easier than deduction. A generic nilpotent is $PM(1-P)$, where P is an idempotent, and M is a generic non-commutating multivector. As an example, one nilpotent is

$$\frac{1+e_x}{2}(e_{xyzw})\frac{1-e_x}{2} = \frac{e_{yzw}+e_{xyzw}}{2}$$

Minkowski Algebra

Minkowski algebra is a four dimensional Clifford algebra where the e_x, e_y and e_z basis vector square to positive one, while the time axis e_t squares to negative one. This negative metric on time allows contact transformations, where the effective separation between two events is zero when the space and time magnitudes balance. Exchange of energy in electromagnetism, and the Lorentz transform are consequences of this metric. The Minkowski spacetime models as the direct product of two copies of two dimensional Euclidean geometric algebra. I feel this is very significant.

The multiplication table among Minkowski geometric elements is shown in sidewise Table 1.

Minkowski Determinant

In my computer code, the Minkowski geometric algebra data structures are named Mink. We can easily calculate determinants using either the Reverse function and geometric product, or the Clifford Conjugation function and geometric product.

The Reverse function changes the sign of the bivector and trivector elements, leaving other terms unchanged. The resulting product of the original multivector and the Reverse has ten zeroed elements, and only six nonzero terms, with the bivector and trivector elements eliminated. The code below illustrates this calculation. Notice the sign on the time related term e.

```
//  r = Mink( a, b,c,d,e, f,g,h,i,j,k, l,m,n,o, p)
//  Reverse(r) = Mink( a, b,c,d,e, -f,-g,-h,-i,-j,-k, -l,-m,-n,-o, p)
Mink B, C;
ex a, b, c, d, e, p, det;

B = Reverse(A);
C = A*B;
a = C.q;
b = C.x;
c = C.y;
d = C.z;
e = C.t;

p = C.xyzt;
det = expand(a*a - b*b - c*c - d*d + e*e + p*p);
```

q	x	y	z	t	xy	xz	yz	xt	yt	zt	xyz	xyt	xzt	yzt	xyzt
x	q	xy	xz	xt	y	z	xyz	t	xyt	xzt	yz	yt	zt	xyzt	yzt
y	-xy	q	yz	yt	-x	-xyz	z	-xyt	t	yzt	-xz	-xt	-xyzt	zt	-xzt
z	-xz	-yz	q	zt	xyz	-x	-y	-xzt	-yzt	t	xy	xyzt	-xt	-yt	xyt
t	-xt	-yt	-zt	-q	xyt	xzt	yzt	x	y	z	-xyzt	-xy	-xz	-yz	xyz
xy	-y	x	xyz	xyt	-q	-yz	xz	-yt	xt	xyzt	-z	-t	-yzt	xzt	-zt
xz	-z	-xyz	x	xzt	yz	-q	-xy	-zt	-xyzt	xt	y	yzt	-t	-xyt	yt
yz	xyz	-z	y	yzt	-xz	xy	-q	xyzt	-zt	yt	-x	-xzt	xyt	-t	-xt
xt	-t	-xyt	-xzt	-x	yt	zt	xyzt	q	xy	xz	-yzt	-y	-z	-xyz	yz
yt	xyt	-t	-yzt	-y	-xt	-xyzt	zt	-xy	q	yz	xzt	x	xyz	-z	-xz
zt	xzt	yzt	-t	-z	xyzt	-xt	-yt	-xz	-yz	q	-xyt	-xyz	x	y	xy
xyz	yz	-xz	xy	xyzt	-z	y	-x	yzt	-xzt	xyt	-q	-zt	yt	-xt	-t
xyt	yt	-xt	-xyzt	-xy	-t	-yzt	xzt	-y	x	xyz	zt	q	yz	-xz	-z
xzt	zt	xyzt	-xt	-xz	yzt	-t	-xyt	-z	-xyz	x	-yt	-yz	q	xy	y
yzt	-xyzt	zt	-yt	-yz	-xzt	xyt	-t	xyz	-z	y	xt	xz	-xy	q	-x
xyzt	-yzt	xzt	-xyt	-xyz	-zt	yt	-xt	yz	-xz	xy	t	z	-y	x	-q

Table 1: Minkowski Geometric Algebra Multiplication Table

In Euclidean space, we had four methods of symmetrically embedding a three dimensional multivector. In Minkowski space, I do not find any such embeddings, probably due to the time metric being negative.

Minkowski Square

The Minkowski square, in component form, is given by

```
// Fields = q,  x,y,z,t, xy,xz,yz,xt,yt,zt,  xyz,xyt,xzt,yzt,  xyzt
r = Mink( a, b,c,d,e, f,g,h,i,j,k, l,m,n,o, p) ;
s = r*r ;

s.q = +a^2 +b^2+c^2+d^2-e^2 -f^2-g^2-h^2+i^2+j^2+k^2 -l^2+m^2+n^2+o^2 -p^2 ;

s.x = +2*a*b-2*h*l+2*j*m+2*k*n ;
s.y = +2*a*c+2*g*l-2*i*m+2*k*o ;
s.z = +2*a*d-2*f*l-2*i*n-2*j*o ;
s.t = +2*a*e-2*f*m-2*g*n-2*h*o ;

s.xy = +2*a*f+2*d*l-2*e*m+2*k*p ;
s.xz = +2*a*g-2*c*l-2*e*n-2*j*p ;
s.yz = +2*a*h+2*b*l-2*e*o+2*i*p ;
s.xt = +2*a*i-2*c*m-2*d*n-2*h*p ;
s.yt = +2*a*j+2*b*m-2*d*o+2*g*p ;
s.zt = +2*a*k+2*b*n+2*c*o-2*f*p ;

s.xyz = +2*a*l+2*b*h-2*c*g+2*d*f ;
s.xyt = +2*a*m+2*b*j-2*c*i+2*e*f ;
s.xzt = +2*a*n+2*b*k-2*d*i+2*e*g ;
s.yzt = +2*a*o+2*c*k-2*d*j+2*e*h ;

s.xyzt = +2*a*p+2*f*k-2*g*j+2*h*i ;
```

Minkowski Idempotent

We inherit the three dimensional idempotents without a change. We also have a few Minkowski forms inspired by Reverse and Clifford Conjugation expressions.

Look at `r = Mink( a, b,c,d,e, 0,0,0,0,0,0, 0,0,0,0, p)`.

$$s = r^2 = (+a^2+b^2+c^2+d^2-e^2-p^2, 2ab, 2ac, 2ad, 2ae, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2ap)$$

We see $a = 1/2$, and $b^2 + c^2 + d^2 = 1/4 + e^2 + p^2$ is one idempotent.

Likewise, look at `r = Mink( a, 0,0,0,0, 0,0,0,0,0,0, l,m,n,o, p)`.

$$s = r^2 = (+a^2-l^2+m^2+n^2+o^2-p^2, 0,0,0,0,0,0,0,0,0,0, 2al, 2am, 2an, 2ao, 2ap)$$

We see $a = 1/2$, and $+m^2 + n^2 + o^2 = 1/4 + l^2 + p^2$ is another idempotent.

In Mikowski spacetime, we have a new method for creating idempotents from existing idempotents. We can write idempotents as sums of symmetrical and antisymmetrical terms from quadratic equation solutions with the following properties.

$$\begin{aligned} P_+ &= S + A \\ P_- &= S - A \\ S^2 &= A^2 = S/2 \\ AS &= SA = A/2 \end{aligned}$$

We see from above that S can be a scaled idempotent ($S = P/2$), with $A = S * C$, where C is a commutating factor which squares to one. As an example, $P = (1 + e_x)/2$ is an existing idempotent. A commutating factor with P which squares to one is $C = e_{yt}$. We have $S = (1 + e_x)/4$, $A = Se_{yt} = (e_{yt} + e_{xyt})/4$, and a new set of potents

$$P = (1 + e_x \pm (e_{yt} + e_{xyt}))/4$$

These are interesting in that the scalar term is no longer 1/2, but rather 1/4.

If we do a similar process, but with a non-commutating factor, we get a scaling potent like S above. Adjusting the scale factor for idempotentcy, we have

$$\begin{aligned} r &= P + Pe_{zt} = P(1 + e_{zt}) \\ r &= (e_q + e_x + e_{yz} + e_{yt} + e_{zt} + e_{xyz} + e_{xyt} + e_{xzt})/4 \\ r^2 &= r = (e_q + e_x + e_{yz} + e_{yt} + e_{zt} + e_{xyz} + e_{xyt} + e_{xzt})/4 \end{aligned}$$

From this, we see that $(e_{yz} + e_{zt} + e_{xyz} + e_{xzt})$ is nilpotent, and a null factor for $(1 + e_x + e_{yt} + e_{xyt})$.

Minkowski Nilpotent

Nilpotents square to zero. In component form, this means that

```
s.q = 0 = +a^2 +b^2+c^2+d^2-e^2 -f^2-g^2-h^2+i^2+j^2+k^2 -l^2+m^2+n^2+o^2 -p^2 ;

s.x = 0 = +2*a*b-2*h*l+2*j*m+2*k*n ;
s.y = 0 = +2*a*c+2*g*l-2*i*m+2*k*o ;
s.z = 0 = +2*a*d-2*f*l-2*i*n-2*j*o ;
s.t = 0 = +2*a*e-2*f*m-2*g*n-2*h*o ;

s.xy = 0 = +2*a*f+2*d*l-2*e*m+2*k*p ;
s.xz = 0 = +2*a*g-2*c*l-2*e*n-2*j*p ;
s.yz = 0 = +2*a*h+2*b*l-2*e*o+2*i*p ;
s.xt = 0 = +2*a*i-2*c*m-2*d*n-2*h*p ;
s.yt = 0 = +2*a*j+2*b*m-2*d*o+2*g*p ;
s.zt = 0 = +2*a*k+2*b*n+2*c*o-2*f*p ;

s.xyz = 0 = +2*a*l+2*b*h-2*c*g+2*d*f ;
s.xyt = 0 = +2*a*m+2*b*j-2*c*i+2*e*f ;
s.xzt = 0 = +2*a*n+2*b*k-2*d*i+2*e*g ;
s.yzt = 0 = +2*a*o+2*c*k-2*d*j+2*e*h ;

s.xyzt = 0 = +2*a*p+2*f*k-2*g*j+2*h*i ;
```

Synthesizing nilpotents from idempotents is easy enough. Likewise, making nilpotents from mainly zero multivectors is straightforward as well. First, eliminate the bivector terms.

```
s.q = 0 = +a^2 +b^2+c^2+d^2-e^2 -l^2+m^2+n^2+o^2 -p^2 ;

s.x = 0 = +2*a*b;
s.y = 0 = +2*a*c ;
s.z = 0 = +2*a*d ;
s.t = 0 = +2*a*e ;

s.xy = 0 = +2*d*l-2*e*m ;
s.xz = 0 = -2*c*l-2*e*n ;
s.yz = 0 = +2*b*l-2*e*o ;
s.xt = 0 = -2*c*m-2*d*n ;
s.yt = 0 = +2*b*m-2*d*o ;
s.zt = 0 = +2*b*n+2*c*o ;

s.xyz = 0 = +2*a*l ;
s.xyt = 0 = +2*a*m ;
s.xzt = 0 = +2*a*n ;
s.yzt = 0 = +2*a*o ;

s.xyzt = 0 = +2*a*p ;
```

We see that setting $a = 0$, and further eliminating either the vector or the trivector achieve a nilpotent. One example is

$$\begin{aligned} N_1 &= be_x + ce_y + de_z + ee_t + pe_{xyzt} \\ p^2 + e^2 &= b^2 + c^2 + d^2 \end{aligned}$$

Another example is

$$\begin{aligned} N_2 &= le_{xyz} + me_{xyt} + ne_{xzt} + oe_{yzt} + pe_{xyzt} \\ p^2 + l^2 &= m^2 + n^2 + o^2 \end{aligned}$$

Five Dimensional Minkowksi Algebra

The fifth dimension is profoundly interesting to me, as it is the natural space for the Dirac equation of quantum mechanics, as well as the electro-gravity unification space of Nordstrom, Klein and Kaluza.

Ordinarily, I don't present the matrix representation of geometric algebra, as it is far more efficient to compute in geometric algebra, rather than matrices. Further, matrices, at my current level of understanding, obscure rather than illuminate the physics. However, to help convince others that quantum mechanics is not arcane, but rather poorly presented geometric algebra, I will do a little side track before continuing with my various potents.

In quantum mechanics, in the Dirac basis using a (1,4) signature, the Gamma matrices are defined as

$$\Gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \Gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \Gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \Gamma^5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

These matrices represent a spatial basis for a five dimensional geometric algebra with signature `(- - - - +)`.

I find it easier to write these arrays in computer code format, as

```
Gamma0 = e_t = [[ 1, 0, 0, 0], [ 0, 1, 0, 0], [ 0, 0,-1, 0], [ 0, 0, 0,-1 ]]
Gamma1 = e_x = [[ 0, 0, 0, 1], [ 0, 0, 1, 0], [ 0,-1, 0, 0], [-1, 0, 0, 0 ]]
Gamma2 = e_y = [[ 0, 0, 0,-I], [ 0, 0, I, 0], [ 0, I, 0, 0], [-I, 0, 0, 0 ]]
Gamma3 = e_z = [[ 0, 0, 1, 0], [ 0, 0, 0,-1], [-1, 0, 0, 0], [ 0, 1, 0, 0 ]]
Gamma5 = e_w = [[ 0, 0, 1, 0], [ 0, 0, 0, 1], [ 1, 0, 0, 0], [ 0, 1, 0, 0 ]]
```

From these five matrices, 32 total matrices are created which represent the multivector blades for five dimensional geometric algebra. These 32 matrices, within an overall possible sign change per matrix, support a wide variety of geometric algebras in five space via re-assignments. For example, there are 192 different sets of five dimensional anti-commutating blades among these matrices. The differences between Bjorken Drell, Dirac, Weyl and Majorana equations are due to the signature choices for the space being used to factor the Klein Gordon equation. These choices can be re-interpreted as a choice over which multivector components from which the energy momentum equation arises. Likewise, there are 480 different four dimensional subspaces, and 640 three dimensional subspaces representable by these matrices.

My preferred basis uses a `(+ + + + -)` signature, has real components in the matrices representing Minkowki spacetime, and has a commutating pseudoscalar which maps to $i = \sqrt{-1}$.

```
     w              x              y              z              t
[ 0  0  0 -I]  [ 1  0  0  0]  [ 0  1  0  0]  [ 0  0  0 -1]  [ 0 -1  0  0]
[ 0  0  I  0]  [ 0 -1  0  0]  [ 1  0  0  0]  [ 0  0  1  0]  [ 1  0  0  0]
[ 0 -I  0  0]  [ 0  0  1  0]  [ 0  0  0  1]  [ 0  1  0  0]  [ 0  0  0  1]
[ I  0  0  0]  [ 0  0  0 -1]  [ 0  0  1  0]  [-1  0  0  0]  [ 0  0 -1  0]
```

My blade representations are derived from the five basis above by matrix multiplication.

```
*********** Basis Matrices ************

q     = [[ 1, 0, 0, 0], [ 0, 1, 0, 0], [ 0, 0, 1, 0], [ 0, 0, 0, 1 ]]

w     = [[ 0, 0, 0,-I], [ 0, 0, I, 0], [ 0,-I, 0, 0], [ I, 0, 0, 0 ]]
x     = [[ 1, 0, 0, 0], [ 0,-1, 0, 0], [ 0, 0, 1, 0], [ 0, 0, 0,-1 ]]
y     = [[ 0, 1, 0, 0], [ 1, 0, 0, 0], [ 0, 0, 0, 1], [ 0, 0, 1, 0 ]]
z     = [[ 0, 0, 0,-1], [ 0, 0, 1, 0], [ 0, 1, 0, 0], [-1, 0, 0, 0 ]]
t     = [[ 0,-1, 0, 0], [ 1, 0, 0, 0], [ 0, 0, 0, 1], [ 0, 0,-1, 0 ]]

wx    = [[ 0, 0, 0, I], [ 0, 0, I, 0], [ 0, I, 0, 0], [ I, 0, 0, 0 ]]
```

```
wy    = [[ 0, 0,-I, 0], [ 0, 0, 0, I], [-I, 0, 0, 0], [ 0, I, 0, 0 ]]
wz    = [[ I, 0, 0, 0], [ 0, I, 0, 0], [ 0, 0,-I, 0], [ 0, 0, 0,-I ]]
wt    = [[ 0, 0, I, 0], [ 0, 0, 0, I], [-I, 0, 0, 0], [ 0,-I, 0, 0 ]]
xy    = [[ 0, 1, 0, 0], [-1, 0, 0, 0], [ 0, 0, 0, 1], [ 0, 0,-1, 0 ]]
xz    = [[ 0, 0, 0,-1], [ 0, 0,-1, 0], [ 0, 1, 0, 0], [ 1, 0, 0, 0 ]]
xt    = [[ 0,-1, 0, 0], [-1, 0, 0, 0], [ 0, 0, 0, 1], [ 0, 0, 1, 0 ]]
yz    = [[ 0, 0, 1, 0], [ 0, 0, 0,-1], [-1, 0, 0, 0], [ 0, 1, 0, 0 ]]
yt    = [[ 1, 0, 0, 0], [ 0,-1, 0, 0], [ 0, 0,-1, 0], [ 0, 0, 0, 1 ]]
zt    = [[ 0, 0, 1, 0], [ 0, 0, 0, 1], [ 1, 0, 0, 0], [ 0, 1, 0, 0 ]]

wxy   = [[ 0, 0, I, 0], [ 0, 0, 0, I], [ I, 0, 0, 0], [ 0, I, 0, 0 ]]
wxz   = [[-I, 0, 0, 0], [ 0, I, 0, 0], [ 0, 0, I, 0], [ 0, 0, 0,-I ]]
wxt   = [[ 0, 0,-I, 0], [ 0, 0, 0, I], [ I, 0, 0, 0], [ 0,-I, 0, 0 ]]
wyz   = [[ 0,-I, 0, 0], [-I, 0, 0, 0], [ 0, 0, 0, I], [ 0, 0, I, 0 ]]
wyt   = [[ 0, 0, 0,-I], [ 0, 0,-I, 0], [ 0, I, 0, 0], [ I, 0, 0, 0 ]]
wzt   = [[ 0,-I, 0, 0], [ I, 0, 0, 0], [ 0, 0, 0,-I], [ 0, 0, I, 0 ]]
xyz   = [[ 0, 0, 1, 0], [ 0, 0, 0, 1], [-1, 0, 0, 0], [ 0,-1, 0, 0 ]]
xyt   = [[ 1, 0, 0, 0], [ 0, 1, 0, 0], [ 0, 0,-1, 0], [ 0, 0, 0,-1 ]]
xzt   = [[ 0, 0, 1, 0], [ 0, 0, 0,-1], [ 1, 0, 0, 0], [ 0,-1, 0, 0 ]]
yzt   = [[ 0, 0, 0, 1], [ 0, 0, 1, 0], [ 0, 1, 0, 0], [ 1, 0, 0, 0 ]]

wxyz  = [[ 0, I, 0, 0], [-I, 0, 0, 0], [ 0, 0, 0,-I], [ 0, 0, I, 0 ]]
wxyt  = [[ 0, 0, 0, I], [ 0, 0,-I, 0], [ 0,-I, 0, 0], [ I, 0, 0, 0 ]]
wxzt  = [[ 0, I, 0, 0], [ I, 0, 0, 0], [ 0, 0, 0, I], [ 0, 0, I, 0 ]]
wyzt  = [[-I, 0, 0, 0], [ 0, I, 0, 0], [ 0, 0,-I, 0], [ 0, 0, 0, I ]]
xyzt  = [[ 0, 0, 0, 1], [ 0, 0,-1, 0], [ 0, 1, 0, 0], [-1, 0, 0, 0 ]]

wxyzt  = [[ I, 0, 0, 0], [ 0, I, 0, 0], [ 0, 0, I, 0], [ 0, 0, 0, I ]]
```

The generic multivector in matrix form in (4,1) metric is

```
Multivector component order, grouped by grade,

q,   w,x,y,z,t,  wx,wy,wz,wt,xy,xz,xt,yz,yt,zt,
wxy,wxz,wxt,wyz,wyt,wzt,xyz,xyt,xzt,yzt,
wxyz,wxyt,wxzt,wyzt,xyzt,  wxyzt;

Generic MV = (a, b,c,d,e,f, g,h,j,k,l,m,n,p,r,s, S,R,P,N,M,L,K,J,H,G, F,E,D,C,B, A)

Matrix form for Generic MV =
[(+a+c+J+r)+I(+A-C+j-R), (+d-f+l-n)+I(+D+F-L-N), (+H+K+p+s)+I(-h+k-P+S), (+B-e+G-m)+I(-b+E+g-M)],
[(+d+f-l-n)+I(+D-F+L-N), (+a-c+J-r)+I(+A+C+j+R), (-B+e+G-m)+I(+b-E+g-M), (-H+K-p+s)+I(+h+k+P+S)],
[(+H-K-p+s)+I(-h-k+P+S), (+B+e+G+m)+I(-b-E+g+M), (+a+c-J-r)+I(+A-C-j+R), (+d+f+l+n)+I(+D-F-L+N)],
[(-B-e+G+m)+I(+b+E+g+M), (-H-K+p+s)+I(+h-k-P+S), (+d-f-l+n)+I(+D+F+L+N), (+a-c-J+r)+I(+A+C-j-R)]
```

Five Dimensional Minkowksi Determinant

The matrix represention of the five dimensional multivector has a determinant. The expanded formula for this determinant fills a page of text, is inefficient to calculate, and is not very informative. The computer code below calculates the same determinant much faster, and is much more compact.

```
MV1 = GA5_4_1(a, b,c,d,e,f,  g, h, j, k, l, m, n, p, r, s,  S, R, P, N, M, L, K, J, H, G, F,E,D,C,B, A);
MV2 = GA5_4_1(a, b,c,d,e,f, -g,-h,-j,-k,-l,-m,-n,-p,-r,-s, -S,-R,-P,-N,-M,-L,-K,-J,-H,-G, F,E,D,C,B, A);
MV3 = MV1*MV2; // result has zero in bivector and trivector components (20 zeroes for middle terms!)

MV4 = Reverse_Vector_Quad(MV3); // change sign of vector and quadvector components
MV5 = MV3*MV4;  // product has only scalar and pseudoscalar terms (10 more middle symmetric zeroes)
det = MV5.q + I*MV5.wxyzt;
```

Five Dimensional Minkowksi Square

The multivector square component equations are

```
MV1 = GA5_4_1(a, b,c,d,e,f, g,h,j,k,l,m,n,p,r,s, S,R,P,N,M,L,K,J,H,G, F,E,D,C,B, A)
MV2 = MV1*MV1

MV2.q      = +a^2+b^2+c^2+d^2+e^2-f^2-g^2-h^2-j^2+k^2-l^2-m^2+n^2-p^2+r^2+s^2
             -S^2-R^2+P^2-N^2+M^2+L^2-K^2+J^2+H^2+G^2+F^2-E^2-D^2-C^2-B^2-A^2

MV2.w      = +2*a*b-2*l*S-2*m*R+2*n*P-2*p*N+2*r*M+2*s*L-2*B*A
MV2.x      = +2*a*c+2*h*S+2*j*R-2*k*P-2*p*K+2*r*J+2*s*H+2*C*A
MV2.y      = +2*a*d-2*g*S+2*j*N-2*k*M+2*m*K-2*n*J+2*s*G-2*D*A
MV2.z      = +2*a*e-2*g*R-2*h*N-2*k*L-2*l*K-2*n*H-2*r*G+2*E*A
MV2.t      = +2*a*f-2*g*P-2*h*M-2*j*L-2*l*J-2*m*H-2*p*G+2*F*A

MV2.wx     = +2*a*g+2*d*S+2*e*R-2*f*P+2*r*E-2*p*F+2*s*D+2*G*A
MV2.wy     = +2*a*h-2*c*S+2*e*N-2*f*M+2*m*F-2*n*E+2*s*C-2*H*A
MV2.wz     = +2*a*j-2*c*R-2*d*N-2*f*L-2*l*F-2*n*D-2*r*C+2*J*A
MV2.wt     = +2*a*k-2*c*P-2*d*M-2*e*L-2*l*E-2*m*D-2*p*C+2*K*A
MV2.xy     = +2*a*l+2*b*S+2*e*K-2*f*J-2*j*F+2*k*E+2*s*B+2*L*A
MV2.xz     = +2*a*m+2*b*R-2*d*K-2*f*H+2*h*F+2*k*D-2*r*B-2*M*A
MV2.xt     = +2*a*n+2*b*P-2*d*J-2*e*H+2*h*E+2*j*D-2*p*B-2*N*A
MV2.yz     = +2*a*p+2*b*N+2*c*K-2*f*G-2*g*F+2*k*C+2*n*B+2*P*A
MV2.yt     = +2*a*r+2*b*M+2*c*J-2*e*G-2*g*E+2*j*C+2*m*B+2*R*A
MV2.zt     = +2*a*s+2*b*L+2*c*H+2*d*G-2*g*D-2*h*C-2*l*B-2*S*A

MV2.wxy    = +2*a*S+2*b*l-2*c*h+2*d*g+2*G*D-2*H*C+2*L*B+2*s*A
MV2.wxz    = +2*a*R+2*b*m-2*c*j+2*e*g-2*G*E+2*J*C-2*M*B-2*r*A
MV2.wxt    = +2*a*P+2*b*n-2*c*k+2*f*g-2*F*G+2*K*C-2*N*B-2*p*A
MV2.wyz    = +2*a*N+2*b*p-2*d*j+2*e*h+2*H*E-2*J*D+2*P*B+2*n*A
MV2.wyt    = +2*a*M+2*b*r-2*d*k+2*f*h+2*H*F-2*K*D+2*B*R+2*m*A
MV2.wzt    = +2*a*L+2*b*s-2*e*k+2*f*j-2*J*F+2*E*K-2*B*S-2*l*A
MV2.xyz    = +2*a*K+2*c*p-2*d*m+2*e*l-2*E*L+2*M*D-2*P*C-2*k*A
MV2.xyt    = +2*a*J+2*c*r-2*d*n+2*f*l-2*L*F+2*N*D-2*R*C-2*j*A
MV2.xzt    = +2*a*H+2*c*s-2*e*n+2*f*m+2*M*F-2*E*N+2*C*S+2*h*A
MV2.yzt    = +2*a*G+2*d*s-2*e*r+2*f*p-2*P*F+2*R*E-2*S*D-2*g*A
```

```
MV2.wxyz  = +2*a*F+2*g*p+2*j*l-2*h*m+2*H*M-2*L*J-2*P*G-2*f*A
MV2.wxyt  = +2*a*E+2*g*r-2*h*n+2*k*l-2*L*K+2*N*H-2*R*G-2*e*A
MV2.wxzt  = +2*a*D+2*g*s-2*j*n+2*k*m+2*M*K-2*N*J+2*S*G+2*d*A
MV2.wyzt  = +2*a*C+2*h*s-2*j*r+2*k*p-2*P*K+2*R*J-2*H*S-2*c*A
MV2.xyzt  = +2*a*B+2*l*s-2*m*r+2*n*p+2*N*P-2*R*M+2*S*L+2*b*A

MV2.wxyzt = +2*a*A+2*b*B-2*c*C+2*d*D-2*e*E+2*f*F+2*g*G-2*h*H
            +2*j*J-2*k*K+2*l*L-2*m*M+2*n*N+2*p*P-2*r*R+2*s*S
```

Explicitly grouping the dual terms, we have

```
MV1 = GA5_4_1(a, b,c,d,e,f, g,h,j,k,l,m,n,p,r,s, S,R,P,N,M,L,K,J,H,G, F,E,D,C,B, A)
MV2 = MV1*MV1

MV2.q     = +a^2-A^2+b^2-B^2+c^2-C^2+d^2-D^2+e^2-E^2-f^2+F^2-g^2+G^2-h^2+H^2
            -j^2+J^2+k^2-K^2-l^2+L^2-m^2+M^2+n^2-N^2-p^2+P^2+r^2-R^2+s^2-S^2

MV2.w     = 2*(+a*b-B*A -l*S+s*L -m*R+r*M +n*P-p*N)
MV2.x     = 2*(+a*c+C*A +h*S+s*H +j*R+r*J -k*P-p*K)
MV2.y     = 2*(+a*d-D*A -g*S+s*G +j*N-n*J -k*M+m*K)
MV2.z     = 2*(+a*e+E*A -g*R-r*G -h*N-n*H -k*L-l*K)
MV2.t     = 2*(+a*f+F*A -g*P-p*G -h*M-m*H -j*L-l*J)

MV2.wx    = 2*(+a*g+G*A +d*S+s*D +e*R+r*E -f*P-p*F)
MV2.wy    = 2*(+a*h-H*A -c*S+s*C +e*N-n*E -f*M+m*F)
MV2.wz    = 2*(+a*j+J*A -c*R-r*C -d*N-n*D -f*L-l*F)
MV2.wt    = 2*(+a*k+K*A -c*P-p*C -d*M-m*D -e*L-l*E)
MV2.xy    = 2*(+a*l+L*A +b*S+s*B +e*K+k*E -f*J-j*F)
MV2.xz    = 2*(+a*m-M*A +b*R-r*B -d*K+k*D -f*H+h*F)
MV2.xt    = 2*(+a*n-N*A +b*P-p*B -d*J+j*D -e*H+h*E)
MV2.yz    = 2*(+a*p+P*A +b*N+n*B +c*K+k*C -f*G-g*F)
MV2.yt    = 2*(+a*r+R*A +b*M+m*B +c*J+j*C -e*G-g*E)
MV2.zt    = 2*(+a*s-S*A +b*L-l*B +c*H-h*C +d*G-g*D)

MV2.wxy   = 2*(+a*S+s*A +b*l+L*B -c*h-H*C +d*g+G*D)
MV2.wxz   = 2*(+a*R-r*A +b*m-M*B -c*j+J*C +e*g-G*E)
MV2.wxt   = 2*(+a*P-p*A +b*n-N*B -c*k+K*C +f*g-F*G)
MV2.wyz   = 2*(+a*N+n*A +b*p+P*B -d*j-J*D +e*h+H*E)
MV2.wyt   = 2*(+a*M+m*A +b*r+B*R -d*k-K*D +f*h+H*F)
MV2.wzt   = 2*(+a*L-l*A +b*s-B*S -e*k+E*K +f*j-J*F)
MV2.xyz   = 2*(+a*K-k*A +c*p-P*C -d*m+M*D +e*l-E*L)
MV2.xyt   = 2*(+a*J-j*A +c*r-R*C -d*n+N*D +f*l-L*F)
MV2.xzt   = 2*(+a*H+h*A +c*s+C*S -e*n-E*N +f*m+M*F)
MV2.yzt   = 2*(+a*G-g*A +d*s-S*D -e*r+R*E +f*p-P*F)

MV2.wxyz  = 2*(+a*F-f*A +g*p-P*G +j*l-L*J -h*m+H*M)
MV2.wxyt  = 2*(+a*E-e*A +g*r-R*G -h*n+N*H +k*l-L*K)
MV2.wxzt  = 2*(+a*D+d*A +g*s+S*G -j*n-N*J +k*m+M*K)
MV2.wyzt  = 2*(+a*C-c*A +h*s-H*S -j*r+R*J +k*p-P*K)
MV2.xyzt  = 2*(+a*B+b*A +l*s+S*L -m*r-R*M +n*p+N*P)

MV2.wxyzt = 2*(+a*A+b*B-c*C+d*D-e*E+f*F+g*G-h*H
            +j*J-k*K+l*L-m*M+n*N+p*P-r*R+s*S)
```

Like Minkowski spacetime, there is no easy exploit of this symmetry.

Of special interest in relation to the Dirac equations in quantum mechanics are multivectors which square to a simple scalar. An example is the pure vector $M = be_w + ce_x + de_y + ee_e + fe_t$ which squares to $b^2 + c^2 + d^2 + e^2 - f^2$. This vector, and the 191 other basis vector remappings, can be used to factor the Klein-Gordon equation.

Five Dimensional Minkowksi Idempotent

```
Multivector Idempotent component equations

a = +a^2+b^2+c^2+d^2+e^2-f^2-g^2-h^2-j^2+k^2-l^2-m^2+n^2-p^2+r^2+s^2
    -S^2-R^2+P^2-N^2+M^2+L^2-K^2+J^2+H^2+G^2+F^2-E^2-D^2-C^2-B^2-A^2

b = +2*a*b-2*l*S-2*m*R+2*n*P-2*p*N+2*r*M+2*s*L-2*B*A
c = +2*a*c+2*h*S+2*j*R-2*k*P-2*p*K+2*r*J+2*s*H+2*C*A
d = +2*a*d-2*g*S+2*j*N-2*k*M+2*m*K-2*n*J+2*s*G-2*D*A
e = +2*a*e-2*g*R-2*h*N-2*k*L-2*l*K-2*n*H-2*r*G+2*E*A
f = +2*a*f-2*g*P-2*h*M-2*j*L-2*l*J-2*m*H-2*p*G+2*F*A

g = +2*a*g+2*d*S+2*e*R-2*f*P+2*r*E-2*p*F+2*s*D+2*G*A
h = +2*a*h-2*c*S+2*e*N-2*f*M+2*m*F-2*n*E+2*s*C-2*H*A
j = +2*a*j-2*c*R-2*d*N-2*f*L-2*l*F-2*n*D-2*r*C+2*J*A
k = +2*a*k-2*c*P-2*d*M-2*e*L-2*l*E-2*m*D-2*p*C+2*K*A
l = +2*a*l+2*b*S+2*e*K-2*f*J-2*j*F+2*k*E+2*s*B+2*L*A
m = +2*a*m+2*b*R-2*d*K-2*f*H+2*h*F+2*k*D-2*r*B-2*M*A
n = +2*a*n+2*b*P-2*d*J-2*e*H+2*h*E+2*j*D-2*p*B-2*N*A
p = +2*a*p+2*b*N+2*c*K-2*f*G-2*g*F+2*k*C+2*n*B+2*P*A
r = +2*a*r+2*b*M+2*c*J-2*e*G-2*g*E+2*j*C+2*m*B+2*R*A
s = +2*a*s+2*b*L+2*c*H+2*d*G-2*g*D-2*h*C-2*l*B-2*S*A

S = +2*a*S+2*b*l-2*c*h+2*d*g+2*G*D-2*H*C+2*L*B+2*s*A
R = +2*a*R+2*b*m-2*c*j+2*e*g-2*G*E+2*J*C-2*M*B-2*r*A
P = +2*a*P+2*b*n-2*c*k+2*f*g-2*F*G+2*K*C-2*N*B-2*p*A
N = +2*a*N+2*b*p-2*d*j+2*e*h+2*H*E-2*J*D+2*P*B+2*n*A
M = +2*a*M+2*b*r-2*d*k+2*f*h+2*H*F-2*K*D+2*B*R+2*m*A
L = +2*a*L+2*b*s-2*e*k+2*f*j-2*J*F+2*E*K-2*B*S-2*l*A
K = +2*a*K+2*c*p-2*d*m+2*e*l-2*E*L+2*M*D-2*P*C-2*k*A
J = +2*a*J+2*c*r-2*d*n+2*f*l-2*L*F+2*N*D-2*R*C-2*j*A
H = +2*a*H+2*c*s-2*e*n+2*f*m+2*M*F-2*E*N+2*C*S+2*h*A
G = +2*a*G+2*d*s-2*e*r+2*f*p-2*P*F+2*R*E-2*S*D-2*g*A

F = +2*a*F+2*g*p+2*j*l-2*h*m+2*H*M-2*L*J-2*P*G-2*f*A
E = +2*a*E+2*g*r-2*h*n+2*k*l-2*L*K+2*N*H-2*R*G-2*e*A
D = +2*a*D+2*g*s-2*j*n+2*k*m+2*M*K-2*N*J+2*S*G+2*d*A
C = +2*a*C+2*h*s-2*j*r+2*k*p-2*P*K+2*R*J-2*H*S-2*c*A
B = +2*a*B+2*l*s-2*m*r+2*n*p+2*N*P-2*R*M+2*S*L+2*b*A

A = +2*a*A+2*b*B-2*c*C+2*d*D-2*e*E+2*f*F+2*g*G-2*h*H
    +2*j*J-2*k*K+2*l*L-2*m*M+2*n*N+2*p*P-2*r*R+2*s*S
```

All smaller idempotents still work, but I have no new forms specifically for the fifth dimension.

Five Dimensional Minkowksi Nilpotent

```
Multivector Nilpotent component equations

0 = +a^2+b^2+c^2+d^2+e^2-f^2-g^2-h^2-j^2+k^2-l^2-m^2+n^2-p^2+r^2+s^2
    -S^2-R^2+P^2-N^2+M^2+L^2-K^2+J^2+H^2+G^2+F^2-E^2-D^2-C^2-B^2-A^2

0 = +2*a*b-2*l*S-2*m*R+2*n*P-2*p*N+2*r*M+2*s*L-2*B*A
0 = +2*a*c+2*h*S+2*j*R-2*k*P-2*p*K+2*r*J+2*s*H+2*C*A
0 = +2*a*d-2*g*S+2*j*N-2*k*M+2*m*K-2*n*J+2*s*G-2*D*A
0 = +2*a*e-2*g*R-2*h*N-2*k*L-2*l*K-2*n*H-2*r*G+2*E*A
0 = +2*a*f-2*g*P-2*h*M-2*j*L-2*l*J-2*m*H-2*p*G+2*F*A

0 = +2*a*g+2*d*S+2*e*R-2*f*P+2*r*E-2*p*F+2*s*D+2*G*A
0 = +2*a*h-2*c*S+2*e*N-2*f*M+2*m*F-2*n*E+2*s*C-2*H*A
0 = +2*a*j-2*c*R-2*d*N-2*f*L-2*l*F-2*n*D-2*r*C+2*J*A
0 = +2*a*k-2*c*P-2*d*M-2*e*L-2*l*E-2*m*D-2*p*C+2*K*A
0 = +2*a*l+2*b*S+2*e*K-2*f*J-2*j*F+2*k*E+2*s*B+2*L*A
0 = +2*a*m+2*b*R-2*d*K-2*f*H+2*h*F+2*k*D-2*r*B-2*M*A
0 = +2*a*n+2*b*P-2*d*J-2*e*H+2*h*E+2*j*D-2*p*B-2*N*A
0 = +2*a*p+2*b*N+2*c*K-2*f*G-2*g*F+2*k*C+2*n*B+2*P*A
0 = +2*a*r+2*b*M+2*c*J-2*e*G-2*g*E+2*j*C+2*m*B+2*R*A
0 = +2*a*s+2*b*L+2*c*H+2*d*G-2*g*D-2*h*C-2*l*B-2*S*A

0 = +2*a*S+2*b*l-2*c*h+2*d*g+2*G*D-2*H*C+2*L*B+2*s*A
0 = +2*a*R+2*b*m-2*c*j+2*e*g-2*G*E+2*J*C-2*M*B-2*r*A
0 = +2*a*P+2*b*n-2*c*k+2*f*g-2*F*G+2*K*C-2*N*B-2*p*A
0 = +2*a*N+2*b*p-2*d*j+2*e*h+2*H*E-2*J*D+2*P*B+2*n*A
0 = +2*a*M+2*b*r-2*d*k+2*f*h+2*H*F-2*K*D+2*B*R+2*m*A
0 = +2*a*L+2*b*s-2*e*k+2*f*j-2*J*F+2*E*K-2*B*S-2*l*A
0 = +2*a*K+2*c*p-2*d*m+2*e*l-2*E*L+2*M*D-2*P*C-2*k*A
0 = +2*a*J+2*c*r-2*d*n+2*f*l-2*L*F+2*N*D-2*R*C-2*j*A
0 = +2*a*H+2*c*s-2*e*n+2*f*m+2*M*F-2*E*N+2*C*S+2*h*A
0 = +2*a*G+2*d*s-2*e*r+2*f*p-2*P*F+2*R*E-2*S*D-2*g*A

0 = +2*a*F+2*g*p+2*j*l-2*h*m+2*H*M-2*L*J-2*P*G-2*f*A
0 = +2*a*E+2*g*r-2*h*n+2*k*l-2*L*K+2*N*H-2*R*G-2*e*A
0 = +2*a*D+2*g*s-2*j*n+2*k*m+2*M*K-2*N*J+2*S*G+2*d*A
0 = +2*a*C+2*h*s-2*j*r+2*k*p-2*P*K+2*R*J-2*H*S-2*c*A
0 = +2*a*B+2*l*s-2*m*r+2*n*p+2*N*P-2*R*M+2*S*L+2*b*A

0 = +2*a*A+2*b*B-2*c*C+2*d*D-2*e*E+2*f*F+2*g*G-2*h*H
    +2*j*J-2*k*K+2*l*L-2*m*M+2*n*N+2*p*P-2*r*R+2*s*S
```

All smaller nilpotents still work, but I have no new forms specifically for the fifth dimension.