3D Euclidean Geometric Algebra Matrix Representation

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Abstract

I investigate 3D geometric algebra matrix representation. From the multiplication properties of determinants, 3x3 matrices cannot represent bivectors (which square to negative values), and so I look at 4x4 (16 component) matrices, knowing that these will only be half-utilized in representing 3D (8 component) multivectors.

3D Euclidean Symbolic Geometric Algebra

Three dimensional Euclidean geometrical algebra has a scalar (1), three vectors (e_x, e_y and e_z), three bivectors (e_xe_y, e_ze_x, and e_ye_z), and one trivector ($e_xe_ye_z$) defining the geometry. In multiplication table format, the order-sensitive multiplication among these elements is

	1	e_x	e_y	e_z	e_xe_y	e_ze_x	e_ye_z	$e_xe_ye_z$
1	1	e_x	e_y	e_z	e_xe_y	e_ze_x	e_ye_z	$e_xe_ye_z$
e_x	e_x	1	e_xe_y	-e_ze_x	e_y	$-e_z$	$e_xe_ye_z$	e_ye_z
e_y	e_y	$-e_xe_y$	1	e_ye_z	$-e_x$	$e_xe_ye_z$	e_z	e_ze_x
e_z	e_z	e_ze_x	$-e_ye_z$	1	$e_xe_ye_z$	e_x	$-e_y$	e_xe_y
e_xe_y	e_xe_y	$-e_y$	e_x	$e_xe_ye_z$	-1	e_ye_z	$-e_ze_x$	$-e_z$
e_ze_x	e_ze_x	e_z	$e_xe_ye_z$	$-e_x$	$-e_ye_z$	-1	e_xe_y	$-e_y$
e_ye_z	e_ye_z	$e_xe_ye_z$	$-e_z$	e_y	e_ze_x	$-e_xe_y$	-1	$-e_x$
$e_xe_ye_z$	$e_xe_ye_z$	e_ye_z	e_ze_x	e_xe_y	$-e_z$	$-e_y$	$-e_x$	-1

In this algebra, scalar multiplication is commutative and associative, basis vectors square to scalar one, and the product of two vectors resulting in a bivector is anti-commutative, associative, squares to negative one, and trivector basis commute with everything, yet square to negative one.

Matrix Representation

An associative algebra, matrix multiplication can provide a faithful representation for a geometric algebra. Looking at our basis set, we have eight basii, and would initially expect 3x3 matrices to be adequate for the representation, with one redundant slot. However, we find initially experimentally, that 3x3 matrices cannot support unit bivectors, which square to negative one. Upon reflection, we note that the determinant of -1 in 3x3 space is -1, which cannot be achieved as the square of a real number. In general, odd number sized matrices cannot represent imaginary terms without imaginary entries. By contrast, even number size matrices can, as the determinant of the negative unit matrix is positive 1, resulting from the square of a matrix with determinant -1.

Given all that, to represent our eight multivector basis will require 4x4 matrices, whose 16 entries are twice the required number to represent our eight basis. Consequently, I expect multiple, redundant representations.

Methodolgy

To identify vectors, which square to positive one, I wrote a program treating each cell of the 4x4 matrix as a location which could be -1, 0 or +1. In effect, this defines each possible matrix as a sixteen digit trinary number. In this fashion, I identified 5436 matrices which square to one. To be more selective, I then further required each candidate have a non-zero determinant, and twelve zeroes. This reduced the vector list to 76, but looking at the list, there were a number of entries with trace 2 or -2, which would not be orthogonal to the unit vector. Consequently, I further restricted the list by requiring trace = 0, which lead to the short list of 42 vector candidates. (Douglas Adams fans should take note.) Of these 42 entries, half are negatives of the other half, so we really have 21 unique entries.

Next, I formed all 21 x 21 products, looking for antisymmetric pairs whose product squared to negative one. These products can be either bivectors or trivectors, depending upon the context of the basis vectors chosen. Examing the feeders for each bivector/trivector candidate, I identified six sets of positive representations for 3D Euclidean Clifford algebras. Due to the fact that each vector can be multiplied by -1 and still be a faithful representation, we have 48 unique sets available. I am only showing the postive set, leaving the $\pm x$, $\pm y$ and $\pm z$ variations for the reader.

Set 1

```
       1               x               y               z
[ 1  0  0  0 ]  [ 0 -1  0  0 ]  [-1  0  0  0 ]  [ 0  0  0  1 ]
[ 0  1  0  0 ]  [-1  0  0  0 ]  [ 0  1  0  0 ]  [ 0  0  1  0 ]
[ 0  0  1  0 ]  [ 0  0  0  1 ]  [ 0  0 -1  0 ]  [ 0  1  0  0 ]
[ 0  0  0  1 ]  [ 0  0  1  0 ]  [ 0  0  0  1 ]  [ 1  0  0  0 ]

      xy              zx              yz              xyz
[ 0 -1  0  0 ]  [ 0  0  1  0 ]  [ 0  0  0 -1 ]  [ 0  0 -1  0 ]
[ 1  0  0  0 ]  [ 0  0  0  1 ]  [ 0  0  1  0 ]  [ 0  0  0  1 ]
[ 0  0  0  1 ]  [-1  0  0  0 ]  [ 0 -1  0  0 ]  [ 1  0  0  0 ]
[ 0  0 -1  0 ]  [ 0 -1  0  0 ]  [ 1  0  0  0 ]  [ 0 -1  0  0 ]
```

Set 2

```
       1               x               y               z
[ 1  0  0  0 ]  [ 1  0  0  0 ]  [ 0  1  0  0 ]  [ 0  0 -1  0 ]
[ 0  1  0  0 ]  [ 0 -1  0  0 ]  [ 1  0  0  0 ]  [ 0  0  0  1 ]
[ 0  0  1  0 ]  [ 0  0 -1  0 ]  [ 0  0  0  1 ]  [-1  0  0  0 ]
[ 0  0  0  1 ]  [ 0  0  0  1 ]  [ 0  0  1  0 ]  [ 0  1  0  0 ]

      xy              zx              yz              xyz
[ 0  1  0  0 ]  [ 0  0  1  0 ]  [ 0  0  0  1 ]  [ 0  0  0  1 ]
[-1  0  0  0 ]  [ 0  0  0  1 ]  [ 0  0 -1  0 ]  [ 0  0  1  0 ]
[ 0  0  0 -1 ]  [-1  0  0  0 ]  [ 0  1  0  0 ]  [ 0 -1  0  0 ]
[ 0  0  1  0 ]  [ 0 -1  0  0 ]  [-1  0  0  0 ]  [-1  0  0  0 ]
```

Set 3

```
      1             x             y             z
[ 1  0  0  0 ] [ 0  0  1  0 ] [ 0  0  0  1 ] [-1  0  0  0 ]
[ 0  1  0  0 ] [ 0  0  0  1 ] [ 0  0 -1  0 ] [ 0 -1  0  0 ]
[ 0  0  1  0 ] [ 1  0  0  0 ] [ 0 -1  0  0 ] [ 0  0  1  0 ]
[ 0  0  0  1 ] [ 0  1  0  0 ] [ 1  0  0  0 ] [ 0  0  0  1 ]

     xy            zx            yz            xyz
[ 0 -1  0  0 ] [ 0  0 -1  0 ] [ 0  0  0  1 ] [ 0  1  0  0 ]
[ 1  0  0  0 ] [ 0  0  0 -1 ] [ 0  0 -1  0 ] [   -1  0  0  0 ]
[ 0  0  0  1 ] [ 1  0  0  0 ] [ 0  1  0  0 ] [ 0  0  0  1 ]
[ 0  0 -1  0 ] [ 0  1  0  0 ] [-1  0  0  0 ] [ 0  0 -1  0 ]
```

Set 4

```
      1             x             y             z
[ 1  0  0  0 ] [ 0 -1  0  0 ] [ 1  0  0  0 ] [ 0  0  1  0 ]
[ 0  1  0  0 ] [-1  0  0  0 ] [ 0 -1  0  0 ] [ 0  0  0  1 ]
[ 0  0  1  0 ] [ 0  0  0  1 ] [ 0  0 -1  0 ] [ 1  0  0  0 ]
[ 0  0  0  1 ] [ 0  0  1  0 ] [ 0  0  0  1 ] [ 0  1  0  0 ]

     xy            zx            yz            xyz
[ 0  1  0  0 ] [ 0  0  0  1 ] [ 0  0  1  0 ] [ 0  0  0  1 ]
[-1  0  0  0 ] [ 0  0  1  0 ] [ 0  0  0 -1 ] [ 0  0 -1  0 ]
[ 0  0  0  1 ] [ 0 -1  0  0 ] [-1  0  0  0 ] [ 0  1  0  0 ]
[ 0  0 -1  0 ] [-1  0  0  0 ] [ 0  1  0  0 ] [-1  0  0  0 ]
```

Set 5

```
        1                x                y                z
[ 1  0  0  0 ]  [-1  0  0  0 ]  [ 0  1  0  0 ]  [ 0  0  0  1 ]
[ 0  1  0  0 ]  [ 0  1  0  0 ]  [ 1  0  0  0 ]  [ 0  0 -1  0 ]
[ 0  0  1  0 ]  [ 0  0 -1  0 ]  [ 0  0  0  1 ]  [ 0 -1  0  0 ]
[ 0  0  0  1 ]  [ 0  0  0  1 ]  [ 0  0  1  0 ]  [ 1  0  0  0 ]

       xy               zx               yz               xyz
[ 0 -1  0  0 ]  [ 0  0  0  1 ]  [ 0  0 -1  0 ]  [ 0  0  1  0 ]
[ 1  0  0  0 ]  [ 0  0  1  0 ]  [ 0  0  0  1 ]  [ 0  0  0  1 ]
[ 0  0  0 -1 ]  [ 0 -1  0  0 ]  [ 1  0  0  0 ]  [-1  0  0  0 ]
[ 0  0  1  0 ]  [-1  0  0  0 ]  [ 0 -1  0  0 ]  [ 0 -1  0  0 ]
```

Set 6

```
        1                x                y                z
[ 1  0  0  0 ]  [ 0  0  0  1 ]  [ 0  0 -1  0 ]  [-1  0  0  0 ]
[ 0  1  0  0 ]  [ 0  0  1  0 ]  [ 0  0  0  1 ]  [ 0 -1  0  0 ]
[ 0  0  1  0 ]  [ 0  1  0  0 ]  [-1  0  0  0 ]  [ 0  0  1  0 ]
[ 0  0  0  1 ]  [ 1  0  0  0 ]  [ 0  1  0  0 ]  [ 0  0  0  1 ]

       xy               zx               yz               xyz
[ 0  1  0  0 ]  [ 0  0  0 -1 ]  [ 0  0 -1  0 ]  [ 0 -1  0  0 ]
[-1  0  0  0 ]  [ 0  0 -1  0 ]  [ 0  0  0  1 ]  [ 1  0  0  0 ]
[ 0  0  0  1 ]  [ 0  1  0  0 ]  [ 1  0  0  0 ]  [ 0  0  0  1 ]
[ 0  0 -1  0 ]  [ 1  0  0  0 ]  [ 0 -1  0  0 ]  [ 0  0 -1  0 ]
```

Commentary

References

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