A Matter of Ideals

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Abstract

Generalized ideals partition a space under some multiplication operation, and seem to be based upon projection operators. After a term has been multiplied by an ideal, it remains in that subspace regardless of future products. This note first looks at column and row ideals in 2x2 and 4x4 complex matrices and their associated three dimensional and five dimensional geometric algebras, and identifies the idempotent factors involved. Next, sandwich product spinor ideals are examined, and again, idempotent post-factor terms are found to be involved.

Matrix Left Ideals

Historically, the left ideal in matrices was a square matrix replacement for a column vector. The matrix was all zero, with the exception of the left column, which held the former column vector.

$$\begin{pmatrix} 3 & 5 & 7 & 11 \\ 13 & 17 & 19 & 23 \\ 29 & 31 & 37 & 41 \\ 43 & 47 & 53 & 59 \end{pmatrix} \begin{pmatrix} 61 \\ 67 \\ 71 \\ 73 \end{pmatrix} = \begin{pmatrix} 1818 \\ 4960 \\ 9466 \\ 13842 \end{pmatrix}$$

becomes

$$\begin{pmatrix} 3 & 5 & 7 & 11 \\ 13 & 17 & 19 & 23 \\ 29 & 31 & 37 & 41 \\ 43 & 47 & 53 & 59 \end{pmatrix} \begin{pmatrix} 61 & 0 & 0 & 0 \\ 67 & 0 & 0 & 0 \\ 71 & 0 & 0 & 0 \\ 73 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1818 & 0 & 0 & 0 \\ 4960 & 0 & 0 & 0 \\ 9466 & 0 & 0 & 0 \\ 13842 & 0 & 0 & 0 \end{pmatrix}$$

In the case of the left ideal, the subset is the left column only subspace, and the multiplication operation is matrix post-multiplication. We also clearly see that there are three more column which operate in a similar fashion. The set of these four spans the whole matrix space.

Samples of the other three ideals are

$$\begin{pmatrix} 3 & 5 & 7 & 11 \\ 13 & 17 & 19 & 23 \\ 29 & 31 & 37 & 41 \\ 43 & 47 & 53 & 59 \end{pmatrix} \begin{pmatrix} 0 & 61 & 0 & 0 \\ 0 & 67 & 0 & 0 \\ 0 & 71 & 0 & 0 \\ 0 & 73 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1818 & 0 & 0 \\ 0 & 4960 & 0 & 0 \\ 0 & 9466 & 0 & 0 \\ 0 & 13842 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 5 & 7 & 11 \\ 13 & 17 & 19 & 23 \\ 29 & 31 & 37 & 41 \\ 43 & 47 & 53 & 59 \end{pmatrix} \begin{pmatrix} 0 & 0 & 61 & 0 \\ 0 & 0 & 67 & 0 \\ 0 & 0 & 71 & 0 \\ 0 & 0 & 73 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1818 & 0 \\ 0 & 0 & 4960 & 0 \\ 0 & 0 & 9466 & 0 \\ 0 & 0 & 13842 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 5 & 7 & 11 \\ 13 & 17 & 19 & 23 \\ 29 & 31 & 37 & 41 \\ 43 & 47 & 53 & 59 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 61 \\ 0 & 0 & 0 & 67 \\ 0 & 0 & 0 & 71 \\ 0 & 0 & 0 & 73 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1818 \\ 0 & 0 & 0 & 4960 \\ 0 & 0 & 0 & 9466 \\ 0 & 0 & 0 & 13842 \end{pmatrix}$$

In a similar fashion, using pre-multiplication as the multiplication operation, we can identify four row based ideals.

$$\begin{pmatrix} 61 & 67 & 71 & 73 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 5 & 7 & 11 \\ 13 & 17 & 19 & 23 \\ 29 & 31 & 37 & 41 \\ 43 & 47 & 53 & 59 \end{pmatrix} = \begin{pmatrix} 6252 & 7076 & 8196 & 9430 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 61 & 67 & 71 & 73 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 5 & 7 & 11 \\ 13 & 17 & 19 & 23 \\ 29 & 31 & 37 & 41 \\ 43 & 47 & 53 & 59 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 6252 & 7076 & 8196 & 9430 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 61 & 67 & 71 & 73 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 5 & 7 & 11 \\ 13 & 17 & 19 & 23 \\ 29 & 31 & 37 & 41 \\ 43 & 47 & 53 & 59 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6252 & 7076 & 8196 & 9430 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 61 & 67 & 71 & 73 \end{pmatrix} \begin{pmatrix} 3 & 5 & 7 & 11 \\ 13 & 17 & 19 & 23 \\ 29 & 31 & 37 & 41 \\ 43 & 47 & 53 & 59 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6252 & 7076 & 8196 & 9430 \end{pmatrix}$$

How about a composite case? No surprise, both criteria must be simultaneously satisfied.

$$\begin{pmatrix} 61 & 67 & 71 & 73 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 13 & 0 & 0 & 0 \\ 29 & 0 & 0 & 0 \\ 43 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 6252 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 61 & 67 & 71 & 73 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 13 & 0 & 0 \\ 0 & 29 & 0 & 0 \\ 0 & 43 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 6252 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

How about a sandwich product? Same logic applies.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 5 & 7 & 11 \\ 13 & 17 & 19 & 23 \\ 29 & 31 & 37 & 41 \\ 43 & 47 & 53 & 59 \end{pmatrix} \begin{pmatrix} 0 & 5 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 7 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 11068 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

With the row and column ideals, we can easily see the subgrouping and membership. Many other ideals exist, such as multiplication by zero, rotation at a fixed radius, and others. I now want to look at the row and column ideals from the perspective of geometric algebra.

Sigma Matrix ($C\ell_3$) Ideals

Complex 2x2 matrices (Pauli sigma matrices) faithfully implement three dimensional Euclidean geometric algebra. The mapping that I prefer to use is

```
q = [ 1, 0]    x = [ 0, 1]     y = [ 1, 0]     xy = [ 0,-1]
    [ 0, 1]        [ 1, 0]         [ 0,-1]          [ 1, 0]

z = [ 0, I]   xz = [-I, 0]    yz = [ 0, I]    xyz = [ I, 0]
    [-I, 0]        [ 0, I]         [ I, 0]          [ 0, I]
```

A generic 3D Euclidean multivector maps to complex 2x2 matrices as

$$\mathrm{MV} = ae_q + be_x + ce_y + de_z + ee_{xy} + fe_{xz} + ge_{yz} + he_{xyz}$$

$$\mathrm{M} = \begin{pmatrix} [(a+c) - I(f-h)] & [(b-e) + I(d+g)] \\ [(b+e) - I(d-g)] & [(a-c) + I(f+h)] \end{pmatrix}$$

Recovering the multivector from the matrix representation is done via

$$\begin{aligned}
a &= \texttt{real_part}(+M(0,0) + M(1,1))/2 \\
b &= \texttt{real_part}(+M(0,1) + M(1,0))/2 \\
c &= \texttt{real_part}(+M(0,0) - M(1,1))/2 \\
d &= \texttt{imag_part}(+M(0,1) - M(1,0))/2 \\
e &= \texttt{real_part}(-M(0,1) + M(1,0))/2 \\
f &= \texttt{imag_part}(-M(0,0) + M(1,1))/2 \\
g &= \texttt{imag_part}(+M(0,1) + M(1,0))/2 \\
h &= \texttt{imag_part}(+M(0,0) + M(1,1))/2
\end{aligned}$$

We see by inspection that the left ideal requires $b = e$, $a = c$, $d = -g$ and $f = -h$. The left ideal multivector is thus of the format

$$\mathrm{MV_L} = (ae_q + be_x + ae_y + de_z + be_{xy} + fe_{xz} - de_{yz} - fe_{xyz})/2$$

Likewise, the right column ideal requires $a = -c$, $f = h$, $b = -e$, and $d = g$. The right column ideal multivector is thus of the format

$$\mathrm{MV_R} = (ae_q + be_x - ae_y + de_z - be_{xy} + fe_{xz} + de_{yz} + fe_{xyz})/2$$

Together, these two ideals span the multivector space. Post-multiplication by these multivectors traps the output in their respective halves.

In a similar fashion, the prefactor up ideal requires $b = -e$, $d = g$, $a = c$, and $f = -h$.

$$\mathrm{MV_U} = (ae_q + be_x + ae_y + de_z - be_{xy} + fe_{xz} + de_{yz} - fe_{xyz})/2$$

The down ideal requires $a=-c$, $f=h$, $b=e$, and $d=-g$.

$$\mathrm{MV_D} = (ae_q + be_x - ae_y + de_z + be_{xy} + fe_{xz} - de_{yz} + fe_{xyz})/2$$

As easily seen from the matrix representation, the determinant of each of these ideals is zero, as expected for projection operators.

Grouping by Scalar Factors

It is instructive to group these expressions by common factor in order to see the embedded projection operator. Using $\mathrm{MV_L}$ as an example,

$$\begin{aligned}\mathrm{MV_L} &= (ae_q + be_x + ae_y + de_z + be_{xy} + fe_{xz} - de_{yz} - fe_{xyz})/2 \\ &= (ae_q + ae_y + be_x + be_{xy} + de_z - de_{yz} + fe_{xz} - fe_{xyz})/2 \\ &= \left(a\left(1+e_y\right) + be_x\left(1+e_y\right) + de_z\left(1+e_y\right) + fe_{xz}\left(1+e_y\right)\right)/2 \\ &= \left(a + be_x + de_z + fe_{xz}\right)\left(\frac{1+e_y}{2}\right)\end{aligned}$$

In a similar fashion,

$$\mathrm{MV_R} = \left(a + be_x + de_z + fe_{xz}\right)\left(\frac{1-e_y}{2}\right)$$

We see we have the product of a planar 2D multivector times a projection operator $(1+e_y)/2$. Any multivector postmultiplied by this expression will experience the projection operator.

Now we look at the case of a row ideal (prefactor multiplication).

$$\begin{aligned}\mathrm{MV_U} &= (ae_q + be_x + ae_y + de_z - be_{xy} + fe_{xz} + de_{yz} - fe_{xyz})/2 \\ &= (ae_q + ae_y + be_x - be_{xy} + de_z + de_{yz} + fe_{xz} - fe_{xyz})/2 \\ &= \left(\left(1+e_y\right)a + \left(1+e_y\right)be_x + \left(1+e_y\right)de_z + \left(1+e_y\right)fe_{xz}\right)/2 \\ &= \left(\frac{1+e_y}{2}\right)\left(a + be_x + de_z + fe_{xz}\right)\end{aligned}$$

In a similar fashion,

$$\mathrm{MV_D} = \left(\frac{1-e_y}{2}\right)2\left(a + be_x + de_z + fe_{xz}\right)$$

I can create ideals of my own by embedding a projector at the end, for the case of the column ideals, or at the beginning, for the case of the row ideals.

Dirac Gamma Matrix ($C\ell_{4,1}$) Ideals

Complex 4x4 matrices (Dirac gamma matrices) faithfully implement five dimensional Minkowski geometric algebra with a 4,1 signature. The mapping that I prefer to use is

```
      q             wxyzt
[ 1  0  0  0]  [ I  0  0  0]
[ 0  1  0  0]  [ 0  I  0  0]
[ 0  0  1  0]  [ 0  0  I  0]
[ 0  0  0  1]  [ 0  0  0  I]

      w              x              y              z              t
[ 0  0  0 -I]  [ 0  0  1  0]  [ 1  0  0  0]  [ 0  0  0  1]  [ 0  0 -1  0]
[ 0  0 -I  0]  [ 0  0  0  1]  [ 0  1  0  0]  [ 0  0 -1  0]  [ 0  0  0  1]
[ 0  I  0  0]  [ 1  0  0  0]  [ 0  0 -1  0]  [ 0 -1  0  0]  [ 1  0  0  0]
[ I  0  0  0]  [ 0  1  0  0]  [ 0  0  0 -1]  [ 1  0  0  0]  [ 0 -1  0  0]

      wx             wy             wz             wt             xy
[ 0 -I  0  0]  [ 0  0  0  I]  [-I  0  0  0]  [ 0  I  0  0]  [ 0  0 -1  0]
[-I  0  0  0]  [ 0  0  I  0]  [ 0  I  0  0]  [-I  0  0  0]  [ 0  0  0 -1]
[ 0  0  0  I]  [ 0  I  0  0]  [ 0  0 -I  0]  [ 0  0  0  I]  [ 1  0  0  0]
[ 0  0  I  0]  [ I  0  0  0]  [ 0  0  0  I]  [ 0  0 -I  0]  [ 0  1  0  0]

      xz             xt             yz             yt             zt
[ 0 -1  0  0]  [ 1  0  0  0]  [ 0  0  0  1]  [ 0  0 -1  0]  [ 0 -1  0  0]
[ 1  0  0  0]  [ 0 -1  0  0]  [ 0  0 -1  0]  [ 0  0  0  1]  [-1  0  0  0]
[ 0  0  0  1]  [ 0  0 -1  0]  [ 0  1  0  0]  [-1  0  0  0]  [ 0  0  0 -1]
[ 0  0 -1  0]  [ 0  0  0  1]  [-1  0  0  0]  [ 0  1  0  0]  [ 0  0 -1  0]

     wxy            wxz            wxt            wyz            wyt
[ 0 -I  0  0]  [ 0  0  I  0]  [ 0  0  0 -I]  [ I  0  0  0]  [ 0 -I  0  0]
[-I  0  0  0]  [ 0  0  0 -I]  [ 0  0  I  0]  [ 0 -I  0  0]  [ I  0  0  0]
[ 0  0  0 -I]  [ I  0  0  0]  [ 0 -I  0  0]  [ 0  0 -I  0]  [ 0  0  0  I]
[ 0  0 -I  0]  [ 0 -I  0  0]  [ I  0  0  0]  [ 0  0  0  I]  [ 0  0 -I  0]

     wzt            xyz            xyt            xzt            yzt
[ 0  0  I  0]  [ 0  1  0  0]  [-1  0  0  0]  [ 0  0  0 -1]  [ 0 -1  0  0]
[ 0  0  0  I]  [-1  0  0  0]  [ 0  1  0  0]  [ 0  0 -1  0]  [-1  0  0  0]
[-I  0  0  0]  [ 0  0  0  1]  [ 0  0 -1  0]  [ 0 -1  0  0]  [ 0  0  0  1]
[ 0 -I  0  0]  [ 0  0 -1  0]  [ 0  0  0  1]  [-1  0  0  0]  [ 0  0  1  0]

     wxyz           wxyt           wxzt           wyzt           xyzt
[ 0  0  I  0]  [ 0  0  0 -I]  [ I  0  0  0]  [ 0  0 -I  0]  [ 0  0  0  1]
[ 0  0  0 -I]  [ 0  0  I  0]  [ 0  I  0  0]  [ 0  0  0 -I]  [ 0  0  1  0]
[-I  0  0  0]  [ 0  I  0  0]  [ 0  0 -I  0]  [-I  0  0  0]  [ 0 -1  0  0]
[ 0  I  0  0]  [-I  0  0  0]  [ 0  0  0 -I]  [ 0 -I  0  0]  [-1  0  0  0]
```

A generic 5D Minkowski multivector maps to complex 4x4 matrices as

```
MV1  = + a*q + b*w + c*x + d*y + e*z + f*t
MV1 += + g*wx + h*wy + j*wz + k*wt + l*xy + m*xz + n*xt + p*yz + r*yt + s*zt
MV1 += + S*wxy + R*wxz + P*wxt + N*wyz + M*wyt + L*wzt + K*xyz + J*xyt + H*xzt + G*yzt
MV1 += + F*wxyz + E*wxyt + D*wxzt + C*wyzt + B*xyzt + A*wxyzt
```

```
W =
[(+a+d+n-J)+I*(+A+D+N-j),(-m-s-G+K)+I*(-M+S+g-k),(+c-f-l-r)+I*(-C-F-L-R),(+e+p+B-H)+I*(-E+P+b-h)]
[(+m-s-G-K)+I*(+M-S-g-k),(+a+d-n+J)+I*(+A+D-N+j),(-e-p+B-H)+I*(+E+P-b+h),(+c+f-l+r)+I*(-C+F-L+R)]
[(+c+f+l-r)+I*(-C+F+L-R),(-e+p-B-H)+I*(+E-P+b+h),(+a-d-n-J)+I*(+A-D-N-j),(+m-s+G+K)+I*(+M-S+g+k)]
[(+e-B-H-p)+I*(-E-b-h-P),(+c-f+l+r)+I*(-C-F+L+R),(-m-s+G-K)+I*(-M+S-g+k),(+a-d+J+n)+I*(+A-D+j+N)]
```

Recovering the multivector from the matrix representation is done via

```
a = real_part( + W(0,0) + W(1,1) + W(2,2) + W(3,3) )/4;

b = imag_part( - W(0,3) - W(1,2) + W(2,1) + W(3,0) )/4;
c = real_part( + W(0,2) + W(1,3) + W(2,0) + W(3,1) )/4;
d = real_part( + W(0,0) + W(1,1) - W(2,2) - W(3,3) )/4;
e = real_part( + W(0,3) - W(1,2) - W(2,1) + W(3,0) )/4;
f = real_part( - W(0,2) + W(1,3) + W(2,0) - W(3,1) )/4;

g = imag_part( - W(0,1) - W(1,0) + W(2,3) + W(3,2) )/4;
h = imag_part( + W(0,3) + W(1,2) + W(2,1) + W(3,0) )/4;
j = imag_part( - W(0,0) + W(1,1) - W(2,2) + W(3,3) )/4;
k = imag_part( + W(0,1) - W(1,0) + W(2,3) - W(3,2) )/4;
l = real_part( - W(0,2) - W(1,3) + W(2,0) + W(3,1) )/4;
m = real_part( - W(0,1) + W(1,0) + W(2,3) - W(3,2) )/4;
n = real_part( + W(0,0) - W(1,1) - W(2,2) + W(3,3) )/4;
p = real_part( + W(0,3) - W(1,2) + W(2,1) - W(3,0) )/4;
r = real_part( - W(0,2) + W(1,3) - W(2,0) + W(3,1) )/4;
s = real_part( - W(0,1) - W(1,0) - W(2,3) - W(3,2) )/4;

S = imag_part( - W(0,1) - W(1,0) - W(2,3) - W(3,2) )/4;
R = imag_part( + W(0,2) - W(1,3) + W(2,0) - W(3,1) )/4;
P = imag_part( - W(0,3) + W(1,2) - W(2,1) + W(3,0) )/4;
N = imag_part( + W(0,0) - W(1,1) - W(2,2) + W(3,3) )/4;
M = imag_part( - W(0,1) + W(1,0) + W(2,3) - W(3,2) )/4;
L = imag_part( + W(0,2) + W(1,3) - W(2,0) - W(3,1) )/4;
K = real_part( + W(0,1) - W(1,0) + W(2,3) - W(3,2) )/4;
J = real_part( - W(0,0) + W(1,1) - W(2,2) + W(3,3) )/4;
H = real_part( - W(0,3) - W(1,2) - W(2,1) - W(3,0) )/4;
G = real_part( - W(0,1) - W(1,0) + W(2,3) + W(3,2) )/4;

F = imag_part( + W(0,2) - W(1,3) - W(2,0) + W(3,1) )/4;
E = imag_part( - W(0,3) + W(1,2) + W(2,1) - W(3,0) )/4;
D = imag_part( + W(0,0) + W(1,1) - W(2,2) - W(3,3) )/4;
C = imag_part( - W(0,2) - W(1,3) - W(2,0) - W(3,1) )/4;
B = real_part( + W(0,3) + W(1,2) - W(2,1) - W(3,0) )/4;

A = imag_part( + W(0,0) + W(1,1) + W(2,2) + W(3,3) )/4;
```

Translating the four column ideals into geometric algebra, and factoring, we have

```
MV1 = (a*q + b*w + c*x + e*z - g*wx - j*wz + m*xz - R*wzt)*(q + y)*(q + xt)/4
MV2 = (a*q + b*w + c*x - e*z - g*wx + j*wz - m*xz - R*wzt)*(q + y)*(q - xt)/4
MV3 = (a*q - b*w + c*x - e*z + g*wx - j*wz - m*xz + R*wzt)*(q - y)*(q - xt)/4
MV4 = (a*q - b*w + c*x + e*z + g*wx + j*wz + m*xz + R*wzt)*(q - y)*(q + xt)/4
```

Each of these ideals has a pair of commutating projection operators on the tail end. The leading term is simply the necessary factor to obtain a specific column vector result.

In a similar fashion, I can set up four row ideals, translate and factor to get the prefactor row ideals.

```
MV5 = (q + y)*(q + xt)*(a*q - b*w + c*x + e*z - g*wx - j*wz - m*xz + R*wzt)/4
MV6 = (q + y)*(q - xt)*(a*q - b*w + c*x - e*z - g*wx + j*wz + m*xz + R*wzt)/4
MV7 = (q - y)*(q - xt)*(a*q + b*w + c*x - e*z + g*wx - j*wz + m*xz - R*wzt)/4
MV8 = (q - y)*(q + xt)*(a*q + b*w + c*x + e*z + g*wx + j*wz - m*xz - R*wzt)/4
```

Again, we see pairs of commutating projection operators, now in the leading position.

Looking at the matrix format for these projection operators, we have

```
(q + y)/2   => [[1,0,0,0],        (q - y)/2   => [[0,0,0,0],
                [0,1,0,0],                        [0,0,0,0],
                [0,0,0,0],                        [0,0,1,0],
                [0,0,0,0]]                        [0,0,0,1]]

(q + xt)/2  => [[1,0,0,0],        (q - xt)/2  => [[0,0,0,0],
                [0,0,0,0],                        [0,1,0,0],
                [0,0,0,0],                        [0,0,1,0],
                [0,0,0,1]]                        [0,0,0,0]]

(q + y)*(q + xt)/4  => [[1,0,0,0],
                        [0,0,0,0],
                        [0,0,0,0],
                        [0,0,0,0]]

(q + y)*(q - xt)/4  => [[0,0,0,0],
                        [0,1,0,0],
                        [0,0,0,0],
                        [0,0,0,0]]

(q - y)*(q - xt)/4  => [[0,0,0,0],
                        [0,0,0,0],
                        [0,0,1,0],
                        [0,0,0,0]]

(q - y)*(q + xt)/4  => [[0,0,0,0],
                        [0,0,0,0],
                        [0,0,0,0],
                        [0,0,0,1]]
```

From the matrix representation of the dual projector product, the mechanism for row and column ideals becomes clear. From the geometric algebra representation, the projector basis for the ideals becomes clear.

Examples

Begin with my favorite prime number matrix.

$$M = \begin{pmatrix} 3 & 5 & 7 & 11 \\ 13 & 17 & 19 & 23 \\ 29 & 31 & 37 & 41 \\ 43 & 47 & 53 & 59 \end{pmatrix}$$

Initially, use the project pair $P_1 = (1 + e_y)/2$ and $P_2 = (1 - e_y)/2$

$$P_1 = \frac{1 + e_y}{2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P_2 = \frac{1 - e_y}{2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Together, these two projectors span all of the matrix space. We have the tautology

$$M = M * (P_1) + M * (P_2) = M * (P_1 + P_2) = M * (1) = M$$

$$M * P_1 = \begin{pmatrix} 3 & 5 & 0 & 0 \\ 13 & 17 & 0 & 0 \\ 29 & 31 & 0 & 0 \\ 43 & 47 & 0 & 0 \end{pmatrix}$$

$$M * P_2 = \begin{pmatrix} 0 & 0 & 7 & 11 \\ 0 & 0 & 19 & 23 \\ 0 & 0 & 37 & 41 \\ 0 & 0 & 53 & 59 \end{pmatrix}$$

M is not totally in either subspace, as $M * P_1 \neq M$ and $M * P_2 \neq M$. However, $M * P_1$ is totally in P_1 space, as $(M * P_1) * P_1 = (M * P_1)$. Likewise, $M * P_2$ is totally in P_2 space, as $(M * P_2) * P_2 = (M * P_2)$.

Writing our tautology for M in a fashion reminescent of vectors, we have

$$M = (M * P_1) * P_1 + (M * P_2) * P_2$$

Now lets look at the set of four column ideals

$$P_3 = \frac{1 + e_{xt}}{2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P_4 = \frac{1 - e_{xt}}{2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Q_1 = \left(\frac{1 + e_y}{2}\right)\left(\frac{1 + e_{xt}}{2}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Q_2 = \left(\frac{1 + e_y}{2}\right)\left(\frac{1 - e_{xt}}{2}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Q_3 = \left(\frac{1 - e_y}{2}\right)\left(\frac{1 - e_{xt}}{2}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Q_4 = \left(\frac{1 - e_y}{2}\right)\left(\frac{1 + e_{xt}}{2}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In terms of Q, we have

$$M * Q_1 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 13 & 0 & 0 & 0 \\ 29 & 0 & 0 & 0 \\ 43 & 0 & 0 & 0 \end{pmatrix}$$

$$M * Q_2 = \begin{pmatrix} 0 & 5 & 0 & 0 \\ 0 & 17 & 0 & 0 \\ 0 & 31 & 0 & 0 \\ 0 & 47 & 0 & 0 \end{pmatrix}$$

$$M * Q_3 = \begin{pmatrix} 0 & 0 & 7 & 0 \\ 0 & 0 & 19 & 0 \\ 0 & 0 & 37 & 0 \\ 0 & 0 & 53 & 0 \end{pmatrix}$$

$$M * Q_4 = \begin{pmatrix} 0 & 0 & 0 & 11 \\ 0 & 0 & 0 & 23 \\ 0 & 0 & 0 & 41 \\ 0 & 0 & 0 & 59 \end{pmatrix}$$

Our tautology is

$$M = (M * Q_1) * Q_1 + (M * Q_2) * Q_2 + (M * Q_3) * Q_3 + (M * Q_4) * Q_4$$

Not Pretty Example

With the mappings and idempotents chosen above, the matrix representations are nice, pretty and organized in a fashion where memberships are easy to see. We now look at a different example, which has the same mathematical properties, but which has totally different visual layout.

Look at projectors $(1 \pm e_x)/2, (1 \pm e_{yt})/2$.

$$P1 = \left(\frac{1+e_x}{2}\right) = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}$$

$$P2 = = \left(\frac{1-e_x}{2}\right) = \begin{pmatrix} 1/2 & 0 & -1/2 & 0 \\ 0 & 1/2 & 0 & -1/2 \\ -1/2 & 0 & 1/2 & 0 \\ 0 & -1/2 & 0 & 1/2 \end{pmatrix}$$

$$P3 = = \left(\frac{1-e_{yt}}{2}\right) = \begin{pmatrix} 1/2 & 0 & -1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ -1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}$$

$$P4 = = \left(\frac{1+e_{yt}}{2}\right) = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & -1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & -1/2 & 0 & 1/2 \end{pmatrix}$$

$$MV * P1 = \begin{pmatrix} 5 & 8 & 5 & 8 \\ 31/2 & 20 & 31/2 & 20 \\ 33 & 36 & 33 & 36 \\ 48 & 53 & 48 & 53 \end{pmatrix}$$

$$MV * P2 = \begin{pmatrix} -2 & -3 & 2 & 3 \\ -5/2 & -3 & 5/2 & 3 \\ -4 & -5 & 4 & 5 \\ -5 & -6 & 5 & 6 \end{pmatrix}$$

$$MV * P3 = \begin{pmatrix} -2 & 8 & 2 & 8 \\ -5/2 & 20 & 5/2 & 20 \\ -4 & 36 & 4 & 36 \\ -5 & 53 & 5 & 54 \end{pmatrix}$$

$$MV * P4 = \begin{pmatrix} 5 & -3 & 5 & 3 \\ 31/2 & -3 & 31/2 & 3 \\ 33 & -5 & 33 & 5 \\ 48 & -6 & 48 & 56 \end{pmatrix}$$

Each post-product separates the system into two sections, just like the column ideals of the previous sector. However, it is no longer trivial to assess which part the product resides in by inspection. We could use a visator to determine Chatlanians versus Patsaks status for these products, or we can simply multiply by our complementary, annihilating idempotents, looking for a zero result. As a little sidetrack, an idempotent has the property $P^2 = P$. Each idempotent has a complement $1 - P$, which is also an idempotent, since $(1-P)^2 = 1 + P^2 - 2P = (1-P)$, and which yields a zero result when multiplied by the original. $P * (1-P) = P - P^2 = P - P = 0$.

Using our top example, P1 and P2 are complementary. (MV*P1)*P2 is equal to zero.

$$\begin{pmatrix} 5 & 8 & 5 & 8 \\ 31/2 & 20 & 31/2 & 20 \\ 33 & 36 & 33 & 36 \\ 48 & 53 & 48 & 53 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & -1/2 & 0 \\ 0 & 1/2 & 0 & -1/2 \\ -1/2 & 0 & 1/2 & 0 \\ 0 & -1/2 & 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Pauli Matrices and Spin Basis

The Pauli sigma matrices encode three dimensional Euclidean geometric algebra with a slightly capricious basis.

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\sigma_{xy} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \sigma_{xz} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \sigma_{yz} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \sigma_{xyz} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

The two idempotents $(1+\sigma_z)/2$, $(1+\sigma_z)/2$ and two nilpotents $\sigma_x(1+\sigma_z)/2$, $\sigma_x(1-\sigma_z)/2$, along with the use of complex coefficients, spans the space of complex 2x2 matrices. Any complex 2x2 matrix can be represented as the complex weighted sum of these four matrices.

$$\begin{aligned} (1+\sigma_z)/2 &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = e_{\uparrow\uparrow} \\ (1-\sigma_z)/2 &= \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) = e_{\downarrow\downarrow} \\ \sigma_x(1+\sigma_z)/2 &= \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) = e_{\downarrow\uparrow} \\ \sigma_x(1-\sigma_z)/2 &= \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = e_{\uparrow\downarrow} \end{aligned}$$

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = a e_{\uparrow\uparrow} + b e_{\uparrow\downarrow} + c e_{\downarrow\uparrow} + d e_{\downarrow\downarrow}$$

However, with regards to ideals, both $(1+\sigma_z)/2$ and $\sigma_x(1+\sigma_z)/2$ generate left ideals, and both $(1-\sigma_z)/2$ and $\sigma_x(1-\sigma_z)/2$ generate right ideals. We only have two ideals.

Conclusions

The row and column ideals include embedded projection idempotents which are responsible for the segmentation behavior of the ideal. In turn, these embedded projection idempotents can be used to test non-trivial expressions for group membership by forming the product with the idempotents, and noting the lack of change in group members.

References

[1] C. Furey. Towards a Unified Theory of Ideals. *arXiv:1002.1497v5 [hep-th]*, 2018.