Cartesian Ovals

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Abstract

The Cartesian oval is a generalization of the conic sections, where the distance from any two of three fixed points is a generic linear relationship. In this note, I present the radial coordinate definition of the Cartesian ovals, and the relationships between the three foci, based upon the Differential Calculus book of Benjamin Williamson (1887).

Cartesian Oval

The Cartesian Oval was described by Rene Descartes, as a generalization for the ellipse. For the Cartesian Oval, we have two fixed points, usually labelled F_1 and F_2, separated by a distance d. We have two line segments, r_1 from F_1 and r_2 from F_2, which intersect at a point on the curve where $a * r_1 + b * r_2 = cd$, where c is a fixed constant.

Figure 1 illustrates a Cartesian oval and defining triangle with $a = 1$, $b = 2$, $c = 2.4$, and $d = 5$, with F_2 at the origin.

We have quite a few relationships we can immediately write down. I begin by placing the left fixed point at the origin, and the other fixed point at $x = d$. The linear relationship between r_1 and r_2 is

$$ar_1 + br_2 = cd$$

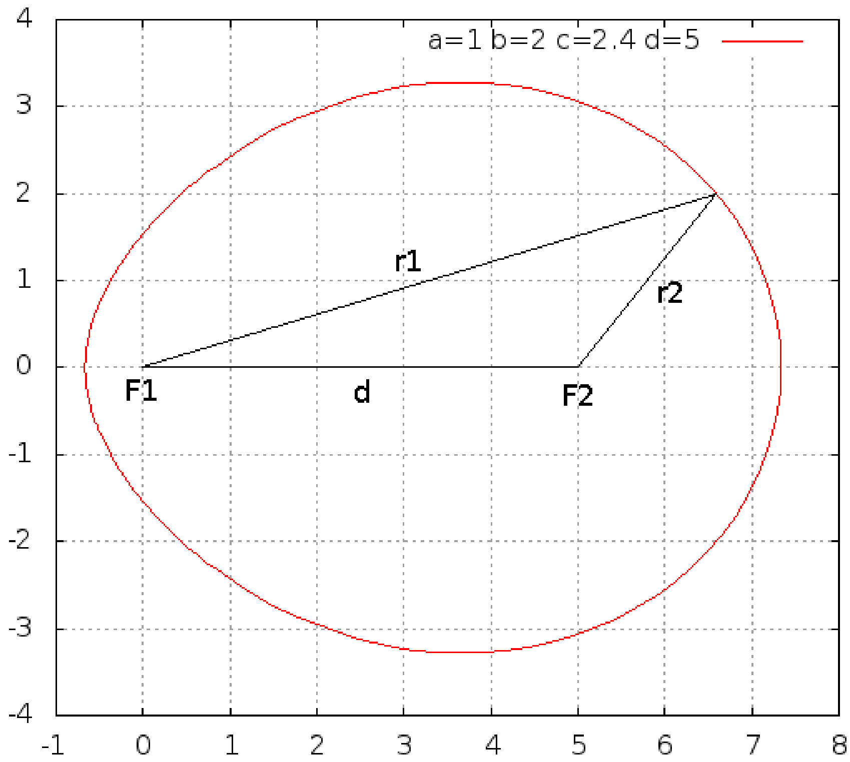

Figure 1: Cartesian Oval with Triangle

This implies

$$\begin{aligned} r_1 &= \frac{cd - br_2}{a} \\ r_2 &= \frac{cd - ar_1}{b} \end{aligned}$$

We can find minimum and maximum values for r_1 and r_2 where the triangle degenerates to a flatten line. Assuming r_1 is the left radii, and $a > b$,

$$\begin{aligned} r_{1max} &= \frac{cd + bd}{a + b} = d\left(\frac{c + b}{a + b}\right) \\ r_{1min} &= \frac{cd - bd}{a + b} = d\left(\frac{c - b}{a + b}\right) \\ r_{2max} &= \frac{cd + ad}{a + b} = d\left(\frac{c + a}{a + b}\right) \\ r_{2min} &= \frac{cd - ad}{a + b} = d\left(\frac{c - a}{a + b}\right) \end{aligned}$$

We notice that $c > b$ and $c > a$ in order to keep our radii positive.

The span in the x direction is

$$\begin{aligned} x_{\text{span}} &= r_{1max} + r_{1min} \\ x_{\text{span}} &= \frac{2cd}{a + b} \end{aligned}$$

One approach for drawing these curves starts with a value for r_1 inside the limited range calculated above. Given d is fixed, and r_2 is calculated from r_1, we then have three sides of a triangle defined, and can then find the angle between r_1 and d.

Using the law of cosines, the angle between r_1 and d is

$$\begin{aligned} r_2^2 = \left(\frac{cd - ar_1}{b}\right)^2 &= r_1^2 + d^2 - 2dr_1\cos\theta \\ \cos\theta &= \frac{r_1^2 - r_2^2 + d^2}{2r_1 d} \end{aligned}$$

In polar coordinates, $r = r_1$ by virtue of placement of r_1 anchored at the origin. Consequently,

$$
\begin{aligned}
r &= r_1 \\
x &= r_1 \cos\theta = \frac{r_1^2 - r_2^2 + d^2}{2d} \\
y &= \pm\sqrt{r^2 - x^2}
\end{aligned}
$$

However, I believe it is preferable to drive the illustration by means of the angle θ. We again start with the equation relating r_2 via the linear relationship, as well as the law of cosines form.

$$
\left(\frac{cd - ar_1}{b}\right)^2 = r_1^2 + d^2 - 2dr_1\cos(\theta)
$$

Organize in power of r_1.

$$
(b^2 - a^2)r_1^2 + (2acd - 2b^2 d\cos(\theta))r_1 + (b^2d^2 - c^2d^2) = 0
$$

Solve this quadratic for $r_1(\theta)$. Define

$$
\begin{aligned}
A &= (b^2 - a^2) \\
B &= (2acd - 2b^2 d\cos(\theta)) \\
C &= (b^2d^2 - c^2d^2) \quad \text{and find} \\
r_1 &= \frac{-B + \sqrt{B^2 - 4AC}}{2A} \\
r_1(\theta) &= d\left[\frac{ac - b^2\cos(\theta) - b\sqrt{c^2 + a^2 - 2ac\cos(\theta) - b^2\sin^2(\theta)}}{a^2 - b^2}\right] \\
&= d\left[\frac{(a/c) - (b/c)^2\cos(\theta) - (b/c)\sqrt{1 + (a/c)^2 - 2(a/c)\cos(\theta) - (b/c)^2\sin^2(\theta)}}{(a/c)^2 - (b/c)^2}\right]
\end{aligned}
$$

where the positive root of the original quadratic is found to be the solution which satisfies $ar_1 + br_2 = cd$. (The negative sign on the square roots of the later equations arises from changing the order of the subtraction in the denominator.) We see that the foci separation d operates as an overall scale factor. We also see that only the ratios a/c and b/c matter as far as shape is concerned. We ordinarily have $c > a > b$, so these ratios are less than one.

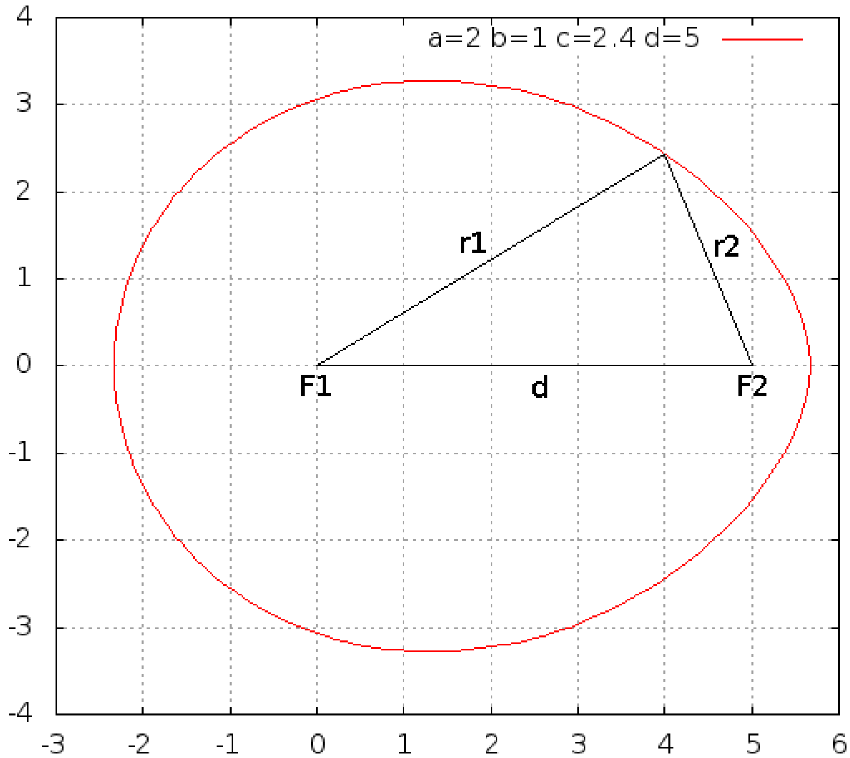

Figure 2: Cartesian Oval with $a = 2$, $b = 1$, $c = 2.4$ and $d = 5$

It is interesting to see the effect of relative values for a and b. Switching relative values of a and b reverses the orientation of the narrow end of the oval. This gives us a certain freedom in our work, with regard to biradial coordinates. Results found for one foci at the origin, versus results found for the other foci at the origin, can be easily translated. This is illustrated in Figure 2.

The negative root of the equation above defines an enveloping 'apple' curve. This curve, show in green in Figure 3, does not satisfy $ar_1 + br_2 = cd$, but rather (using g for the radii to the green curve)

$$\begin{aligned} \text{if } a < b &: \quad ag_1 - bg_2 = -cd \\ \text{if } a = b &: \quad \text{curve goes to infinity} \\ \text{if } a > b &: \quad ag_1 - bg_2 = cd \end{aligned}$$

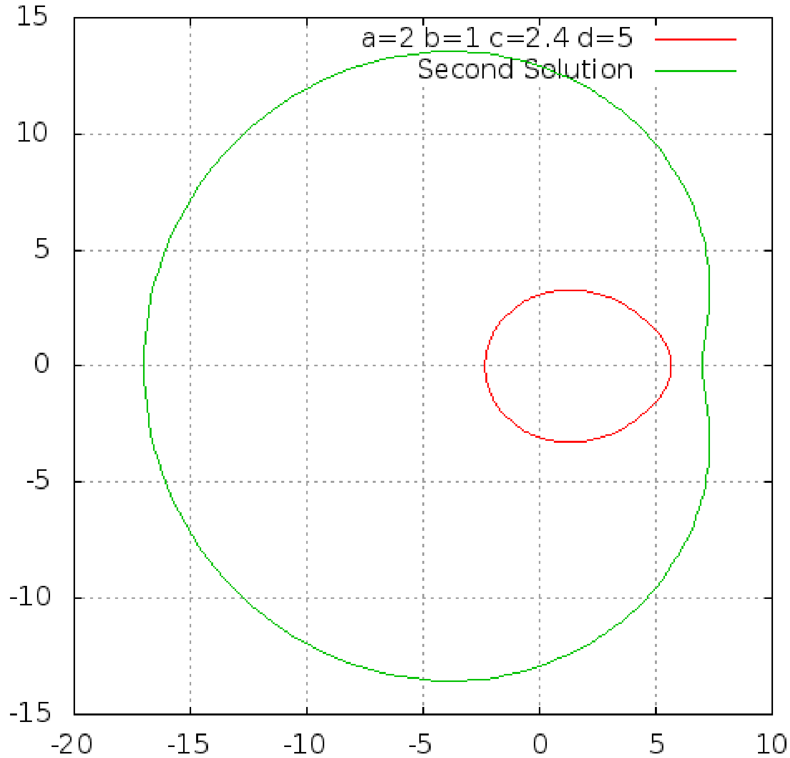

Figure 3: Cartesian Oval with Outer Radial Inverse Curve

This curve is more akin to a hyperbola than an ellipse. Figure 4 illustrates the radial approach to infinity for the outer curve as a and b converge, using $a = 1.02$, $b = 0.98$, $c = 2.4$ and $d = 5$.

Properties of the Outer 'Apple' Curve

At this point, I want to standardize some nomenclature. For the interior red oval, I will label the radii of the large end as r_1, and the radii of the narrow end as r_2. For the exterior, green apple curve, the radii from the large end of the interior oval will be g_1 (g for green in the illustration), and the radii from the narrow end will be g_2.

From `http://mathworld.wolfram.com/CartesianOvals.html`, we learn that these two curves are anallagmatic curves, related through radial inversion, such that $r_1 * g_1 = \text{constant}$, and $r_2 * g_2 = \text{different constant}$.

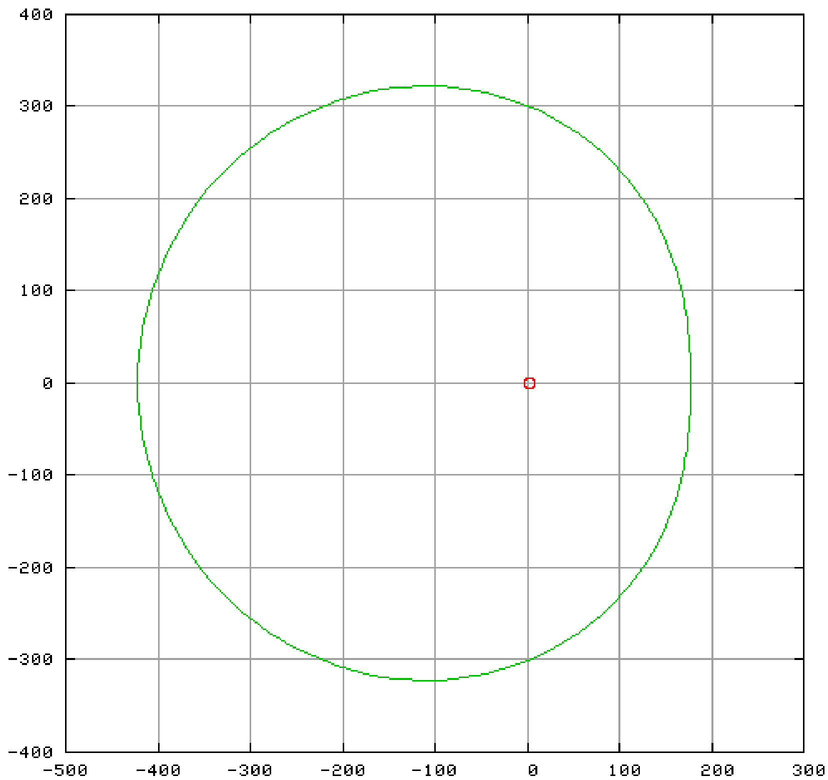

Figure 4: $a = 1.02$, $b = 0.98$, $c = 2.4$ and $d = 5$

The curve constant is seen from the above quadratic equation to be

$$\begin{aligned}
a > b \quad r_1 * g_1 &= \frac{C}{A} = \frac{d^2(c^2 - b^2)}{(a^2 - b^2)} \\
b > a \quad r_2 * g_2 &= -\frac{d^2(c^2 - a^2)}{(b^2 - a^2)} \qquad \text{(just swap a and b)}
\end{aligned}$$

which we see is sensitive to the relative sizes of a and b. For the case $a > b$, we have the large end of the egg to the left of the origin, and the outer radius is positive. This focus will hence be called F_1. For the case $a < b$, we have the narrow end of the egg to the left of the origin, and the outer radius is a negative number, as is the constant product of the two radii. This focus, by the narrow end, will hence be called F_2.

Figure 5 shows the appropriate radii to use with F_1 and F_2.

From earlier, we saw the maximum and minimum x coordinates for the interior oval to be (assuming $a > b$)

$$\begin{aligned}
x_{\text{max}} &= d\left(\frac{c+b}{a+b}\right) \\
x_{\text{min}} &= -d\left(\frac{c-b}{a+b}\right) \\
x_{\text{span}} &= x_{\text{max}} - x_{\text{min}} = \frac{2cd}{a+b}
\end{aligned}$$

Using our scale relationship for the two curves (with $a > b$), we find

$$\begin{aligned}
r_1 * g_1 &= \frac{d^2(c^2 - b^2)}{a^2 - b^2} \\
g_1 &= \frac{d^2(c^2 - b^2)}{r_1(a^2 - b^2)} \\
X_{\text{min}} &= -\left(\frac{c^2d^2 - b^2d^2}{a^2 - b^2}\right)\left(\frac{a+b}{cd - bd}\right) = -d\left(\frac{b+c}{a-b}\right) \\
X_{\text{max}} &= \left(\frac{c^2d^2 - b^2d^2}{a^2 - b^2}\right)\left(\frac{a+b}{cd + bd}\right) = d\left(\frac{c-b}{a-b}\right) \\
X_{\text{span}} &= X_{\text{max}} - X_{\text{min}} = \frac{2cd}{a-b}
\end{aligned}$$

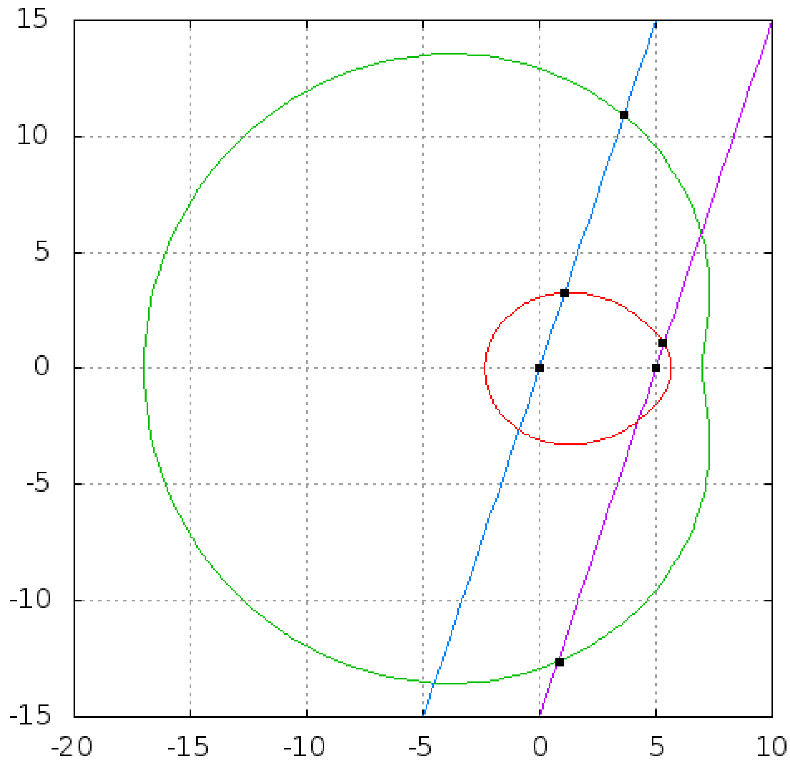

Figure 5: For Constant Product Relationship for F1 (Large end of egg) use standard radii. For F2, use opposing radii.

The Third Focus

From wikipedia and Mathworld, we learn of a third foci on line with F_1 and F_2. We have a reference to James Clerk Maxwell, who at the age of 14 wrote an Essay *Observations on Circumscribed Figures Having a Plurality of Foci, and Radii of Various Proportions.* Courtesy of Google Books, we find a nice geometrical description of this focus in Differential Calculus (1887) by Benjamin Williamson, pp 411-416. A more modern reference is Derek F. Lawden, Elliptic Functions and Applications (1989), Springer-Verlag Applied Mathematical Sciences Vol 80. pp 107-113. Addition references are J. Dennis Lawrence, *A Catalog of Special Plane Curves* pp. 155-157 and E. H. Lockwood, *A Book of Curves* pp.186-190 These last two books are a treasure for all math skill levels.

With the large end of the egg left, $(a > b)$, F_3 is outside the ovals to the right. We have three focii, at x coordinates

$$\begin{aligned} F_1 &= 0 \\ F_2 &= d \\ F_3 &= e = d\frac{(c^2 - b^2)}{(a^2 - b^2)} \end{aligned}$$

Using F_1 and F_2 to describe the ovals, we have

$$\begin{aligned} ar_1 + br_2 &= cd \\ ag_1 - bg_2 &= cd \end{aligned}$$

Using F_1 and F_3 to describe the ovals, we have

$$\begin{aligned} cr_1 + br_3 &= ae \\ cg_1 - bg_3 &= ae \end{aligned}$$

Using F_2 and F_3 to describe the ovals, we have

$$\begin{aligned} cr_2 - ar_3 &= -b(e - d) \\ cg_2 - ag_3 &= +b(e - d) \end{aligned}$$

Taking two of our equations for the inner oval, we can cross multiply by the constant right hand sides, subtract equations, then eliminate static constants, resulting in a weighted sum of three radii going to zero. Start with

$$\begin{aligned} ar_1 + br_2 &= cd \\ cr_1 + br_3 &= ae \end{aligned}$$

Cross multiply

$$\begin{aligned} a^2er_1 + aber_2 &= acde \\ c^2dr_1 + bcdr_3 &= acde \end{aligned}$$

Subtract

$$\begin{aligned} a^2er_1 + aber_2 - c^2dr_1 - bcdr_3 &= 0 \\ (a^2e - c^2d)r_1 + aber_2 - bcdr_3 &= 0 \end{aligned}$$

Substitute for e above, then simplify.

$$e = d\left(\frac{c^2 - b^2}{a^2 - b^2}\right)$$

The result for the inner oval, in my preferred form, is

$$b(c^2 - a^2)r_1 + a(c^2 - b^2)r_2 - c(a^2 - b^2)r_3 = 0$$

In a similar fashion, for the outer curve, we find

$$b(c^2 - a^2)g_1 - a(c^2 - b^2)g_2 + c(a^2 - b^2)g_3 = 0$$

Product Formula per Third Focus

With F_1, we saw that the product r_1g_1 was constant.

With F_2, we saw that the product r_2g_2 with g_2 being the opposing outer oval across from r_2 was another constant. Due to the negative radius, this product will be negative.

With F_3, we have a slightly different setup, as this focus is exterior to the curves. Let a line from F_3 pass through the outer curve at g_{3a} and g_{3b}. The product $g_{3a} * g_{3b}$ is constant. Likewise, when appropriate, let a line from F_3 pass through the inner oval at r_{3a} and r_{3b}. The product $r_{3a} * r_{3b}$ is constant.

We have the following product.

$$\begin{aligned} g_{3a} * g_{3b} &= d^2 \left[\frac{(c^2 - b^2)(c^2 - a^2)}{(a^2 - b^2)^2} \right] \\ r_{3a} * r_{3b} &= d^2 \left[\frac{(c^2 - b^2)(c^2 - a^2)}{(a^2 - b^2)^2} \right] \end{aligned}$$

Notice we have the same constant for both curves.

Summarizing the product constants for the three foci, we have

$$\begin{aligned} F_1 &: K_1 = d^2 \frac{(c^2 - b^2)}{(a^2 - b^2)} \\ F_2 &: K_2 = d^2 \frac{(c^2 - a^2)}{(a^2 - b^2)} \\ F_3 &: K_3 = d^2 \left[\frac{(c^2 - b^2)(c^2 - a^2)}{(a^2 - b^2)^2} \right] \end{aligned}$$

Figure 6 illustrates the points involved in the constant products along lines through F_3 intersection green outer and red inner ovals.

Figure 7 illustrates the exterior focus product as applied to the inner oval. Choose an angle at F_1, and draw the radial line to the curve at point P. Draw the line to this point on the curve from F_3, extending if necessary to cut the curve twice. Scale this length via the product formula above to find $r_3 = K_3/PF_3$. A circle of this radius at F_3 intersects the oval at the second crossing of the the line PF_3 and the oval.

Orthogonality Between Inner and Outer Curves

In problem 17, on page 112 of Derek Lawden's book Elliptic Functions and Applications, we are given a nice parameterization for these curves, intended for showing orthogonality. Given the separation between interior focii $d = 1$, and given $k^2 + k'^2 = 1$, and given $-1 < \mu < 1$, and finally also given

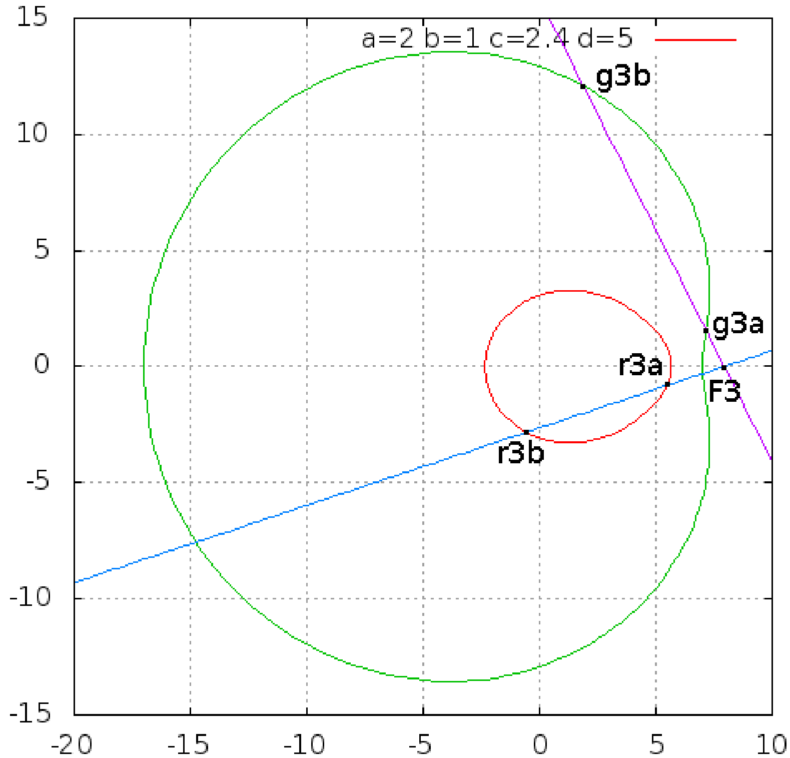

Figure 6: For Constant Product Relationship for F3, use both radii on the same curve.

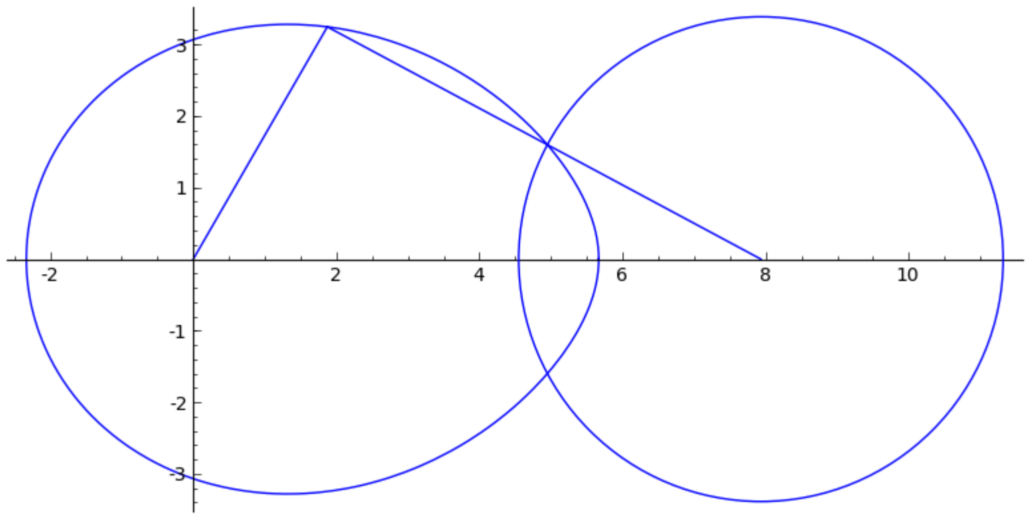

Figure 7: The Circle Radius is K_3/PF_3

$-1 < \lambda < 1$, the curves given by the following are orthogonal

$$\begin{aligned} \mu r + r'\sqrt{k^2 + k'^2\mu^2} &= 1 \\ r - r'\sqrt{k'^2 + k^2\lambda^2} &= \lambda \end{aligned}$$

Naturally, seeing all these unitary constraints, I am inclined to see three angles and some trig being useful. Using my preferred notation of r_1, r_2, g_1, g_2, I write

$$\begin{aligned} r_1\cos(\gamma) + r_2\sqrt{\sin^2(\psi) + \cos^2(\psi)\cos^2(\gamma)} &= 1 \\ g_1 - g_2\sqrt{\cos^2(\psi) + \sin^2(\psi)\cos^2(\phi)} &= \cos(\phi) \end{aligned}$$

In the first equation for the interior oval, we clearly see γ governing our deviation from an ellipse. When $\gamma = 0$, we have a flattened ellipse, being a flat line between the two interior foci, and as γ increases, we increase our ovalization.

Likewise, in the second equation, for the outer oval, ϕ governs our deviation from one branch of a hyperbola.

Figure 8 shows a family of inner ovals as γ changes from 15 to 90 degrees in 15 degree increments. Using my notation,

$$\begin{aligned} a &= \cos(\gamma) \\ b &= \sqrt{\sin^2(\psi) + \cos^2(\psi)\cos^2(\gamma)} \\ c &= 1 \\ d &= 1 \end{aligned}$$

Figure 9 shows a family of ovals as ψ changes from 15 to 90 degrees. We see that ψ affects how clamshell-like the inner curve appears.

Figure 10 shows the related family of outer curves as ϕ changes from 15 to 90 degrees in 15 degree increments.

Figure 11 shows the related family of outer curves as ψ changes from 15 to 90 degrees in 15 degree increments.

For the family orthogonal to the original ovals, I am using

$$\begin{aligned} a &= 1 \\ b &= \sqrt{\cos^2(\psi) + \sin^2(\psi)\cos^2(\phi)} \\ c &= \cos(\phi) \\ d &= 1 \end{aligned}$$

Figure 12 shows select members of both curves, illustrating the orthogonality between families.

References

[1] Benjamin Williamson, *An Elementary Treatise on the Differential Calculus, Containing the Theory of Plane Curves*, Chapter XX, pp. 375-384.

[2] Derek Lawden, *Elliptic Functions and Applications*, p. 112

[3] J. Dennis Lawrence, *A Catalog of Special Plane Curves*, pp. 155-157

[4] E. H. Lockwood, *A Book of Curves*, pp. 186-190

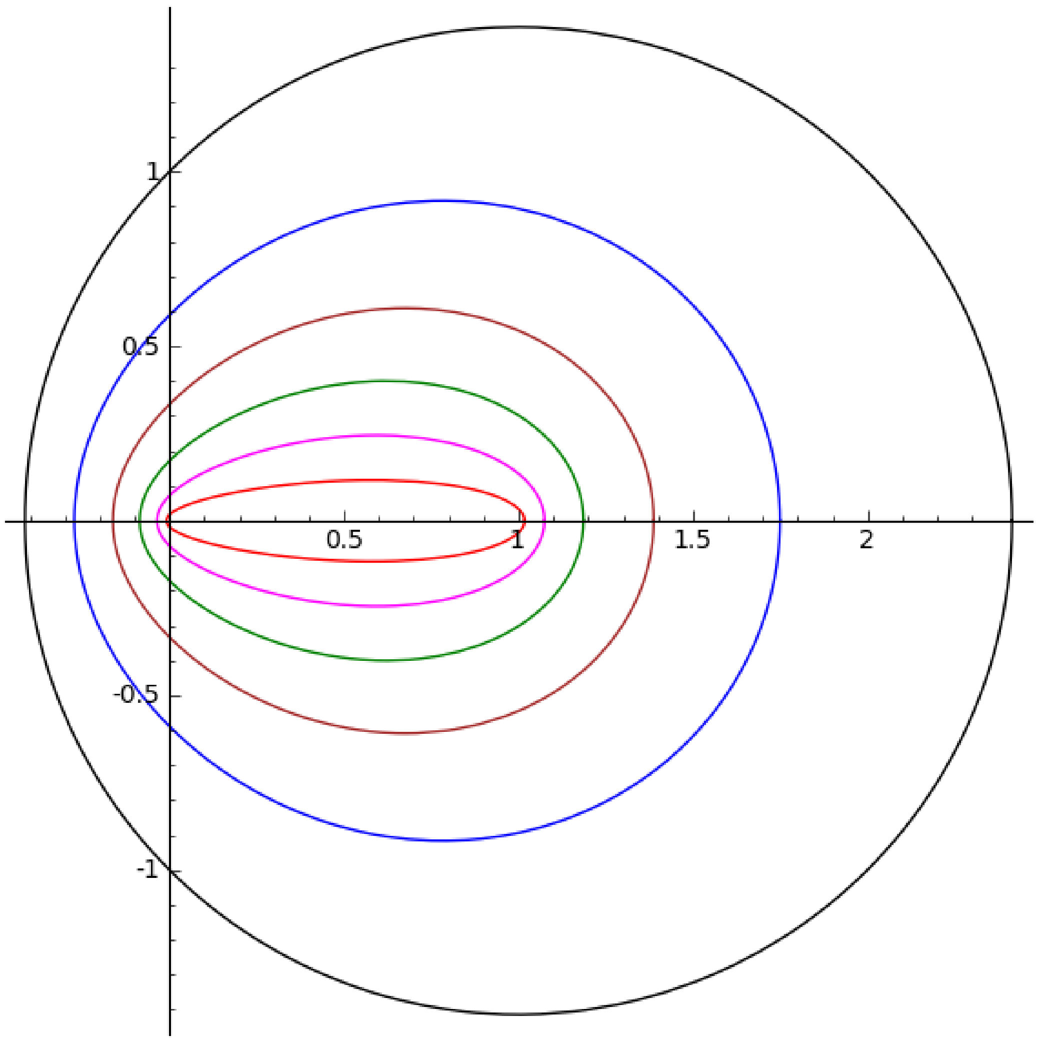

Figure 8: Oval Family at 15 Degree Increments in γ

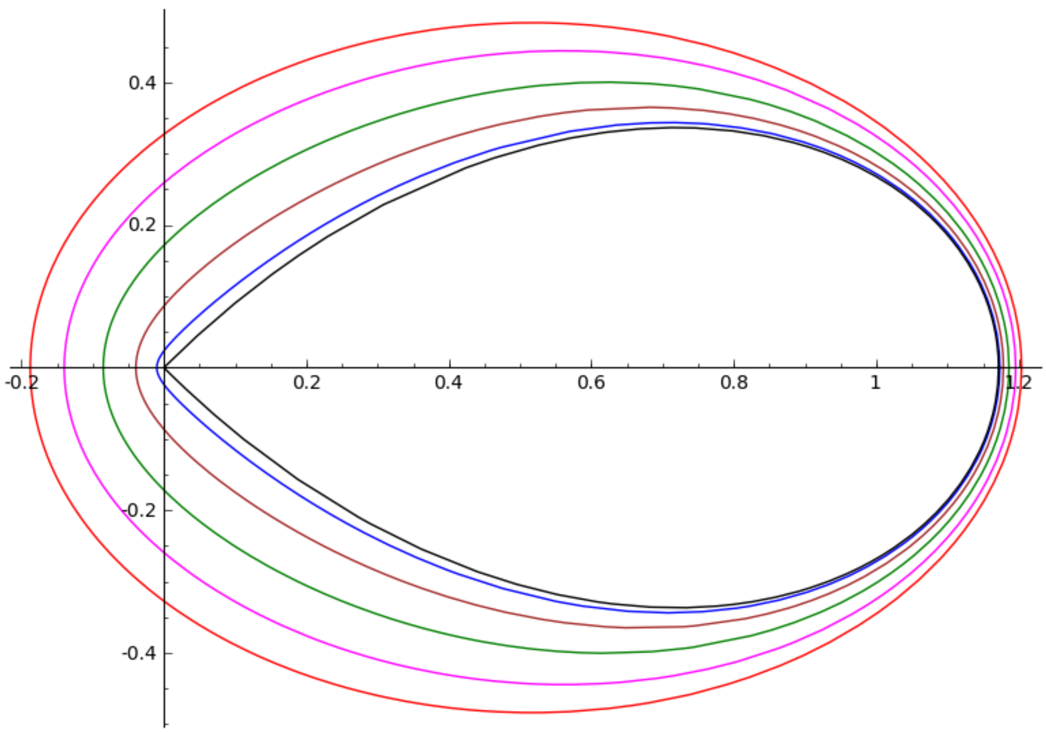

Figure 9: Oval Family at 15 Degree Increments in ψ

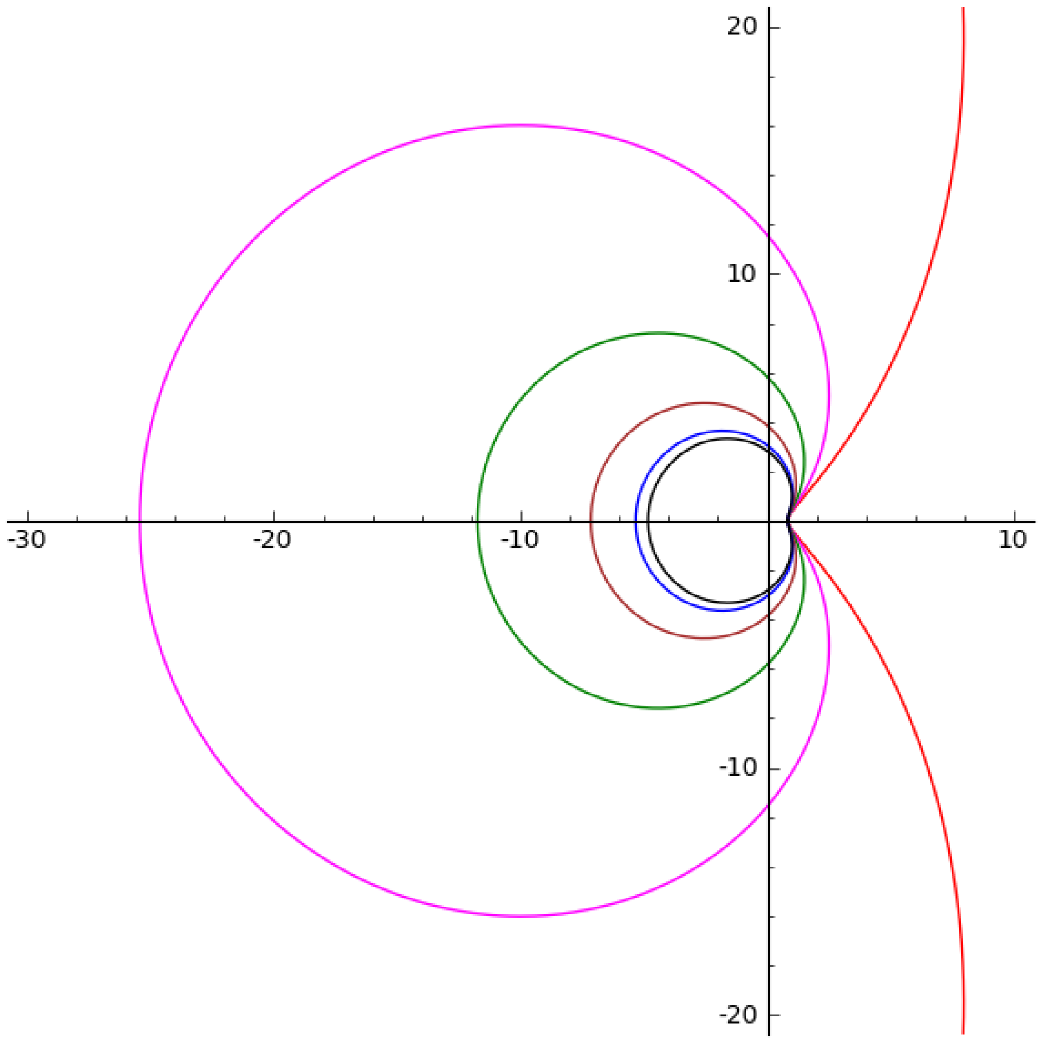

Figure 10: Apple Family at 15 Degree Increments in ϕ

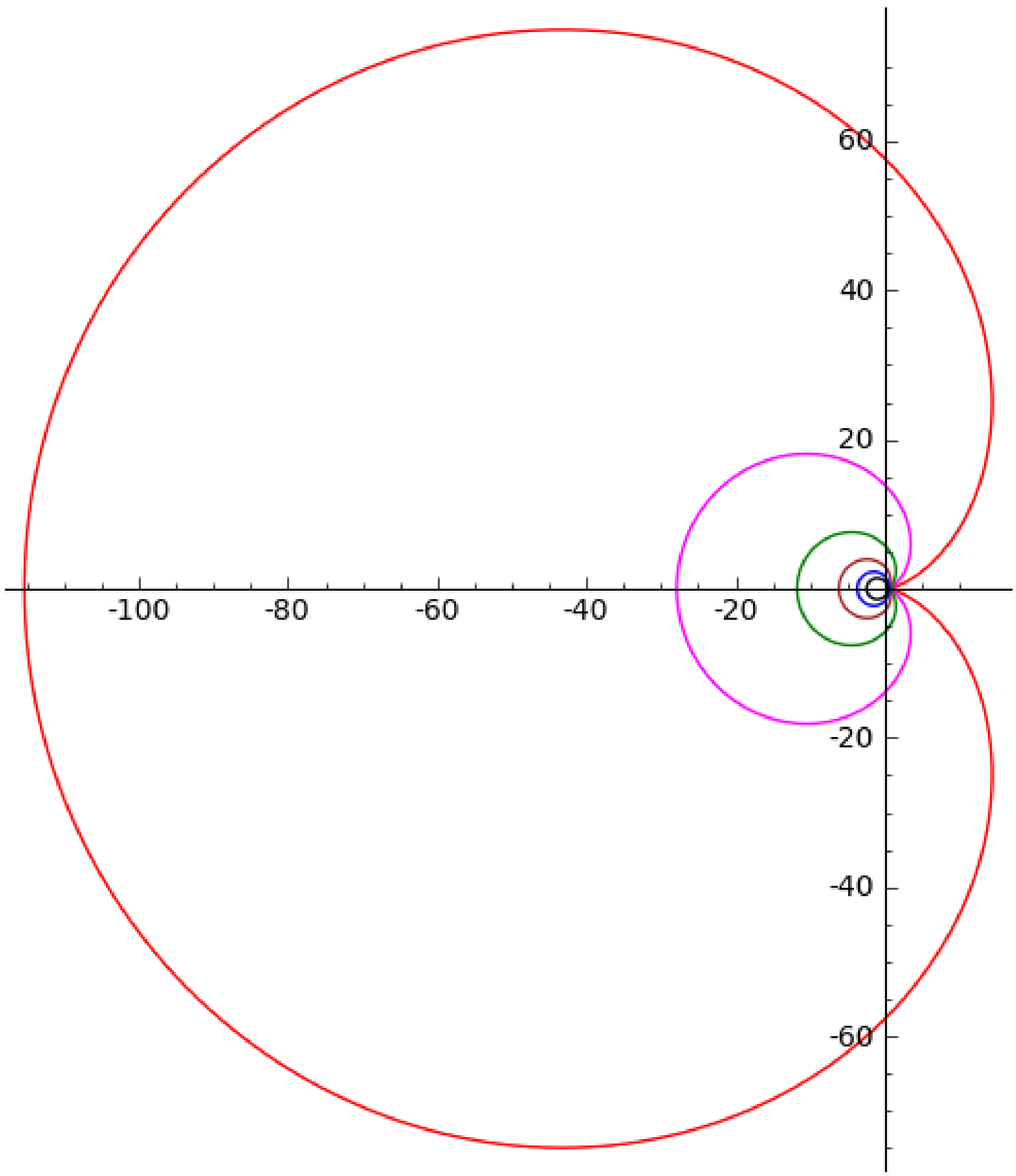

Figure 11: Apple Family at 15 Degree Increments in γ

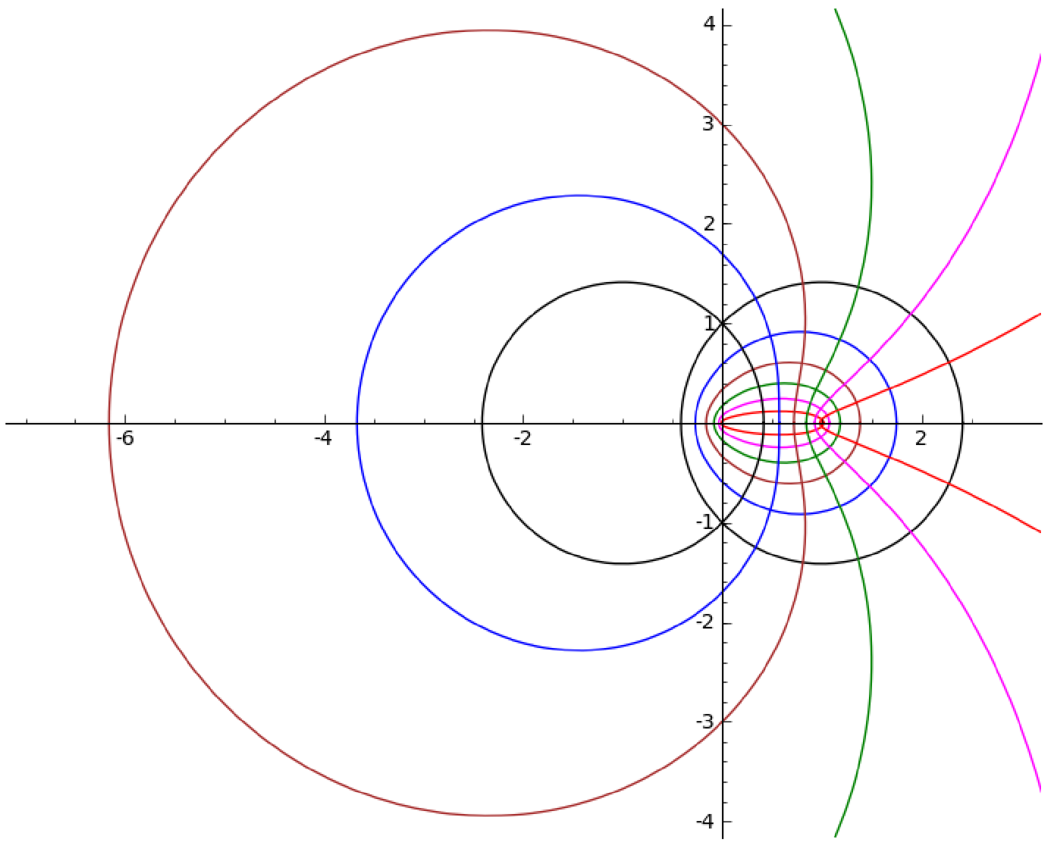

Figure 12: Orthogonality Demonstrated