2D Euclidean Geometric Algebra Matrix Representation

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Abstract

I present the well-known matrix representation of 2D Euclidean Geometric Algebra, and suggest a literal geometric interpretation.

2D Euclidean Symbolic Geometric Algebra

Two dimensional Euclidean geometrical algebra has a scalar (1), two vectors (e_x and e_y) and one bivector ($e_x e_y$) defining the geometry. In multiplication table format, the order-sensitive multiplication among these elements is

	1	e_x	e_y	$e_x e_y$
1	1	e_x	e_y	$e_x e_y$
e_x	e_x	1	$e_x e_y$	e_y
e_y	e_y	$-e_x e_y$	1	$-e_x$
$e_x e_y$	$e_x e_y$	$-e_y$	e_x	-1

In this algebra, scalar multiplication is commutative and associative, vectors square to scalar one, and the product of two vectors resulting in a bivector is anti-commutative, associative and squares to negative one.

Matrix Representation

An associative algebra, matrix multiplication can provide a faithful representation for a geometric algebra. Two by two matrices can support four independent information elements. Restricting to matrices with elements of +1, 0 and -1, we find 14 matrices which square to unity, and two matrices which square to -1. Further restricting each matrix to have two zeroes, we get the following set of interesting, (and well known) matrix representation for the 2D Euclidean geometric algebras.

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$e_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$e_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$e_x e_y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Equally valid are the negatives of the above.

$$-1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$-e_x = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$-e_y = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$-e_x e_y = e_y e_x = (-e_y)(-e_x) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Within each of these basis sets, the mutual matrix dot product

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \cdot \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = (a_{00}b_{00} + a_{01}b_{01} + a_{10}b_{10} + a_{11}b_{11}) = 0,$$

demonstrating the linear independence of each basis from the other.

A general 2D multivector, $q + ae_x + be_y + ce_x e_y$ in matrix format becomes

$$\begin{bmatrix} q+a & b-c \\ b+c & q-a \end{bmatrix}$$

This matrix has trace $2q$ and determinant $q^2 - a^2 - b^2 + c^2$.

Dotting this matrix with each of the basis representation picks up twice the component values times the basis squared. The factor of two arises naturally from the 2x2 matrix representation. The negative factor on the bivector term reflects that the bivector basis squares to negative one.

$$\begin{bmatrix} q+a & b-c \\ b+c & q-a \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2q$$

$$\begin{bmatrix} q+a & b-c \\ b+c & q-a \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 2a$$

$$\begin{bmatrix} q+a & b-c \\ b+c & q-a \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2b$$

$$\begin{bmatrix} q+a & b-c \\ b+c & q-a \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -2c$$

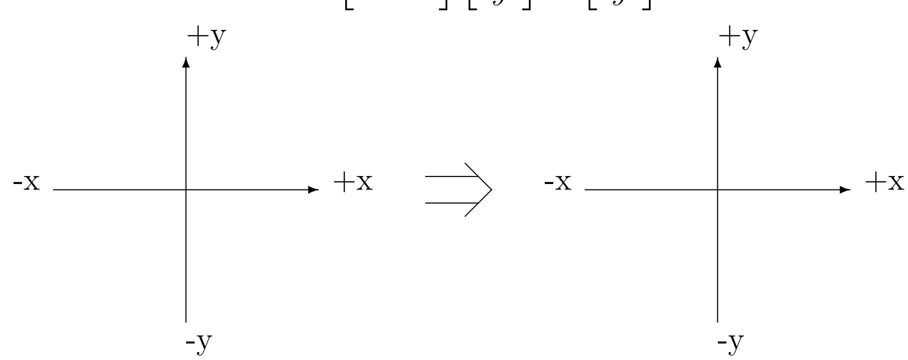
Literal Interpretation of the Basis Matrices

Usually, matrix representations are considered simply as a formal representation, with no intrinsic geometrical content. In the spirit of geometrical algebra, I want to look at the geometric transformational properties of the eight matrix representations shown above.

Positive Unity Matrix

The unity matrix leaves the orientation unchanged.

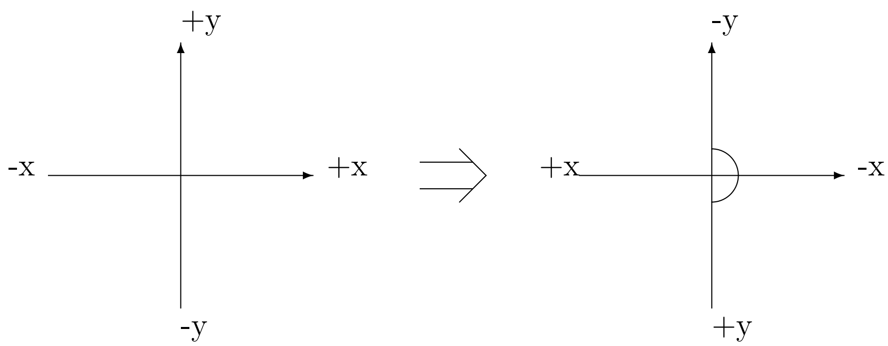
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$



Negative Unity Matrix

This matrix inverts both x and y axis. This is same effect as rotating 180 degrees about the z axis. Repeating this operation twice restores the original configuration. This operator thus squares to unity.

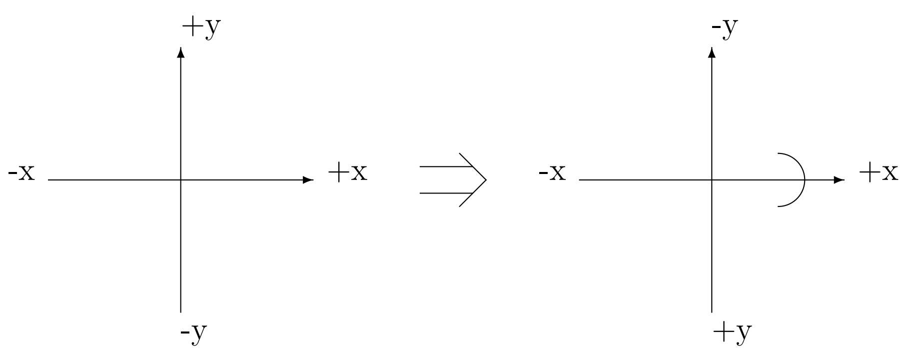
$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$



Positive e_x Matrix

This matrix inverts the y axis. This is same effect as rotating the plane 180 degrees around the x axis. Repeating this operation twice restores the original configuration. This operator thus squares to unity.

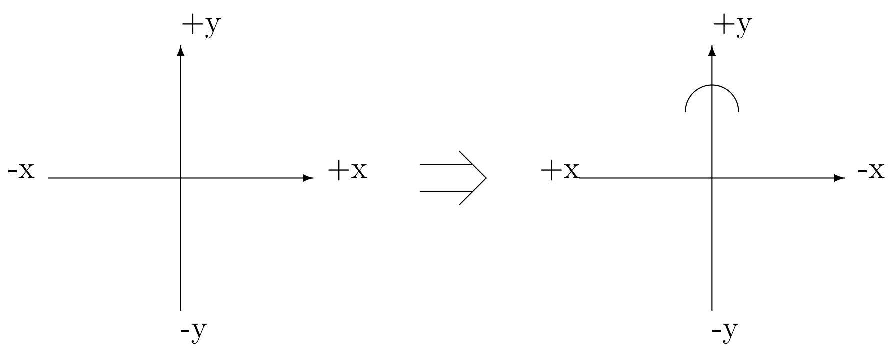
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$



Negative e_x Matrix

This matrix inverts the x axis. This is same effect as rotating the plane 180 degrees around the y axis. Notice that the minus sign on the matrix has changed which axis is rotated. Repeating this operation twice restores the original configuration. This operator thus squares to unity.

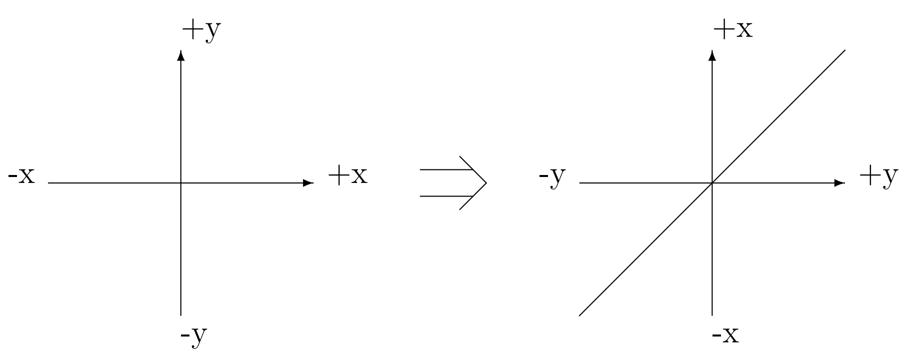
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$



Positive e_y Matrix

This matrix interchanges the x and y axii. This is same effect as rotating the plane 180 degrees around the 45 degree positive diagonal through the origin. Repeating this operation twice restores the original configuration. This operator thus squares to unity.

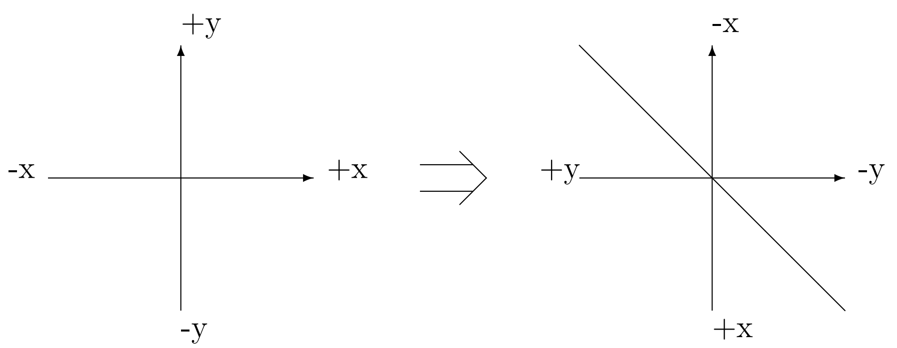
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$



Negative e_y Matrix

This matrix interchanges the x and $-y$ axii. This is same effect as rotating the plane 180 degrees around the negative 45 degree positive diagonal through the origin. Repeating this operation twice restores the original configuration. This operator thus squares to unity.

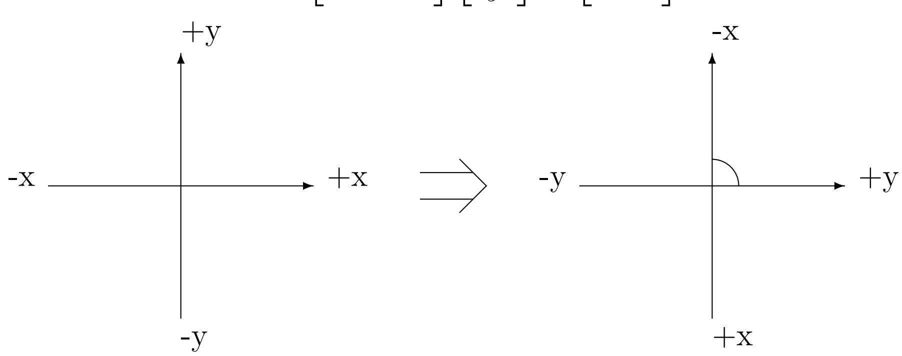
$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$$



Positive e_xe_y Matrix

This matrix first does the e_y transform, then the e_x. This is same effect as rotating the plane 90 degrees clockwise around the origin. Repeating this operation twice results in a rotation of 180 degrees about the origin, or equivalently, inversion through the origin. Thus this operator corresponds to $-i = -\sqrt{-1}$.

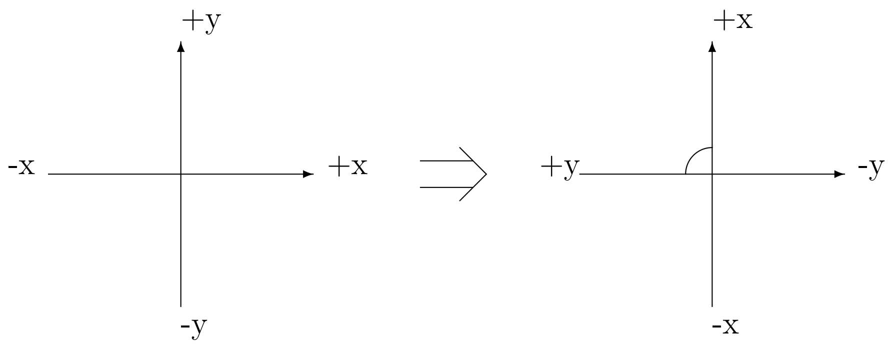
$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$$



Negative $e_x e_y$ Matrix

This matrix first does the e_x transform, then the e_y. This is same effect as rotating the plane 90 counter-clockwise degrees around the origin. Repeating this operation twice results in a rotation of 180 degrees about the origin, or equivalently, inversion through the origin. Thus this operator corresponds to $i = \sqrt{-1}$.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$



Commentary

So what exactly are the basis vectors 1, e_x, e_y and $e_x e_y$? These are mutually orthogonal decompositions for a non-translational two dimension linear transformation about the origin. Any non-translational 2D linear operation can be implemented using combinations of these terms.

So what exactly does a multivector do? A multivector implements a non-translational, two dimensional linear transformation of a point in the plane to another point in the plane using radial scaling (scalar term), reflections about x or y (e_x term), reflections about 45 degree diagonal (e_y term), and 90 degree rotations ($e_x e_y$ term) about the origin.

References

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[4] Anthony Lasenby and Chris Doran, *Lectures and Handouts 1999* `www.mrao.cam.ac.uk/clifford/ptIIIcourse/`