Curvature State Multivectors in Geometric Algebra

Kurt Nalty

January 19, 2015

Abstract

The Frenet-Serret formulas in classical 3D analytic geometry define a curve in terms of pathlength, curvature (deviation from a line) and torsion (deviation from a plane). These formulas easily extend to higher dimensions, with the addition of higher order curvatures. This note takes these concepts of deviation, and uses geometric algebra to find a surprisingly simple multivector which encodes the instantaneous state of a curve.

Geometric Algebra versus Vector Algebra

Vector algebra, as developed by Hamilton, Gibbs and Heaviside, has wide acceptance in the engineering and undergraduate physics areas, but has a few, very interesting problems which indicate a lack of generality. The most important is the distinction between polar vectors, such as momentum or electric field, which change sign under a parity transformation (replacement of x by -x, y by -y and z by -z), versus axial vectors such as angular momentum, or magnetic field, which are invariant under a parity transformation.

In geometric algebra, this is recognized as being due to the confusion between vectors, and their dual planar elements in three dimensional space. Using the analogy of the hub and the wheel, we can see that rotation of the wheel, being a planar relationship, can be correlated with an axle normal to the plane of rotation. For example, rotation in the xy plane can be associated with an axle along the z axis.

The cross product only applies in three-space, and is the source of major confusion. Instead of the cross product, geometric algebra, as developed by Hermann Grassman Sr., William Clifford, and David Hestenes, suggests the use of the wedge product.

The wedge product, is an associative, anti-commutative product between basis vectors, where the product is zero if the two vectors are colinear. However, if the two vectors are not colinear, the product is a planar composite.

Wedge products of basis vectors anti-commute: $\mathbf{e}_1 \wedge \mathbf{e}_2 = -\mathbf{e}_2 \wedge \mathbf{e}_1$. This implies vectors self-wedge to zero. $\mathbf{e}_1 \wedge \mathbf{e}_1 = 0$.

Higher order wedge products define oriented volumes, hypervolumes, and so forth, up to the limit of the dimensionality of the space. In the higher products, all multiplication is associative: $(\mathbf{e}_1\mathbf{e}_2)\mathbf{e}_3 = \mathbf{e}_1(\mathbf{e}_2\mathbf{e}_3)$. Due to the freedom in the ordering of terms, we need to pay attention to the ordering used by different authors when comparing products.

Left Hand Frenet-Serret Formulas in 3D Space

The Frenet-Serret formulas parameterize a curve using pathlength, and specify the curve using signed scalar curvature (a measure of deviation from a line) and signed scalar torsion (deviation from a plane), with a local orthonomal (left hand) frame with unit tangent $\vec{u}$, normal $\vec{n}$, and binormal $\vec{b}$. The left hand coordinates have a simpler sign convention, especially in higher dimensions, than a right hand coordinate frame.

Using a left-handed local frame with unit tangent $\vec{u}$, unit normal $\vec{n}$, and unit binormal $\vec{b}$,

$$\begin{aligned}
(ds)^2 &= d\vec{r} \cdot d\vec{r} \\
\frac{d\vec{r}}{ds} &= \vec{u} \\
\frac{d\vec{u}}{ds} &= \kappa\vec{n} \\
\frac{d\vec{n}}{ds} &= \tau\vec{b} - \kappa\vec{u} \\
\frac{d\vec{b}}{ds} &= -\tau\vec{n}
\end{aligned}$$

Left Hand Frenet-Serret Formulas in 4D Space

The Frenet-Serret equations in left hand format for four dimensional space (with $\tilde{w}$ as the trivector) are

$$\begin{aligned}
\frac{d\tilde{r}}{ds} &= \tilde{u} \\
\frac{d\tilde{u}}{ds} &= \kappa\tilde{n} \\
\frac{d\tilde{n}}{ds} &= \tau\tilde{b} - \kappa\tilde{u} \\
\frac{d\tilde{b}}{ds} &= \gamma\tilde{w} - \tau\tilde{n} \\
\frac{d\tilde{w}}{ds} &= -\gamma\tilde{b}
\end{aligned}$$

The left hand sign convention has the next higher basis with a positive sign in the right hand side of the derivative, while the previous lower basis term has a negative sign.

Commentary on These Classical Forms

In the scalar format for the Frenet equations, the scalar curvature, torsion, boost, and so on have units of inverse distance, or inverse radii of curvature The pattern easily extends to higher dimensions, being a form of Gram-Schmidt orthogonalization.

Given a starting point, and signed values for the curvatures, we can recreate the trajectories.

Likewise, any two curves with the same history of curvature as a function of distance are congruent.

Geometric Algebra and the Trajectory State

In three dimensions, geometric algebra multivectors consist of a scalar, vector, bivector and trivector components. The vector has three components (x, y and z linear elements), the bivector three planar components (xy, xz and yz planar elements), and the trivector is a directed volume element (xyz volume).

Let's examine a proposed state multivector:

$$\begin{aligned}
\text{State} &= s + \vec{u} + \left(\vec{u} \wedge \frac{d\vec{u}}{ds}\right) + \left(\vec{u} \wedge \frac{d\vec{u}}{ds} \wedge \frac{d^2\vec{u}}{ds^2}\right) \\
&= s + \frac{d\vec{r}}{ds} + \left(\frac{d\vec{r}}{ds} \wedge \frac{d^2\vec{r}}{ds^2}\right) + \left(\frac{d\vec{r}}{ds} \wedge \frac{d^2\vec{r}}{ds^2} \wedge \frac{d^3\vec{r}}{ds^3}\right)
\end{aligned}$$

The scalar portion is the distance along the curve. The vector portion is the unit tangent. We note for future reference

$$s = \int ds = \int \vec{u} \cdot d\vec{r} = \int \frac{d\vec{r}}{ds} \cdot d\vec{r} = \int \frac{(ds)^2}{ds}$$

The bivector portion is the curvature bivector, with units of inverse distance. As $\vec{u}$ is a unit vector, it is normal to $d\vec{u}/ds$. The wedge product has the same magnitude as Frenet κ, but is a bivector in the tangent/normal plane.

$$\begin{aligned}
\left(\frac{d\vec{r}}{ds} \wedge \frac{d^2\vec{r}}{ds^2}\right) &= \vec{u} \wedge \frac{d\vec{u}}{ds} \\
&= \vec{u} \wedge (\kappa\vec{n}) = \kappa\,(\vec{u} \wedge \vec{n})
\end{aligned}$$

When $\vec{u}$ and $d\vec{u}/ds$ are colinear, their wedge product is zero, as expected with zero curvature. At any non-zero curvature, the magnitude of the bivector is κ. Consequently, there is no ambiguity in calling this expression the curvature bivector.

Now, let's examine the trivector term. With primes indicating derivatives by s, we find $\vec{u} \wedge \vec{u}' \wedge \vec{u}''$ has units of m^{-3}, and is dimensionally incompatible with the Frenet formula torsion. However, when deviation from a plane is zero, this expression is zero. With a little bit of math, we find

$$
\begin{aligned}
\vec{u} \wedge \vec{u}' \wedge \vec{u}'' &= \left(\vec{u} \wedge \frac{d\vec{u}}{ds} \wedge \frac{d^2\vec{u}}{ds^2}\right) \\
&= \vec{u} \wedge (\kappa\vec{n}) \wedge \frac{d}{ds}(\kappa\vec{n}) \\
&= \vec{u} \wedge (\kappa\vec{n}) \wedge \left(\frac{d\kappa}{ds}\vec{n} + \kappa\frac{d\vec{n}}{ds}\right) \\
&= \vec{u} \wedge (\kappa\vec{n}) \wedge \left(\frac{d\kappa}{ds}\vec{n} + \kappa\left(\tau\vec{b} - \kappa\vec{u}\right)\right)
\end{aligned}
$$

In the wedge product, squared vectors become zero. Consequently, in the parenthesis terms on the right, we lose the $\vec{u}$ and $\vec{n}$ terms, and find

$$
\begin{aligned}
\vec{u} \wedge \vec{u}' \wedge \vec{u}'' &= \vec{u} \wedge (\kappa\vec{n}) \wedge \left(\frac{d\kappa}{ds}\vec{n} + \kappa\left(\tau\vec{b} - \kappa\vec{u}\right)\right) \\
&= \vec{u} \wedge (\kappa\vec{n}) \wedge \kappa\left(\tau\vec{b}\right) \\
&= \kappa^2\tau(\vec{u} \wedge \vec{n} \wedge \vec{b})
\end{aligned}
$$

At the expense of abuse of terminology, and with the potential for user frustration, I am quite willing to call this the torsion trivector, even though it is not numerically the same as the Frenet torsion, but rather κ^2 times the Frenet torsion.

Summarizing, in three dimensions, our state is

$$
\begin{aligned}
\text{State} &= s + \vec{u} + \left(\vec{u} \wedge \frac{d\vec{u}}{ds}\right) + \left(\vec{u} \wedge \frac{d\vec{u}}{ds} \wedge \frac{d^2\vec{u}}{ds^2}\right) \\
&= s + \vec{u} + \kappa(\vec{u} \wedge \vec{n}) + \kappa^2\tau(\vec{u} \wedge \vec{n} \wedge \vec{b})
\end{aligned}
$$

Four Dimensions

In four dimensions, using a tilde to indicate four vectors rather than three vectors, we have

$$
\text{State} = s + \tilde{u} + (\tilde{u} \wedge \tilde{u}') + (\tilde{u} \wedge \tilde{u}' \wedge \tilde{u}'') + (\tilde{u} \wedge \tilde{u}' \wedge \tilde{u}'' \wedge \tilde{u}''')
$$

Attemping a wedge of four terms in three space always yields a zero result, so we don't see the last term in 3D.

Our tangent has four components, x, y, z and t. Our curvature now has six planar components, xy, xz, xt, yz, yt, and zt. We have four torsion components, xyz, xyt, xzt, and yzt. We also have a pseudoscalar boost component, $xyzt$.

We repeat the previous development, repeating the Frenet-Serret equations here for convenience.

$$\begin{aligned}
\frac{d\tilde{r}}{ds} &= \tilde{u} \\
\frac{d\tilde{u}}{ds} &= \kappa\tilde{n} \\
\frac{d\tilde{n}}{ds} &= \tau\tilde{b} - \kappa\tilde{u} \\
\frac{d\tilde{b}}{ds} &= \gamma\tilde{w} - \tau\tilde{n} \\
\frac{d\tilde{w}}{ds} &= -\gamma\tilde{b}
\end{aligned}$$

We find the curvature, just as before.

$$\begin{aligned}
\tilde{u}' &= \kappa\tilde{n} \\
(\tilde{u} \wedge \tilde{u}') &= \kappa(\tilde{u} \wedge \tilde{n})
\end{aligned}$$

Now we find the torsion. Notice all the terms which drop out, due to the wedge product. This is the same expression as found in the three dimensional case, which is good.

$$\begin{aligned}
\tilde{u}'' &= \kappa'\tilde{n} + \kappa\tilde{n}' \\
\tilde{n}' &= \tau\tilde{b} - \kappa\tilde{u} \\
\tilde{u}'' &= \kappa'\tilde{n} + \kappa(\tau\tilde{b} - \kappa\tilde{u}) \\
(\tilde{u} \wedge \tilde{u}' \wedge \tilde{u}'') &= \kappa(\tilde{u} \wedge \tilde{n}) \wedge \Big(\kappa'\tilde{n} + \kappa(\tau\tilde{b} - \kappa\tilde{u})\Big) \\
&= \kappa^2\tau(\tilde{u} \wedge \tilde{n} \wedge \tilde{b})
\end{aligned}$$

Now we find the boost.

$$\begin{aligned}
\tilde{u}''' &= \kappa''\tilde{n} + 2\kappa'\tilde{n}' + \kappa\tilde{n}'' \\
&= \kappa''\tilde{n} + 2\kappa'(\tau\tilde{b} - \kappa\tilde{u}) + \kappa\tilde{n}'' \\
\tilde{n}'' &= \tau'\tilde{b} + \tau\tilde{b}' - \kappa'\tilde{u} - \kappa\tilde{u}' \\
&= \tau'\tilde{b} + \tau(\gamma\tilde{w} - \tau\tilde{n}) - \kappa'\tilde{u} - \kappa(\kappa\tilde{n}) \\
\tilde{u}''' &= \kappa''\tilde{n} + 2\kappa'(\tau\tilde{b} - \kappa\tilde{u}) + \kappa\left[\tau'\tilde{b} + \tau(\gamma\tilde{w} - \tau\tilde{n}) - \kappa'\tilde{u} - \kappa(\kappa\tilde{n})\right]
\end{aligned}$$

In our wedge product, all the terms except the coefficient of $\tilde{w}$ disappear, leaving

$$\begin{aligned}
(\tilde{u} \wedge \tilde{u}' \wedge \tilde{u}'' \wedge \tilde{u}''') &= \kappa^2\tau(\tilde{u} \wedge \tilde{n} \wedge \tilde{b}) \wedge (\kappa\tau\gamma\tilde{w}) \\
&= \kappa^3\tau^2\gamma(\tilde{u} \wedge \tilde{n} \wedge \tilde{b} \wedge \tilde{w})
\end{aligned}$$

At this point, a clear pattern is forming.

$$\begin{aligned}
\text{State} &= s + \tilde{u} + (\tilde{u} \wedge \tilde{u}') + (\tilde{u} \wedge \tilde{u}' \wedge \tilde{u}'') + (\tilde{u} \wedge \tilde{u}' \wedge \tilde{u}'' \wedge \tilde{u}''') \\
&= s + \vec{u} + \kappa\,(\tilde{u} \wedge \tilde{n}) + \kappa^2\tau\left(\tilde{u} \wedge \tilde{n} \wedge \tilde{b}\right) + \kappa^3\tau^2\gamma\left(\tilde{u} \wedge \tilde{n} \wedge \tilde{b} \wedge \tilde{w}\right)
\end{aligned}$$

Conclusion

For any trajectory, parameterized by pathlength, we can form the curvature state multivector

$$\begin{aligned}
\text{State} &= s + \tilde{u} + (\tilde{u} \wedge \tilde{u}') + (\tilde{u} \wedge \tilde{u}' \wedge \tilde{u}'') + (\tilde{u} \wedge \tilde{u}' \wedge \tilde{u}'' \wedge \tilde{u}''') + ... \\
&= s + \vec{u} + \kappa\,(\tilde{u} \wedge \tilde{n}) + \kappa^2\tau\left(\tilde{u} \wedge \tilde{n} \wedge \tilde{b}\right) + \kappa^3\tau^2\gamma\left(\tilde{u} \wedge \tilde{n} \wedge \tilde{b} \wedge \tilde{w}\right) + ...
\end{aligned}$$

While the derivatives, in general, are not orthogonal, from the cascaded wedge products we can determine the local Frenet frame, as well as the values for the Frenet curvatures.