The AntiWedge Product is Associative

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December 31, 2017

Abstract

This note demonstrates the associativity of Lengyel's antiwedge product, and Hestenes' regressive product.

Eric Lengyel's Two Complements

Eric's introduces the right and left complement operations for multivector elements. Eric's notation uses a horizontal overbar for the right complement, and a horizontal underbar for the left complement. In odd numbered dimensions, such as three dimensional Euclidean space, the right and left complement coincide, while in even dimensional spaces, such as Minkowski spacetime, the two complements differ. The rule for the right complement is to choose the wedge factor, which postmultiplying the basis element, results in the pseudoscalar for the space at hand. Using three dimensional Euclidean space as an example, we have Table 1 for the right complement.

In a similar fashion, for the left complement, choose the factor which premultiplying via wedge product on the left, produces the pseudoscalar element. For three dimensions, we have the same terms as the right complement.

In Table 3, we present the left and right complements for four dimensional Euclidean space, as well as the double complements. (This is Table 4.4 of Eric's *Foundations of Game Engine Development* book.)

Element	Right Complement	Element $\wedge$ (Right Complement)
1	$\overline{1} = e_{xyz}$	$1 \wedge e_{xyz} = e_{xyz}$
e_x	$\overline{e_x} = e_{yz}$	$e_x \wedge e_{yz} = e_{xyz}$
e_y	$\overline{e_y} = e_{zx}$	$e_y \wedge e_{zx} = e_{xyz}$
e_z	$\overline{e_z} = e_{xy}$	$e_z \wedge e_{xy} = e_{xyz}$
e_{yz}	$\overline{e_{yz}} = e_x$	$e_{yz} \wedge e_x = e_{xyz}$
e_{zx}	$\overline{e_{zx}} = e_y$	$e_{zx} \wedge e_y = e_{xyz}$
e_{xy}	$\overline{e_{xy}} = e_z$	$e_{xy} \wedge e_z = e_{xyz}$
e_{xyz}	$\overline{e_{xyz}} = 1$	$e_{xyz} \wedge 1 = e_{xyz}$

Table 1: Right Hand Complement in 3D Euclidean Space

Element	Left Complement	(Left Complement) $\wedge$ Element
1	$\underline{1} = e_{xyz}$	$e_{xyz} \wedge 1 = e_{xyz}$
e_x	$\underline{e_x} = e_{yz}$	$e_{yz} \wedge e_x = e_{xyz}$
e_y	$\underline{e_y} = e_{zx}$	$e_{zx} \wedge e_y = e_{xyz}$
e_z	$\underline{e_z} = e_{xy}$	$e_{xy} \wedge e_z = e_{xyz}$
e_{yz}	$\underline{e_{yz}} = e_x$	$e_x \wedge e_{yz} = e_{xyz}$
e_{zx}	$\underline{e_{zx}} = e_y$	$e_y \wedge e_{zx} = e_{xyz}$
e_{xy}	$\underline{e_{xy}} = e_z$	$e_z \wedge e_{xy} = e_{xyz}$
e_{xyz}	$\underline{e_{xyz}} = 1$	$1 \wedge e_{xyz} = e_{xyz}$

Table 2: Left Hand Complement in 3D Euclidean Space

Element	Right Complement	Left Complement	Double Complement
1	$\overline{1} = e_{xyzt}$	$\underline{1} = e_{xyzt}$	1
e_x	$\overline{e_x} = e_{yzt}$	$\underline{e_x} = -e_{yzt}$	$-e_x$
e_y	$\overline{e_y} = e_{zxt}$	$\underline{e_y} = -e_{zxt}$	$-e_y$
e_z	$\overline{e_z} = e_{xyt}$	$\underline{e_z} = -e_{xyt}$	$-e_z$
e_t	$\overline{e_z} = e_{xzy}$	$\underline{e_z} = -e_{xzy}$	$-e_t$
e_{tx}	$\overline{e_{tx}} = -e_{yz}$	$\underline{e_{tx}} = -e_{yz}$	e_{tx}
e_{ty}	$\overline{e_{ty}} = -e_{zx}$	$\underline{e_{ty}} = -e_{zx}$	e_{ty}
e_{tz}	$\overline{e_{tz}} = -e_{xy}$	$\underline{e_{tz}} = -e_{xy}$	e_{tz}
e_{yz}	$\overline{e_{yz}} = -e_{tx}$	$\underline{e_{yz}} = -e_{tx}$	e_{yz}
e_{zx}	$\overline{e_{zx}} = -e_{ty}$	$\underline{e_{zx}} = -e_{ty}$	e_{zx}
e_{xy}	$\overline{e_{xy}} = -e_{tz}$	$\underline{e_{xy}} = -e_{tz}$	e_{xy}
e_{yzt}	$\overline{e_{yzt}} = -e_x$	$\underline{e_{yzt}} = e_x$	$-e_{yzt}$
e_{zxt}	$\overline{e_{zxt}} = -e_y$	$\underline{e_{zxt}} = e_y$	$-e_{zxt}$
e_{xyt}	$\overline{e_{xyt}} = -e_z$	$\underline{e_{xyt}} = e_z$	$-e_{xyt}$
e_{xzy}	$\overline{e_{xzy}} = -e_t$	$\underline{e_{xzy}} = e_t$	$-e_{xzy}$
e_{xyzt}	$\overline{e_{xyzt}} = 1$	$\underline{e_{xyzt}} = 1$	e_{xyzt}

Table 3: Complements in 4D Euclidean Space

In odd numbered dimensions, double application of the complement recovers the original multivector. In all dimensions, application of left complement followed by application of right complement, or vice versus, recovers the original multivector.

$$\overline{\underline{A}} = A$$

Having the complements at hand, we can define the antiwedge product via DeMorgan style duality relationships.

$$A \vee B = \underline{\overline{A} \wedge \overline{B}} = \overline{\underline{A} \wedge \underline{B}}$$

We can also define the wedge product in terms of the antiwedge product.

$$A \wedge B = \underline{\overline{A} \vee \overline{B}} = \overline{\underline{A} \vee \underline{B}}$$

Demonstration of Associativity

We begin with the DeMorgan style relationship.

$$A \vee B = \underline{\overline{A} \wedge \overline{B}}$$

We extend to three terms.

$$(A \vee B) \vee C \quad = \quad \underline{\left(\overline{\underline{\overline{A} \wedge \overline{B}}}\right) \wedge \overline{C}}$$

The over and underbars in the parenthesis cancel, so

$$(A \vee B) \vee C \quad = \quad \underline{\left(\overline{A} \wedge \overline{B}\right) \wedge \overline{C}}$$

The wedge product is associative, so parenthesis are not needed.

$$(A \vee B) \vee C \quad = \quad \underline{\overline{A} \wedge \overline{B} \wedge \overline{C}}$$

Associativity is thus proven. In a similar fashion, using our other expression for the antiwedge product, we can show

$$(A \vee B) \vee C = \overline{\underline{A} \wedge \underline{B} \wedge \underline{C}}$$

Summarizing

$$A \vee B \vee C = \underline{\overline{A} \wedge \overline{B} \wedge \overline{C}} = \overline{\underline{A} \wedge \underline{B} \wedge \underline{C}}$$

David Hestenes 1986 Regressive Product

David Hestenes, in his (1986) paper "Universal Geometric Algebra" [1] , defined a form of the regressive product which I never understood well enough to implement correctly. Having achieved an understanding of the regressive product through the works of Lengyel and Fernandes, I now revisit Hestenes' work.

Hestenes defines reversion as an operator which reverses the order of multiplication in the basis blades of the multivector. The dagger symbol is used to indicate reversion by Hestenes. An example of reversion is

$$\begin{aligned}(a + be_x + ce_{yz} + de_{xyz})^\dagger &= a + be_x + ce_{zy} + de_{zyx} \\ &= a + be_x - ce_{yz} - de_{xyz}\end{aligned}$$

If k indicates the step of a multivector component, the reverse of that component has a sign change given by $(-1)^k(k-1)/2$, or $+ + - - + + - -$ organized by ascending step for Euclidean space.

The motivation behind the reverse is to have an easy definition for the inverse of a blade. For Euclidean space, a blade M times the reverse is equal to one. $MM^\dagger = 1$. For more generic spacetime, where the dot product square of the time basis is negative, we need to pay attention to sign based upon the actual basis being used.

If I represents the pseudoscalar for a space, and s is number of timelike basis vectors, then $I^{-1} = (-1)^s I^\dagger$.

As the reverse and inverse are unary operations, there is no distinction between prefactor or postfactor formulas. However, duality will occur in two forms. Hestenes only uses the postfactor format for his duality. Using a tilde to indicate the dual of a blade, Hestenes defines

$$\tilde{A} = AI^{-1}$$

Hestenes then defines the regressive product implicitly through a DeMorgan style formula.

$$(A \vee B)\tilde{} = \tilde{A} \wedge \tilde{B}$$

Everything was good to this point. My mistake was to treat double duals as a identity operation. In reality, $(I\tilde{})\tilde{} \neq I$. Geometric algebra is not binary digital logic.

Using the explicit form of Hestenes definition, we can see what the next step should have been.

$$(A \vee B)\tilde{} = \tilde{A} \wedge \tilde{B}$$
$$(A \vee B)I^{-1} = (AI^{-1}) \wedge (BI^{-1})$$

Postmultiplication by I undoes the dual. Thus

$$(A \vee B)I^{-1} = (AI^{-1}) \wedge (BI^{-1})$$
$$A \vee B = \left[\left(AI^{-1}\right) \wedge \left(BI^{-1}\right)\right] I$$

We now test this definition for associativity.

$$\begin{aligned}(A \vee B) \vee C &= \left[\left(\left[\left(AI^{-1}\right) \wedge \left(BI^{-1}\right)\right] I\right) I^{-1} \wedge \left(CI^{-1}\right)\right] I \\ &= \left[\left[\left(AI^{-1}\right) \wedge \left(BI^{-1}\right)\right] \wedge \left(CI^{-1}\right)\right] I \\ &= \left[\left(AI^{-1}\right) \wedge \left(BI^{-1}\right) \wedge \left(CI^{-1}\right)\right] I\end{aligned}$$

Associativity is thus demonstrated courtesy of the associativity of the wedge product.

The next section (to be written) compares component level definitions of Hestenes versus Lengyel.

References

[1] David Hestenes, *Universal Geometric Algebra* in SIMON STEVIN, A Quarterly Journal of Pure and Applied Mathematics, Volume 62 (1988), Number 3-4 (September-December 1988)

[2] John Browne, *Grassmann Algebra*, Barnard Publishing, 9-781479-197637

[3] Hermann Grassmann, *A New Branch of Mathematics*, Translated by Lloyd C. Kannenberg, Open Court Publishing, ISBN 0-8126-9276-4

[4] Chris Doran and Anthony Lasenby, *Geometric Algebra for Physicists*, Cambridge University Press, ISBN 978-0-521-71595-9

[5] Leo Dorst, Daniel Fontune and Stephen Mann, *Geometric Algebra for Computer Science*, Morgan Kaufmann Publishers, ISBN 978-0-12-374942-0

[6] David Hestenes and Garret Sobczyk, *Clifford Algebra to Geometric Calculus*, D. Reidal Publishing Company, ISBN 978-90-277-2581-5

[7] Anthony Lasenby and Chris Doran, *Lectures and Handouts 1999*, `www.mrao.cam.ac.uk/clifford/ptIIIcourse/`

[8] Eric Lengyel, *Foundations of Game Engine Development*, Terathon Software LLC, ISBN 9-780985-811747