The AGM Function

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Abstract

These are some informal notes on the Arithmetic Geometic Mean function, and the connection to elliptic integrals of the first and second kind. The primary references for this material are the entertaining article by Gert Almkvist and Bruce Berndt in Americal Mathematical Monthly, Vol. 95, No. 7 (Aug-Sept, 1988), pp. 585-608, and "The Simple AGM Pendulum" by Mark B. Villarino, as well as the standard wikipedia pages for elliptic integrals.

AGM History

Legendre (1784) seems to be first to describe and use the AGM process when calculating tables of elliptic integrals of the first kind.

Slightly later, Gauss (1791) independently developed the AGM. However, the bulk of Gauss work was not published, till after his death, in 1876. Gauss' work is very highly regarded.

Maxwell used the AGM process for evaluation of magnetic fields.

More recently Carlson (1971) has advocated and taught the use of the AGM process for computer evaluation of elliptic and other, related integrals and functions.

The AGM Algorithm with Code

The AGM process is a series of calculations, iterating upon the arithmetic and geometric means, rapidly converging on a single number.

Start with two numbers, a_0 and g_0. Form the sequence

$$\begin{aligned} a_{n+1} &= \frac{a+g}{2} \\ g_{n+1} &= \sqrt{a*g} \end{aligned}$$

When the absolute value of the difference between the pairs is small enough for your needs, you're done. As shown, a and g are symmetrical. In practice, we will order $a > g$, so that we can estimate the error (and calculate elliptic integrals of the first kind) using

$$c_n = \sqrt{a_n^2 - g_n^2}$$

The following code calculates the AGM of 1 and 1.0e6. It converges after nine iterations.

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>

int main(void)
{
    double long a[30], g[30], c[30];
    int i;
    double eps = 2.220446049250313e-16;

    for (i=0;i<30;i++) {    // clear arrays
        a[i] = 0.0;
        g[i] = 0.0;
        c[i] = 0.0;
    }

 // some initial values

    a[0] = 1.0e6;  // I assume a[0] > g[0]
    g[0] = 1.0;

// assumption check and correction, if necessary
```

```
    if (a[0] < g[0]) {
        c[0] = a[0];  // swap using currently unused c[0]
        a[0] = g[0];
        g[0] = c[0];
    }

    c[0] = sqrt(a[0]*a[0] - g[0]*g[0]);
    printf("%Lf  %Lf  %Lf \n",a[0], g[0], c[0]);

    for (i=1;i<30;i++) {    // open loop, do 30 iterations max
        a[i] = 0.5*(a[i-1] + g[i-1]);
        g[i] = sqrtl(a[i-1]*g[i-1]);
        c[i] = sqrtl(fabs(a[i]*a[i] - g[i]*g[i]));
        printf("%Lf  %Lf  %Lf \n",a[i], g[i], c[i]);
        if (fabsl(c[i]) < eps) break;
    }
    return 0;
}

/*
Results for 1.0e6, 1.0

1000000.000000       1.000000  999999.999999
 500000.500000    1000.000000  499999.500000
 250500.250000   22360.690955  249500.250000
 136430.470478   74842.225211  114069.779522
 105636.347844  101048.305266   30794.122634
 103342.326555  103316.861761    2294.021289
 103329.594158  103329.593373      12.732397
 103329.593766  103329.593766       0.000394
 103329.593766  103329.593766       0.000000

*/
```

Figure 1 provides a plot of $y = \text{AGM}(1,\text{x})$, as well as $y = \sqrt{x}$, $y = (1+x)/2$ and $y = x$.

The cyan and purple traces are the initial values of 1 and x used in the AGM. The green and blue traces are the first step algebraic and geometric means. The red trace is the converged AGM value. I need to increase the point density around zero to better represent the AGM in that region.

Applications of the AGM

My main interest currently is elliptic integrals and the AGM.

AMS55

The AGM is used with descending Landen transformations to quickly calculate $E(\phi, k)$, $F(\phi, k)$, $E(k)$, $F(k)$ and $Z(\phi, k) = E(\phi, k) - (E(k)/K(k))F(\phi, k)$ all concurrently. (Reference AMS55 p599)

AMS55 was a bit annoying to implement, due to the following details. Formula 17.6.8 presents the descending Landen transformation for the angle ϕ_n indirectly, via

$$\tan(\phi_{n+1} - \phi_n) \quad = \quad (b_n/a_n)\tan\phi_n, \quad \phi_0 = \phi$$

As the AGM converges, (b_n/a_n) approaches 1, and ϕ basically doubles on each step. My naive approach, which did not work, was to write

$$\phi_{n+1} \quad = \quad \phi_n + \tan^{-1}\left[\frac{b_n}{a_n}\tan(\phi_n)\right] \qquad \text{Wrong!}$$

The inverse tangent only returns the primary value in the range $-\pi/2..\pi/2$, and loses the bulk of the actual angle. Instead, a better method for implementing this formula is

$$\begin{aligned}\phi_{n+1} &= \phi_n + \tan^{-1}\left[\frac{b_n}{a_n}\tan(\phi_n)\right] \\ &= 2\phi_n + \left(\tan^{-1}\left[\frac{b_n}{a_n}\tan(\phi_n)\right] - \tan^{-1}\left[\tan(\phi_n)\right]\right)\end{aligned}$$

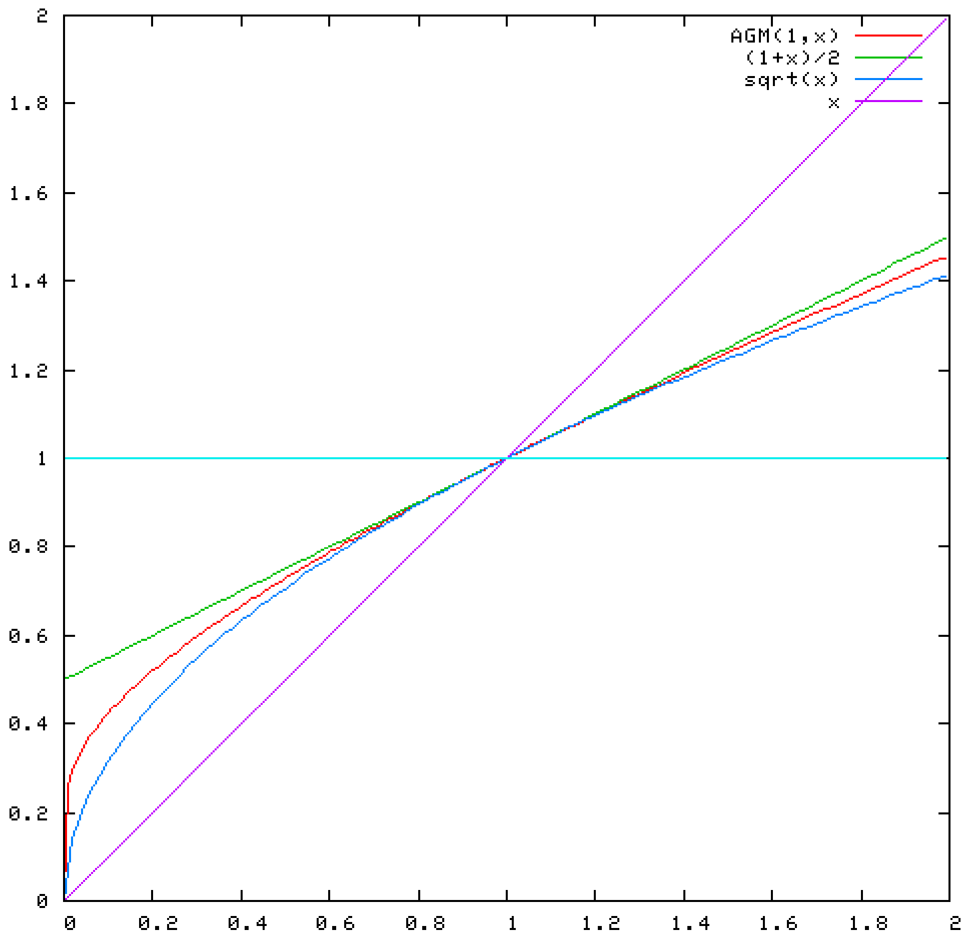

Figure 1: $y = \mathrm{AGM}(1, \mathrm{x})$, $y = \sqrt{x}$, $y = (1 + x)/2$ and $y = x$

We get our angular doubling, and the restricted range for the inverse tangents are no longer a problem.

Begin with the sequence

$$\begin{array}{llll}
a_0 = 1 & b_0 = \sqrt{1-k^2} & c_0 = k & \phi_0 = \phi \\
a_1 = \frac{a_0+b_0}{2} & b_1 = \sqrt{a_0 * b_0} & c_1 = \frac{a_0-b_0}{2} & \phi_1 = 2\phi_0 + \tan^{-1}(\frac{b_0}{a_0}\tan(\phi_0)) - \tan^{-1}(\tan\phi_0) \\
a_2 = \frac{a_1+b_1}{2} & b_2 = \sqrt{a_1 * b_1} & c_2 = \frac{a_1-b_1}{2} & \phi_2 = 2\phi_1 + \tan^{-1}(\frac{b_1}{a_1}\tan(\phi_1)) - \tan^{-1}(\tan\phi_1) \\
a_3 = \frac{a_2+b_2}{2} & b_3 = \sqrt{a_2 * b_2} & c_3 = \frac{a_2-b_2}{2} & \phi_3 = 2\phi_2 + \tan^{-1}(\frac{b_2}{a_2}\tan(\phi_2)) - \tan^{-1}(\tan\phi_2)
\end{array}$$

When the error measured by c_n is low enough, calculate

$$\begin{aligned}
F(\phi,k) &= \frac{\phi_n}{2^n a_n} \\
K(k) &= \frac{\pi}{2a_n} \\
E(k) &= K(k) - \frac{K(k)}{2}\left(2^0c_0^2 + 2^1c_1^2 + 2^2c_2^2 + 2^3c_3^2 + ...\right) \\
Z(\phi,k) &= c_1\sin(\phi_1) + c_2\sin(\phi_2) + ... \\
E(\phi,k) &= Z(\phi,k) + \frac{E(k)}{K(k)}F(\phi,k)
\end{aligned}$$

Richard Hall of MyMathLib.com

Mr. Hall uses a descending Landen transform with the AGM, but deals with the doubling angle problem by separating out the factors of two from the angle ϕ, using $\Phi_n = 2^n\phi_n$ in their formulas instead. (I think Mr. Hall's code is based upon Carlson's 1970's paper, which I've misplaced.)

In the lines which follow, I will use the notation $\Phi_n = 2^n\phi_n$ to describe these related angle. Φ nearly doubles each step, while ϕ will stay bound between $\pm\pi/2$.

Start with

$$\begin{aligned}
\tan(\Phi_{n+1} - \Phi_n) &= \frac{b_n}{a_n}\tan(\Phi_n) \\
\Phi_{n+1} &= \Phi_n + \tan^{-1}\left[\frac{b_n}{a_n}\tan(\Phi_n)\right] \\
&= 2\Phi_n - \tan^{-1}\left[\tan(\Phi_n)\right] + \tan^{-1}\left[\frac{b_n}{a_n}\tan(\Phi_n)\right]
\end{aligned}$$

Using the tangent difference of angles formula in reverse, we have the reference formula

$$\tan^{-1}(x) - \tan^{-1}(y) \quad = \quad \tan^{-1}\left[\frac{x-y}{1+xy}\right]$$

So,

$$\begin{aligned}
\tan(\Phi_{n+1} - \Phi_n) &= \frac{b_n}{a_n}\tan(\Phi_n) \\
\Phi_{n+1} &= \Phi_n + \tan^{-1}\left[\frac{b_n}{a_n}\tan(\Phi_n)\right] \\
&= 2\Phi_n - \tan^{-1}\left[\tan(\Phi_n)\right] + \tan^{-1}\left[\frac{b_n}{a_n}\tan(\Phi_n)\right] \\
&= 2\Phi_n - \left(\tan^{-1}\left[\tan(\Phi_n)\right] - \tan^{-1}\left[\frac{b_n}{a_n}\tan(\Phi_n)\right]\right) \\
&= 2\Phi_n - \tan^{-1}\left[\frac{\tan(\Phi_n) - (b_n/a_n)\tan(\Phi_n)}{1 + (b_n/a_n)\tan^2\Phi_n}\right] \\
&= 2\Phi_n - \tan^{-1}\left[\frac{(a_n - b_n)\tan(\Phi_n)}{a_n + b_n\tan^2\Phi_n}\right]
\end{aligned}$$

Now we change variables to use ϕ.

$$\begin{aligned}
2^{n+1}\phi_{n+1} &= 2\cdot 2^n\phi_n - \tan^{-1}\left[\frac{(a_n - b_n)\tan(2^n\phi_n)}{a_n + b_n\tan^2(2^n\phi_n)}\right] \\
\phi_{n+1} &= \phi_n - 2^{-(n+1)}\tan^{-1}\left[\frac{(a_n - b_n)\tan(2^n\phi_n)}{a_n + b_n\tan^2(2^n\phi_n)}\right]
\end{aligned}$$

Begin with the sequence

$a_0 = 1$	$b_0 = \sqrt{1-k^2}$	$c_0 = k$	$S_0 = 4 - 2k^2$	$I_0 = 0$
$a_1 = \frac{a_0+b_0}{2}$	$b_1 = \sqrt{a_1 * b_1}$	$c_1 = \frac{a_0-b_0}{2}$	$S_1 = S_0 - 2^1c_0^2$	$I_1 = I_0 + c_0\sin(2^1\phi_1)$
$a_2 = \frac{a_1+b_1}{2}$	$b_2 = \sqrt{a_2 * b_2}$	$c_2 = \frac{a_1-b_1}{2}$	$S_2 = S_1 - 2^2c_1^2$	$I_2 = I_1 + c_1\sin(2^2\phi_2)$
$a_3 = \frac{a_2+b_2}{2}$	$b_3 = \sqrt{a_3 * b_3}$	$c_3 = \frac{a_2-b_2}{2}$	$S_3 = S_2 - 2^3c_2^2$	$I_3 = I_2 + c_2\sin(2^3\phi_3)$

When the error measured by c_n is low enough, calculate

$$\begin{aligned}
F(\phi, k) &= \frac{\phi_n}{b_n} \\
K(k) &= \frac{\pi}{2b_n} \\
E(\phi, k) &= \frac{I_n}{2} + \frac{S_n F(\phi, k)}{4} \\
E(k) &= \frac{\pi S_n}{8a_n}
\end{aligned}$$

Code Demonstration for the Two Transforms Above

Here is a small program showing these calculations. http://www.kurtnalty.com/Elliptic_Demos_AGM.c

```
#include <stdio.h>
#include <stdlib.h>
#include "Consolidated_Elliptics.c"

int main(void)
{

    double a[30], b[30], c[30], k, m, phi[30], theta[30];
    double S, I, F, Fk, Ek, E, Z, two_n, tan2nphi;
    int i;

    k = 0.8;
    m = k*k;
    phi[0] = M_PI_4;    // 45 degres

    Landen_Transform( phi[0], m, &F, &Fk, &E, &Ek);
    printf("Reference Values: \n");
    printf("F = %f,   Fk = %f,   E = %f,   Ek = %f \n\n", F, Fk, E, Ek);

//*********************** AMS55 Approach ***************************/
```

```
    a[0] = 1.0;
    b[0] = sqrt(1.0 - k*k);
    c[0] = k;
    phi[0] = M_PI_4;
    S = c[0]*c[0];    two_n = 1.0;    Z = 0.0;
    i = 0;
    printf("\n\n\na = %f   b = %f   c = %f   phi = %f   \n\n",
                                    a[i], b[i], c[i], phi[i]);

    double delta_phi;
    for (i=1;i<8;i++) {
        a[i] = 0.5*(a[i-1] + b[i-1]);
        b[i] = sqrt(a[i-1]*b[i-1]);
        c[i] = 0.5*(a[i-1] - b[i-1]);
        phi[i] = 2.0*phi[i-1] + atan(tan(phi[i-1])*b[i-1]/a[i-1])
                      - atan(tan(phi[i-1])); // This works well

        two_n *= 2.0;
        S += two_n*c[i]*c[i];     // good
        Z += c[i]*sin(phi[i]);    // good
        F = phi[i]/(two_n*a[i]);  // phi[i]/two_n goes fairly constant
        Fk = M_PI/(2.0*a[i]);     // good
        Ek = Fk - 0.5*Fk*S;       // good
        E = Z + ((Ek)/(Fk))*F;

        printf("F = %f,   Fk = %f,   E = %f,   Ek = %f   ", F, Fk, E, Ek);
        printf("delta_phi = %f  phi = %f  phi/2^n = %f   \n",
                              delta_phi, phi[i], phi[i]/two_n);
    }


//********************** Richard Hall Approach *************************/

    a[0] = 1.0;
    b[0] = sqrt(1.0 - k*k);
    c[0] = k;
    phi[0] = M_PI_4;
```

```
    S = 4.0 - 2.0*k*k;
    two_n = 1.0;
    Z = 0.0;
    I = 0.0;
    i = 0;
    printf("\n\n\na = %f   b = %f   c = %f   phi = %f   \n\n",
                                             a[i], b[i], c[i], phi[i]);

    for (i=1;i<6;i++) {
        a[i] = 0.5*(a[i-1] + b[i-1]);
        b[i] = sqrt(a[i-1]*b[i-1]);
        c[i] = 0.5*(a[i] - b[i]);

        tan2nphi = tan(two_n*phi[i-1]);
        phi[i] = phi[i-1] -
             0.5*atan(((a[i-1] - b[i-1])*tan2nphi)/
         (a[i-1] + b[i-1]*tan2nphi*tan2nphi))/two_n;
        two_n *= 2.0;

        S -= 0.5*two_n*(a[i-1] - b[i-1])*(a[i-1] - b[i-1]);
        I += (a[i-1] - b[i-1])*sin(two_n*phi[i]);
        Z += c[i]*sin(phi[i]);
        F = phi[i]/(b[i]);
        Fk = M_PI/(2.0*a[i]);
        Ek = PI*S/(8.0*a[i]);           //  Fk - 0.5*Fk*S;
        E = I*0.5 + 0.25*S*F;   //Z + ((Ek)/(Fk))*F;

        printf("F = %f,   Fk = %f,   E = %f,   Ek = %f   ", F, Fk, E, Ek);
        printf("S = %f   I = %f   phi = %f   \n", S, I, phi[i]);
    }

    return (0);
}
```

Reference Values:
F = 0.839622,   Fk = 1.995303,   E = 0.737136,   Ek = 1.276350

AMS55 Calculation

```
a = 1.000000   b = 0.600000   c = 0.800000   phi = 0.785398

F = 0.828636,   Fk = 1.963495,   E = 0.724356,   Ek = 1.256637
    delta_phi =  0.540420  phi =  1.325818  phi/2^n = 0.662909
F = 0.839559,   Fk = 1.995173,   E = 0.737139,   Ek = 1.276267
    delta_phi =  1.318116  phi =  2.643934  phi/2^n = 0.660983
F = 0.839622,   Fk = 1.995303,   E = 0.737136,   Ek = 1.276350
    delta_phi = -0.497604  phi =  5.287922  phi/2^n = 0.660990
F = 0.839622,   Fk = 1.995303,   E = 0.737136,   Ek = 1.276350
    delta_phi = -0.995263  phi = 10.575844  phi/2^n = 0.660990
F = 0.839622,   Fk = 1.995303,   E = 0.737136,   Ek = 1.276350
    delta_phi =  1.151066  phi = 21.151688  phi/2^n = 0.660990
F = 0.839622,   Fk = 1.995303,   E = 0.737136,   Ek = 1.276350
    delta_phi = -0.839460  phi = 42.303377  phi/2^n = 0.660990
F = 0.839622,   Fk = 1.995303,   E = 0.737136,   Ek = 1.276350
    delta_phi =  1.462672  phi = 84.606753  phi/2^n = 0.660990


Richard Hall Calculation

a = 1.000000   b = 0.600000   c = 0.800000   phi = 0.785398

F = 0.855812,   Fk = 1.963495,   E = 0.741748,   Ek = 1.256637
    S = 2.560000   I = 0.388057   phi = 0.662909
F = 0.839668,   Fk = 1.995173,   E = 0.737209,   Ek = 1.276267
    S = 2.558709   I = 0.400184   phi = 0.660983
F = 0.839622,   Fk = 1.995303,   E = 0.737136,   Ek = 1.276350
    S = 2.558709   I = 0.400098   phi = 0.660990
F = 0.839622,   Fk = 1.995303,   E = 0.737136,   Ek = 1.276350
    S = 2.558709   I = 0.400098   phi = 0.660990
F = 0.839622,   Fk = 1.995303,   E = 0.737136,   Ek = 1.276350
    S = 2.558709   I = 0.400098   phi = 0.660990


Both converge after 3 iterations.
```

Integral Representations of the AGM

$$
\begin{aligned}
\frac{1}{M(a,b)} &= \frac{2}{\pi}\int_0^{\pi/2} \frac{d\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} \\
M(a,b) &= aM(1,b/a) \\
M(1,k) &= \frac{\pi}{2K\left(\sqrt{1-k^2}\right)}
\end{aligned}
$$

Gauss Transform in the AGM

Gauss presents the integral form of the AGM.

$$
\frac{1}{M(a,b)} = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}
$$

and presents a change of variable

$$
\sin\theta = \frac{2a\sin\phi}{(a+b)+(a-b)\sin^2\phi}
$$

and asserts

$$
\begin{aligned}
\frac{1}{M(a,b)} &= \frac{2}{\pi}\int_0^{\pi/2} \frac{d\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{a_1^2\cos^2\theta + b_1^2\sin^2\theta}} \\
&\text{where} \\
a_1 &= \frac{a+b}{2} \quad \text{and} \quad b_1 = \sqrt{ab}
\end{aligned}
$$

To see this development, we show

$$\begin{aligned}
\sin\theta &= \frac{2a\sin\phi}{(a+b)+(a-b)\sin^2\phi} \\
\sin^2\theta &= \frac{(2a\sin\phi)^2}{\left((a+b)+(a-b)\sin^2\phi\right)^2} \\
\cos^2\theta &= 1-\frac{(2a\sin\phi)^2}{((a+b)+(a-b)\sin\phi^2)^2} \\
&= \frac{(a+b)^2+2(a^2-b^2)\sin^2\phi+(a-b)^2\sin^4\phi-4a^2\sin^2\phi}{((a+b)+(a-b)\sin\phi^2)^2} \\
&= \frac{(a+b)^2-2(a^2+b^2)\sin^2\phi+(a-b)^2\sin^4\phi}{((a+b)+(a-b)\sin\phi^2)^2} \\
\cos\theta &= \frac{\sqrt{(a+b)^2-2(a^2+b^2)\sin^2\phi+(a-b)^2\sin^4\phi}}{((a+b)+(a-b)\sin\phi^2)}
\end{aligned}$$

We find

$$\begin{aligned}
a^2\cos^2\theta+b^2\sin^2\theta &= \frac{a^2(a+b)^2-2a^2(a^2+b^2)\sin^2\phi+a^2(a-b)^2\sin^4\phi+4a^2b^2\sin^2\phi}{((a+b)+(a-b)\sin\phi^2)^2} \\
&= a^2\frac{(a+b)^2-2(a^2-b^2)\sin^2\phi+(a-b)^2\sin^4\phi}{((a+b)+(a-b)\sin\phi^2)^2} \\
&= a^2\left[\frac{(a+b)-(a-b)\sin^2\phi}{(a+b)+(a-b)\sin^2\phi}\right]^2 \\
\frac{1}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}} &= \frac{1}{a}\left[\frac{(a+b)+(a-b)\sin^2\phi}{(a+b)-(a-b)\sin^2\phi}\right]
\end{aligned}$$

Now, for the differential,

$$\begin{aligned}
\sin\theta &= \frac{2a\sin\phi}{(a+b)+(a-b)\sin^2\phi} \\
\cos\theta d\theta &= \left[\frac{2a\cos\phi}{(a+b)+(a-b)\sin^2\phi}-\frac{2a\sin\phi}{\left((a+b)+(a-b)\sin^2\phi\right)^2}(a-b)2\sin\phi\cos\phi\right]d\phi \\
&= \left[\frac{(2a\cos\phi)\left((a+b)+(a-b)\sin^2\phi\right)-(2a\sin\phi)(a-b)2\sin\phi\cos\phi}{\left((a+b)+(a-b)\sin^2\phi\right)^2}\right]d\phi \\
&= 2a\cos\phi\left[\frac{(a+b)-(a-b)\sin^2\phi}{\left((a+b)+(a-b)\sin^2\phi\right)^2}\right]d\phi
\end{aligned}$$

We have

$$\begin{aligned}
\frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}} &= \frac{1}{a}\left[\frac{(a+b)+(a-b)\sin^2\phi}{(a+b)-(a-b)\sin^2\phi}\right]\frac{2a\cos\phi}{\cos\theta}\left[\frac{(a+b)-(a-b)\sin^2\phi}{\left((a+b)+(a-b)\sin^2\phi\right)^2}\right]d\phi \\
&= \frac{1}{\cos\theta}\frac{2\cos\phi d\phi}{(a+b)+(a-b)\sin^2\phi} \\
&= \frac{((a+b)+(a-b)\sin\phi^2)}{\sqrt{(a+b)^2-2(a^2+b^2)\sin^2\phi+(a-b)^2\sin^4\phi}}\frac{2\cos\phi d\phi}{(a+b)+(a-b)\sin^2\phi} \\
&= \frac{2\cos\phi d\phi}{\sqrt{(a+b)^2-2(a^2+b^2)\sin^2\phi+(a-b)^2\sin^4\phi}} \\
&= \frac{2\cos\phi d\phi}{\sqrt{\left(1-\sin^2\phi\right)\left((a+b)^2-(a-b)^2\sin^2\phi\right)}} \\
&= \frac{2d\phi}{\sqrt{(a+b)^2-(a-b)^2\sin^2\phi}} \\
&= \frac{d\phi}{\sqrt{\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2\sin^2\phi}} \\
&= \frac{d\phi}{\sqrt{\left(\frac{a+b}{2}\right)^2\cos^2\phi+\left(\frac{a+b}{2}\right)^2\sin^2\phi-\left(\frac{a-b}{2}\right)^2\sin^2\phi}} \\
&= \frac{d\phi}{\sqrt{\left(\frac{a+b}{2}\right)^2\cos^2\phi+ab\sin^2\phi}}
\end{aligned}$$

We thus see our desired relationship.

$$\frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}} = \frac{d\phi}{\sqrt{\left(\frac{a+b}{2}\right)^2\cos^2\phi+ab\sin^2\phi}}$$

When we do integrals from 0 to $\pi/2$, the limits for θ and ϕ agree. For other values, such as incomplete integrals akin to $F(\phi,k)$, we will need to use the formula relating θ and ϕ from above.

$$\theta = \sin^{-1}\left[\frac{2a\sin\phi}{(a+b)+(a-b)\sin^2\phi}\right]$$

Useful AGM Sequence Relationships

Some useful relationships, for future reference, are

$$\begin{aligned}
a + b &= 2a_1 \\
a_1^2 - b_1^2 &= \frac{a^2}{4} + \frac{ab}{2} + \frac{b^2}{4} - ab = \frac{a^2 - b^2}{4} \\
a - b &= 2\sqrt{a_1^2 - b_2^1} \\
a^2 - b^2 &= 4a_1\sqrt{a_1^2 - b_1^2} \\
a &= a_1 + \sqrt{a_1^2 - b_1^2} \\
b &= a_1 - \sqrt{a_1^2 - b_1^2} \\
a^2 + b^2 &= 4a_1^2 - 2b_1^2
\end{aligned}$$

Here is a table for AGM two steps backward to two steps forward.

A	B	n
$\left(\sqrt{a + \sqrt{a^2 - b^2}} + \sqrt{2\sqrt{a}\sqrt[4]{a^2 - b^2}}\right)^2$	$\left(\sqrt{a + \sqrt{a^2 - b^2}} - \sqrt{2\sqrt{a}\sqrt[4]{a^2 - b^2}}\right)^2$	-3
$\left(\sqrt{a} + \sqrt[4]{a^2 - b^2}\right)^2$	$\left(\sqrt{a} - \sqrt[4]{a^2 - b^2}\right)^2$	-2
$a + \sqrt{a^2 - b^2}$	$a - \sqrt{a^2 - b^2}$	-1
a	b	0
$\frac{a+b}{2}$	$\sqrt{ab}$	
$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2$	$\sqrt{\left(\frac{a+b}{2}\right)\sqrt{ab}}$	2
$\left(\frac{\sqrt{\frac{a+b}{2}} + \sqrt[4]{ab}}{2}\right)^2$	$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\sqrt[4]{\left(\frac{a+b}{2}\right)\sqrt{ab}}$	3

Gauss Versus Landen Transformations

The Gauss transform and Landen transform both use the same formula for the modulus $k = (1 - k')/(1 + k')$, but differ in their choice for the angle. Figure 2, 3 and 4 show the relative angles for different values of k. We clearly see the double frequency behavior of the Landen transform, and the same period behavior of the Gauss transform.

Code used to generate these plots follows.

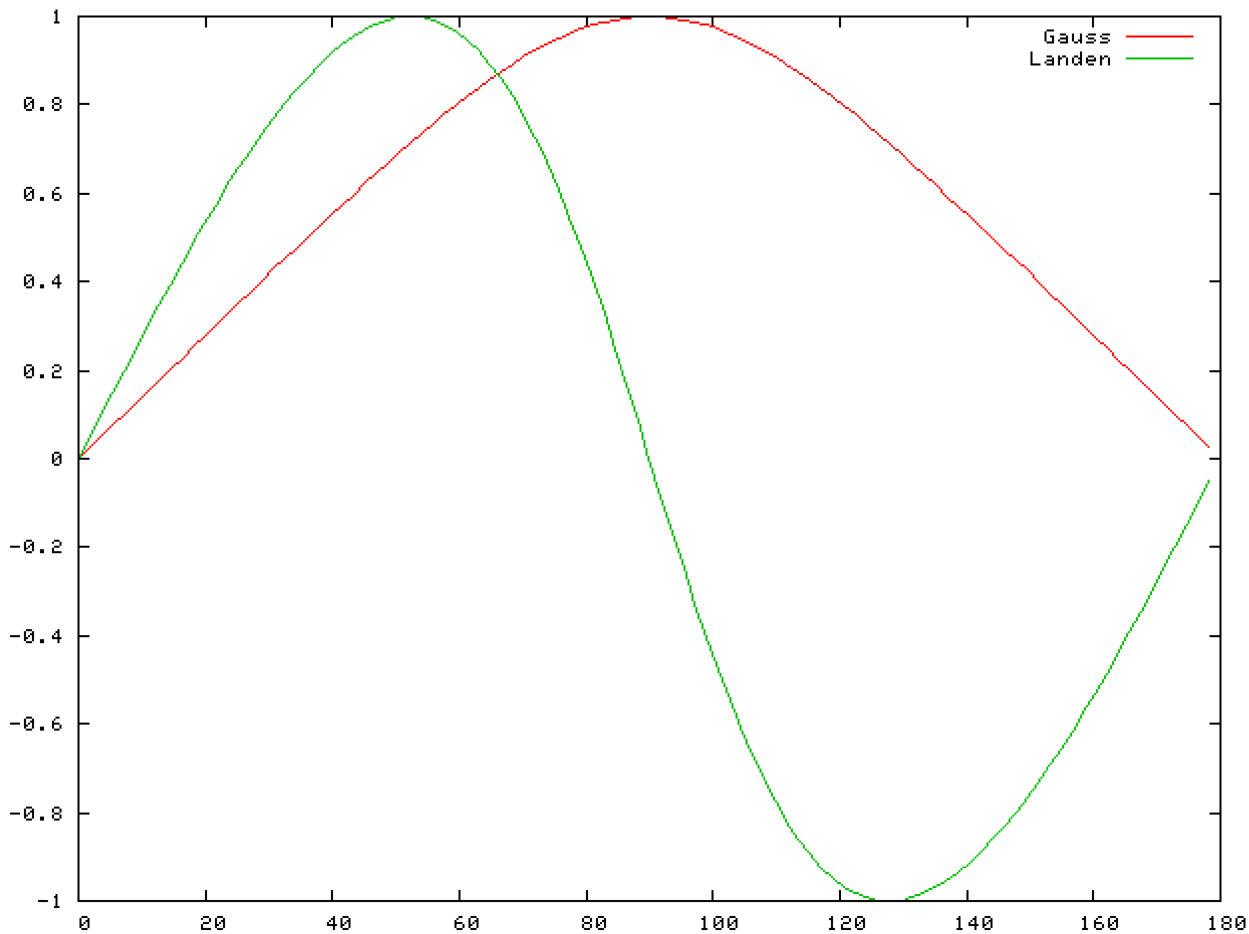

Figure 2: Landen and Gauss versus original angle, $k = 0$

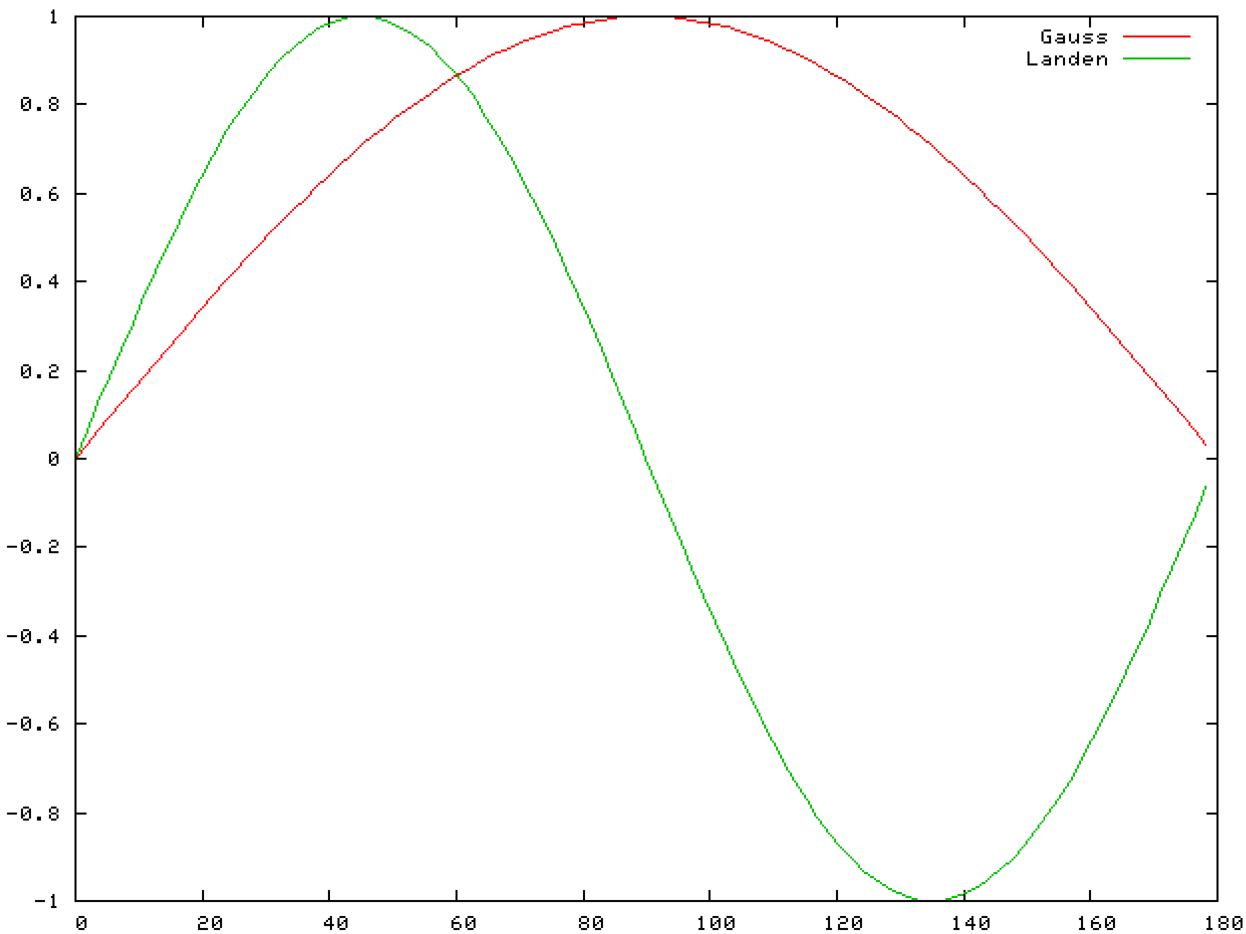

Figure 3: Landen and Gauss versus original angle, $k = 0.1$

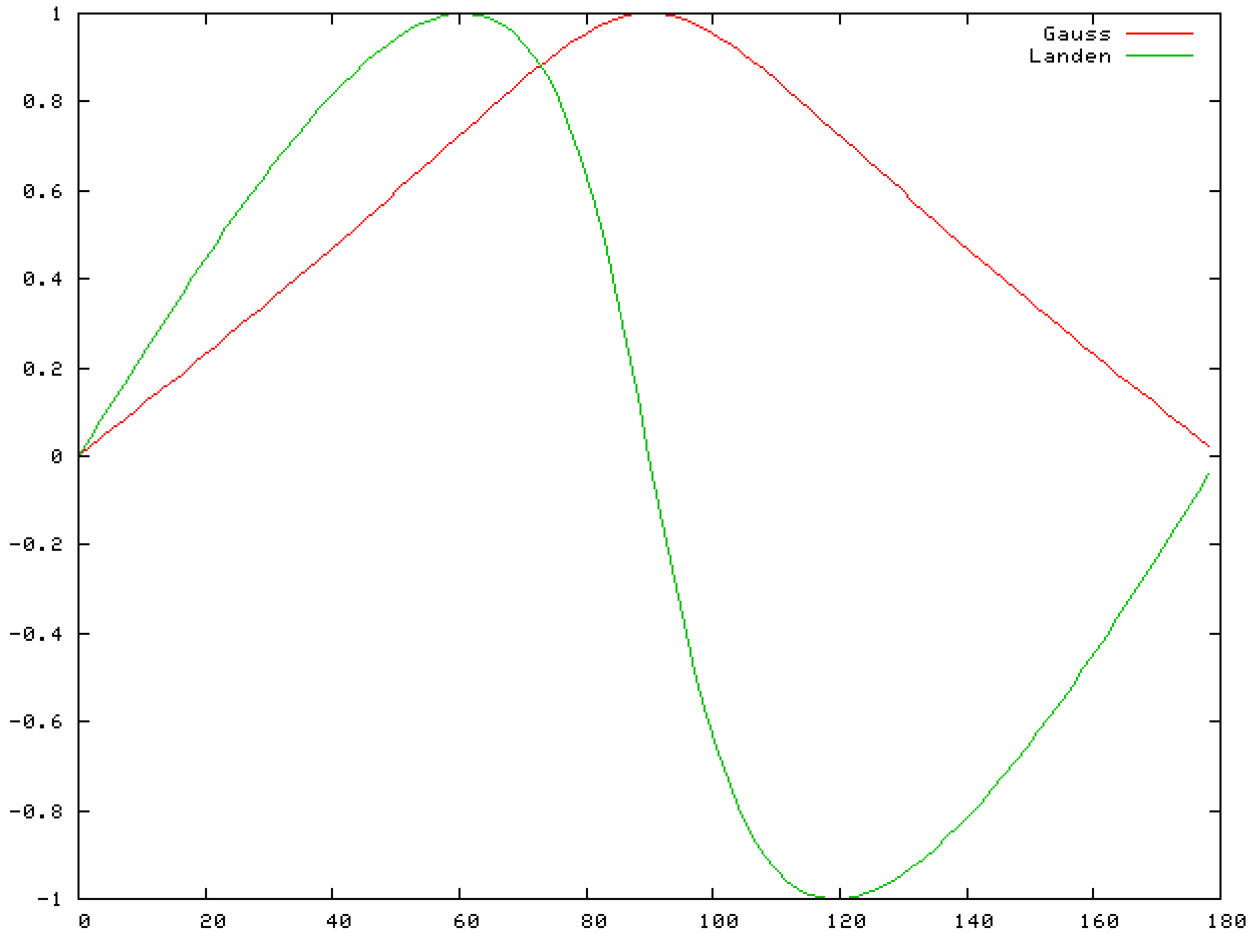

Figure 4: Landen and Gauss versus original angle, $k = 0.95$

```c
#include <stdio.h>
#include <math.h>

int main(void)
{
    int i,j;
    double k, kprime, k1, pi, phi, sin_theta, sin_psi;
    FILE* Output;

    pi = 3.1415926;
    Output = fopen("Gauss.txt","w");
    k = 0.1;
    kprime = sqrt(1.0 - k*k);
    k1 = (1.0 - kprime)/(1.0 + kprime);
    for (i=0;i<100;i++) {
        phi   = (pi*i*0.01);
        sin_psi   = (   ((1.0 + kprime)*sin(phi))          / (1.0 + sqrt(1.0 - k*k
        sin_theta = (   ((1.0 + kprime)*sin(phi)*cos(phi)) / (      sqrt(1.0 - k*k
        fprintf(Output,"%f  %f   %f  \n", phi*180.0/pi, sin_psi, sin_theta);
    }

    fclose(Output);
    return 0;
}
```

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