From A to B with Elliptic Integrals and Circular Wire Loops

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Abstract

This note goes step by step from the A field of a circular current loop to the B field using complete elliptic integrals.

Circular Current Loop

Assume a current loop in the XY plane at Z=0 of radius R carrying a current I. The observation point (ρ, ϕ, z) is in cylindrical coordinates, with $\rho = \sqrt{x^2 + y^2}$, z being the height above the coil.

The $\vec{A}$ field is given by

$$\begin{aligned}
\vec{A}(\rho, \phi, z) &= \vec{a}_\phi \frac{\mu I R}{4\pi} \oint \frac{\cos\theta d\theta}{\sqrt{z^2 + \rho^2 + R^2 - 2R\rho\cos\theta}} \\
A_\phi &= \frac{\mu I R \sqrt{a+b}}{\pi b} \left[\left(1 - \frac{k^2}{2} \right) K(k) - E(k) \right]
\end{aligned}$$

where K and E are complete elliptic integrals, and

$$\begin{aligned}
a &= z^2 + \rho^2 + R^2 \\
b &= 2R\rho \\
k &= \sqrt{\frac{2b}{a+b}} = \sqrt{\frac{4R\rho}{z^2 + (R+\rho)^2}}
\end{aligned}$$

We calculate the $\vec{B}$ field using $\vec{B} = \nabla \times \vec{A}$. Seeing that $\vec{A}$ has only one non-zero component, we have the following simple formulas for the $\vec{B}$ components.

$$
\begin{aligned}
B_\rho &= -\frac{\partial A_\phi}{\partial z} \\
B_\phi &= 0 \\
B_z &= \frac{1}{\rho}\frac{\partial(\rho A_\phi)}{\partial \rho} = \frac{A_\phi}{\rho} + \frac{\partial A_\phi}{\partial \rho}
\end{aligned}
$$

We will have a number of terms in the derivative expression. We should list some of these before we get started.

$$
\begin{aligned}
\frac{\partial K(k)}{\partial \rho} &= \left[\frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k}\right]\frac{\partial k}{\partial \rho} \\
\frac{\partial K(k)}{\partial z} &= \left[\frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k}\right]\frac{\partial k}{\partial z} \\
\frac{\partial E(k)}{\partial \rho} &= \frac{1}{k}\left[E(k) - K(k)\right]\frac{\partial k}{\partial \rho} \\
\frac{\partial E(k)}{\partial z} &= \frac{1}{k}\left[E(k) - K(k)\right]\frac{\partial k}{\partial z}
\end{aligned}
$$

$$
\begin{aligned}
k &= \sqrt{\frac{4R\rho}{z^2 + (R+\rho)^2}} \\
\frac{\partial k}{\partial \rho} &= \frac{1}{2}\sqrt{\frac{4R\rho}{\rho(z^2 + (\rho+R)^2)}} - \frac{1}{2}\frac{\sqrt{4R\rho}(2(R+\rho))}{\left(z^2 + (R+\rho)^2\right)^{3/2}} \\
&= \frac{k}{2}\left[\frac{1}{\rho} - \frac{2(R+\rho)}{z^2 + (R+\rho)^2}\right] \\
&= \frac{k}{2}\left[\frac{z^2 + (R+\rho)^2 - 2(R+\rho)\rho}{\rho(z^2 + (R+\rho)^2)}\right] \\
&= \frac{k}{2\rho}\left[\frac{z^2 + R^2 - \rho^2}{z^2 + (R+\rho)^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
k &= \sqrt{\frac{4R\rho}{z^2 + (R+\rho)^2}} \\
\frac{\partial k}{\partial z} &= \frac{\sqrt{4R\rho}}{\left(z^2 + (R+\rho)^2\right)^{3/2}} \left(-\frac{1}{2}\right) 2z \\
\frac{\partial k}{\partial z} &= -\frac{zk}{z^2 + (R+\rho)^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\sqrt{a+b}}{b} &= \frac{\sqrt{z^2 + \rho^2 + R^2 + 2R\rho}}{2R\rho} = \frac{\sqrt{z^2 + (\rho + R)^2}}{2R\rho} \\
\frac{\partial}{\partial z}\left(\frac{\sqrt{a+b}}{b}\right) &= \frac{\partial}{\partial z}\left(\frac{\sqrt{z^2 + (\rho+R)^2}}{2R\rho}\right) \\
&= \frac{\frac{1}{2}\left(z^2 + (R+\rho)^2\right)^{-1/2}(2z)}{2R\rho} \\
&= \frac{z}{2R\rho\sqrt{z^2 + (R+\rho)^2}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \rho}\left(\frac{\sqrt{a+b}}{b}\right) &= \frac{\partial}{\partial \rho}\left(\frac{\sqrt{z^2 + (\rho+R)^2}}{2R\rho}\right) \\
&= \frac{1}{2}\frac{(z^2 + (R+\rho)^2)^{-1/2}}{2R\rho}(2(R+\rho)) - \frac{\sqrt{z^2 + (R+\rho)^2}}{2R\rho^2} \\
&= \frac{1}{2R\rho^2\sqrt{z^2 + (R+\rho)^2}}\left[(R+\rho)\rho - z^2 - (R+\rho)^2\right] \\
&= -\frac{z^2 + R\rho}{2R\rho^2\sqrt{z^2 + (R+\rho)^2}}
\end{aligned}
$$

Derivation of B_ρ

We begin with

$$
\begin{aligned}
B_\rho &= -\frac{\partial A_\phi}{\partial z} \\
&= -\frac{\partial}{\partial z}\left[\frac{\mu I R\sqrt{a+b}}{\pi b}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right]\right] \\
&= -\frac{\partial}{\partial z}\left[\frac{\mu I R\sqrt{z^2+R^2+\rho^2+2R\rho}}{\pi 2R\rho}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right]\right] \\
-B_\rho\frac{2\pi\rho}{\mu I} &= \frac{\partial}{\partial z}\left[\sqrt{z^2+R^2+\rho^2+2R\rho}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right]\right] \\
&= \frac{\partial}{\partial z}\left[\sqrt{z^2+(R+\rho)^2}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right]\right] \\
&= \frac{z}{\sqrt{z^2+(R+\rho)^2}}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right] \\
&\quad +\sqrt{z^2+(R+\rho)^2}\frac{\partial}{\partial z}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right] \\
\frac{-B_\rho(2\pi\rho)/(\mu I)}{\sqrt{z^2+(R+\rho)^2}} &= \frac{z}{z^2+(R+\rho)^2}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right] \\
&\quad +\frac{\partial}{\partial z}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right] \\
&= \frac{z}{z^2+(R+\rho)^2}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right] \\
&\quad +\left[\frac{\partial}{\partial z}\left(1-\frac{k^2}{2}\right)\right]K(k)+\left(1-\frac{k^2}{2}\right)\frac{\partial}{\partial z}K(k)-\frac{\partial}{\partial z}E(k) \\
&= \frac{z}{z^2+(R+\rho)^2}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right] \\
&\quad +\left[-kK(k)+\left(1-\frac{k^2}{2}\right)\left[\frac{E(k)}{k(1-k^2)}-\frac{K(k)}{k}\right]-\frac{1}{k}\left[E(k)-K(k)\right]\right]\frac{\partial k}{\partial z}
\end{aligned}
$$

Since

$$
\frac{\partial k}{\partial z} = -\frac{zk}{z^2+(R+\rho)^2}
$$

We have

$$\frac{-B_\rho(2\pi\rho)/(\mu I)}{\sqrt{z^2+(R+\rho)^2}} = \frac{z}{z^2+(R+\rho)^2}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right]$$
$$+\left[-kK(k)+\left(1-\frac{k^2}{2}\right)\left[\frac{E(k)}{k(1-k^2)}-\frac{K(k)}{k}\right]-\frac{1}{k}\left[E(k)-K(k)\right]\right]\frac{\partial k}{\partial z}$$
$$=\frac{z}{z^2+(R+\rho)^2}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right]$$
$$+\left[-kK(k)+\left(1-\frac{k^2}{2}\right)\left[\frac{E(k)}{k(1-k^2)}-\frac{K(k)}{k}\right]-\frac{1}{k}\left[E(k)-K(k)\right]\right]\left[-\frac{zk}{z^2+(R+\rho)^2}\right]$$

$$\begin{aligned}
-\frac{B_\rho 2\pi\rho\sqrt{z^2+(R+\rho)^2}}{\mu z I} &= \left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right] \\
&\quad -\left[-k^2K(k)+\left(1-\frac{k^2}{2}\right)\left[\frac{E(k)}{(1-k^2)}-K(k)\right]-\left[E(k)-K(k)\right]\right] \\
&= \left(1-\frac{k^2}{2}\right)K(k)-E(k)+k^2K(k) \\
&\quad -\left(1-\frac{k^2}{2}\right)\frac{E(k)}{(1-k^2)}+\left(1-\frac{k^2}{2}\right)K(k)+E(k)-K(k) \\
&= K(k)-\left(1-\frac{k^2}{2}\right)\frac{E(k)}{(1-k^2)} \\
&= K(k)-E(k)\frac{(1-k^2/2)}{(1-k^2)}
\end{aligned}$$

Now,

$$\begin{aligned}
k^2 &= \frac{4R\rho}{z^2+(R+\rho)^2} \\
1-k^2 &= \frac{z^2+(R+\rho)^2-4R\rho}{z^2+(R+\rho)^2} \\
&= \frac{z^2+R^2+\rho^2-2R\rho}{z^2+(R+\rho)^2} \\
1-k^2/2 &= \frac{z^2+(R+\rho)^2-2R\rho}{z^2+(R+\rho)^2} \\
&= \frac{z^2+R^2+\rho^2}{z^2+(R+\rho)^2} \\
\frac{(1-k^2/2)}{(1-k^2)} &= \frac{z^2+R^2+\rho^2}{z^2+R^2+\rho^2-2R\rho} \\
&= \frac{R^2+\rho^2+z^2}{(R-\rho)^2+z^2}
\end{aligned}$$

Finishing up, we have

$$-\frac{B_\rho 2\pi\rho\sqrt{z^2+(R+\rho)^2}}{\mu z I} = K(k)-E(k)\frac{R^2+\rho^2+z^2}{(R-\rho)^2+z^2}$$

$$B_\rho = \frac{\mu z I}{2\pi\rho\sqrt{z^2+(R+\rho)^2}}\left(K(k)-E(k)\frac{R^2+\rho^2+z^2}{(R-\rho)^2+z^2}\right)$$

Derivation of B_z

We proceed similar to the previous section.

$$\begin{aligned}
\vec{A}(\rho,\phi,z) &= \vec{a}_\phi\frac{\mu I R}{4\pi}\oint\frac{\cos\theta d\theta}{\sqrt{z^2+\rho^2+R^2-2R\rho\cos\theta}} \\
A_\phi &= \frac{\mu I\sqrt{z^2+(R+\rho)^2}}{2\pi\rho}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right] \\
B_z &= \frac{1}{\rho}\frac{\partial(\rho A_\phi)}{\partial\rho} = \frac{A_\phi}{\rho}+\frac{\partial A_\phi}{\partial\rho}
\end{aligned}$$

Our first term is easy.

$$\frac{A_\phi}{\rho} = \frac{\mu I\sqrt{z^2+(R+\rho)^2}}{2\pi\rho^2}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right]$$

For the second term in B_z, we will have several subexpressions.

$$\begin{aligned}\frac{\partial A_\phi}{\partial \rho} &= \left[\frac{\partial}{\partial\rho}\left(\frac{\mu I\sqrt{z^2+(R+\rho)^2}}{2\pi\rho}\right)\right]\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right] \\ &\quad+\frac{\mu I\sqrt{z^2+(R+\rho)^2}}{2\pi\rho}\frac{\partial}{\partial\rho}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right]\end{aligned}$$

In the first term, we have

$$\frac{\partial}{\partial\rho}\left(\frac{\mu I\sqrt{z^2+(R+\rho)^2}}{2\pi\rho}\right) = \left(\frac{\mu I(R+\rho)}{2\pi\rho\sqrt{z^2+(R+\rho)^2}}\right)-\left(\frac{\mu I\sqrt{z^2+(R+\rho)^2}}{2\pi\rho^2}\right)$$

In the second term, we have

$$\begin{aligned}&\frac{\partial}{\partial\rho}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right] = \left[-kK(k)+\left(1-\frac{k^2}{2}\right)\frac{\partial K(k)}{\partial k}-\frac{\partial E(k)}{\partial k}\right]\frac{\partial k}{\partial\rho} \\ &= \left[-kK(k)+\left(1-\frac{k^2}{2}\right)\left[\frac{E(k)}{k(1-k^2)}-\frac{K(k)}{k}\right]-\frac{1}{k}\left[E(k)-K(k)\right]\right]\frac{\partial k}{\partial\rho} \\ &= \left[K(k)\left(-k-\frac{1}{k}\left(1-\frac{k^2}{2}\right)+\frac{1}{k}\right)+E(k)\left(\frac{1-k^2/2}{k(1-k^2)}-\frac{1}{k}\right)\right]\frac{\partial k}{\partial\rho} \\ &= \left[K(k)\left(-\frac{k}{2}\right)+E(k)\left(\frac{1-k^2/2}{k(1-k^2)}-\frac{1(1-k^2)}{k(1-k^2)}\right)\right]\frac{\partial k}{\partial\rho} \\ &= \left[K(k)\left(-\frac{k}{2}\right)+E(k)\left(\frac{1-k^2/2-1+k^2}{k(1-k^2)}\right)\right]\frac{\partial k}{\partial\rho} \\ &= \left[K(k)\left(-\frac{k}{2}\right)+E(k)\left(\frac{k^2/2}{k(1-k^2)}\right)\right]\frac{\partial k}{\partial\rho} \\ &= \left[K(k)\left(-\frac{k}{2}\right)+E(k)\left(\frac{k}{2(1-k^2)}\right)\right]\frac{\partial k}{\partial\rho}\end{aligned}$$

Now we substitute for $\partial k/\partial\rho$, and get

$$\frac{\partial}{\partial \rho}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right]=\left[K(k)\left(-\frac{k}{2}\right)+E(k)\left(\frac{k}{2(1-k^2)}\right)\right]\frac{\partial k}{\partial \rho}$$
$$=\left[K(k)\left(-\frac{k}{2}\right)+E(k)\left(\frac{k}{2(1-k^2)}\right)\right]\frac{k}{2\rho}\left[\frac{z^2+R^2-\rho^2}{z^2+(R+\rho)^2}\right]$$

We are now ready to collect all of our terms and simplify.

$$\begin{aligned}
B_z &= \frac{A_\phi}{\rho}+\frac{\partial A_\phi}{\partial \rho} \\
&= \frac{\mu I\sqrt{z^2+(R+\rho)^2}}{2\pi\rho^2}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right] \\
&\quad +\left[\frac{\partial}{\partial \rho}\left(\frac{\mu I\sqrt{z^2+(R+\rho)^2}}{2\pi\rho}\right)\right]\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right] \\
&\quad +\frac{\mu I\sqrt{z^2+(R+\rho)^2}}{2\pi\rho}\frac{\partial}{\partial \rho}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right] \\
&= \frac{\mu I\sqrt{z^2+(R+\rho)^2}}{2\pi\rho^2}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right] \\
&\quad +\left[\left(\frac{\mu I(R+\rho)}{2\pi\rho\sqrt{z^2+(R+\rho)^2}}\right)-\left(\frac{\mu I\sqrt{z^2+(R+\rho)^2}}{2\pi\rho^2}\right)\right]\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right] \\
&\quad +\frac{\mu I\sqrt{z^2+(R+\rho)^2}}{2\pi\rho}\frac{\partial}{\partial \rho}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right]
\end{aligned}$$

I am running out of typesetting margin here, so I will transpose some terms to reduce the right hand side length.

$$\frac{B_z 2\pi}{\mu I\sqrt{z^2+(R+\rho)^2}}=\frac{1}{\rho^2}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right]$$
$$+\left[\left(\frac{(R+\rho)}{\rho(z^2+(R+\rho)^2)}\right)-\left(\frac{1}{\rho^2}\right)\right]\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right]$$
$$+\frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right]$$

Continuing to substitute,

$$\frac{B_z 2\pi}{\mu I\sqrt{z^2+(R+\rho)^2}} = \frac{1}{\rho^2}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right]$$
$$+\left[\left(\frac{(R+\rho)}{\rho(z^2+(R+\rho)^2)}\right)-\left(\frac{1}{\rho^2}\right)\right]\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right]$$
$$+\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right]$$
$$=\frac{1}{\rho^2}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right]$$
$$+\left[\left(\frac{(R+\rho)}{\rho(z^2+(R+\rho)^2)}\right)-\left(\frac{1}{\rho^2}\right)\right]\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right]$$
$$+\frac{1}{\rho}\left[K(k)\left(-\frac{k}{2}\right)+E(k)\left(\frac{k}{2(1-k^2)}\right)\right]\frac{k}{2\rho}\left[\frac{z^2+R^2-\rho^2}{z^2+(R+\rho)^2}\right]$$
$$=\left[\left(\frac{(R+\rho)}{\rho(z^2+(R+\rho)^2)}\right)\right]\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right]$$
$$-\frac{k^2}{4\rho^2}\left[K(k)-E(k)\left(\frac{1}{(1-k^2)}\right)\right]\left[\frac{z^2+R^2-\rho^2}{z^2+(R+\rho)^2}\right]$$

Multiply both sides by $z^2+(R+\rho)^2$

$$\begin{aligned}\frac{B_z 2\pi\sqrt{z^2+(R+\rho)^2}}{\mu I} &= \left[\left(\frac{(R+\rho)}{\rho}\right)\right]\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right]\\ &\quad -\frac{k^2}{4\rho^2}\left[K(k)-E(k)\left(\frac{1}{(1-k^2)}\right)\right]\left[z^2+R^2-\rho^2\right]\\ &= K(k)\left[\left(\frac{(R+\rho)}{\rho}\right)\left(1-\frac{k^2}{2}\right)-\frac{k^2}{4\rho^2}\left(z^2+R^2-\rho^2\right)\right]\\ &\quad +E(k)\left[-\frac{(R+\rho)}{\rho}+\frac{k^2}{4\rho^2}\frac{z^2+R^2-\rho^2}{(1-k^2)}\right]\end{aligned}$$

We now substitute

$$k^2=\frac{4R\rho}{z^2+(R+\rho)^2}$$

The coefficient of $K(k)$ becomes

$$\left[\left(\frac{(R+\rho)}{\rho}\right)\left(1-\frac{k^2}{2}\right)-\frac{k^2}{4\rho^2}\left(z^2+R^2-\rho^2\right)\right]=$$
$$\left[\left(\frac{(R+\rho)}{\rho}\right)\left(1-\frac{2R\rho}{z^2+(R+\rho)^2}\right)-\frac{k^2}{4\rho^2}\left(z^2+R^2-\rho^2\right)\right]=$$
$$\left[\left(\frac{(R+\rho)}{\rho}\right)\left(\frac{z^2+R^2+\rho^2}{z^2+(R+\rho)^2}\right)-\frac{k^2}{4\rho^2}\left(z^2+R^2-\rho^2\right)\right]=$$
$$\left[\left(\frac{4\rho(R+\rho)}{4\rho^2}\right)\left(\frac{z^2+R^2+\rho^2}{z^2+(R+\rho)^2}\right)-\frac{k^2}{4\rho^2}\left(z^2+R^2-\rho^2\right)\right]=$$
$$\left[\left(\frac{4\rho(R+\rho)}{4\rho^2}\right)\left(\frac{z^2+R^2+\rho^2}{z^2+(R+\rho)^2}\right)-\frac{4R\rho}{z^2+(R+\rho)^2}\frac{1}{4\rho^2}\left(z^2+R^2-\rho^2\right)\right]=$$
$$\left[\left(\frac{4\rho(R+\rho)}{4\rho^2}\right)\left(\frac{z^2+R^2+\rho^2}{z^2+(R+\rho)^2}\right)-\frac{4R\rho}{4\rho^2}\frac{(z^2+R^2-\rho^2)}{z^2+(R+\rho)^2}\right]=$$
$$\left[\left(\frac{(R+\rho)}{\rho}\right)\left(\frac{z^2+R^2+\rho^2}{z^2+(R+\rho)^2}\right)-\frac{R}{\rho}\frac{(z^2+R^2-\rho^2)}{z^2+(R+\rho)^2}\right]=$$
$$\left[\left(\frac{(R+\rho)}{\rho}\right)\left(\frac{z^2+R^2+\rho^2}{z^2+(R+\rho)^2}\right)-\frac{R}{\rho}\frac{(z^2+R^2-\rho^2)}{z^2+(R+\rho)^2}\right]=$$
$$\frac{\rho(z^2+R^2+\rho^2)+R(2\rho^2)}{\rho(z^2+(R+\rho)^2)}$$
$$\frac{(z^2+R^2+\rho^2)+2R\rho)}{(z^2+(R+\rho)^2)}$$
$$=1$$

This is a very nice result.

The coefficient for E becomes

$$\left[-\frac{(R+\rho)}{\rho}+\frac{k^2}{4\rho^2}\frac{z^2+R^2-\rho^2}{(1-k^2)}\right]=$$

$$\left[-\frac{(R+\rho)}{\rho}+\frac{4R\rho}{z^2+(R+\rho)^2}\frac{1}{4\rho^2}\frac{z^2+R^2-\rho^2}{(1-\frac{4R\rho}{z^2+(R+\rho)^2})}\right]=$$

$$\left[-\frac{(R+\rho)}{\rho}+\frac{4R\rho}{4\rho^2}\frac{z^2+R^2-\rho^2}{(z^2+(R+\rho)^2-4R\rho)}\right]=$$

$$\left[-\frac{(R+\rho)}{\rho}+\frac{R}{\rho}\frac{z^2+R^2-\rho^2}{z^2+(R-\rho)^2}\right]=$$

$$\frac{-(R+\rho)(z^2+(R-\rho)^2)+R(z^2+R^2-\rho^2)}{\rho(z^2+(R-\rho)^2)}=$$

$$\frac{-(\rho)(z^2+(R-\rho)^2)+-Rz^2-RR^2+2R^2\rho-R\rho^2+R(z^2+R^2-\rho^2)}{\rho(z^2+(R-\rho)^2)}=$$

$$-1+\frac{2R^2\rho-2R\rho^2}{\rho(z^2+(R-\rho)^2)}=$$

$$-1+\frac{2R^2-2R\rho}{z^2+(R-\rho)^2}=$$

$$\frac{-(z^2+(R-\rho)^2}{z^2+(R-\rho)^2}+\frac{2R^2-2R\rho}{z^2+(R-\rho)^2}=$$

$$\frac{-z^2-R^2+2R\rho-\rho^2+2R^2-2R\rho}{z^2+(R-\rho)^2}=$$

$$-\frac{z^2-R^2+\rho^2}{z^2+(R-\rho)^2}=$$

$$\frac{R^2-\rho^2-z^2}{(R-\rho)^2+z^2}$$

Putting it all together, we get the desired result

$$B_z = \frac{\mu I}{2\pi\sqrt{z^2+(R+\rho)^2}}\left(K(k)+E(k)\frac{R^2-\rho^2-z^2}{(R-\rho)^2+z^2}\right)$$